EXERCISES IN EXPERIMENTAL GEOMETRY

BY

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TRANSLATOR'S PREFACE

This remarkable booklet first appeared in 1931 under the title "Uebungensammlung zu einer geometrischen Propädeuse", in a charmingly eccentric form of German — a well-learnt foreign language to the author. In 1901, Tatyana Alexeyevna Afanasyeva and Paul Ehrenfest were studying mathematics at Göttingen, were married in Vienna three years later, and after several years in St.Petersburg moved to the Netherlands in 1912, to stay. These details are recalled only to explain the quality of Tatjana's German: lucid, even eloquent, but not quite standard. The translator sometimes had to guess, and it can only be hoped that he usually guessed right. Square brackets denote additions made by the translator in an effort to clarify meanings.

Klaus Hoechsmann, March 2003

AUTHOR'S PREFACE

The present exercises differ from those usually found in propaedeutic geometry books in one essential respect: they do *not* aim at visually and intuitively grasping the *theorems* of a systematic course, but at developping the most important geometric *notions* while acquiring a repertory of spatial images and mental representations, which is as large as possible. Therefore the *qualitative* aspect of spatial relations will be emphasised.

Creating this collection was guided by the conviction, that fostering imaginative ability by means of the pictorial proofs in the traditional preparatory courses was, on the one hand, insufficient – and on the other, unnecessary – unnecessary when the imagination is sufficiently developped by the time it encounters any theorems; and that the systematic course would be more valuable in every regard, if the fuzzy preliminaries in previous classes had not made it inevitably tedious.

For the rest, the following introductory sketch may throw some light on the author's position relative to the predominant ideas about teaching geometry, and explain her own conception of the most appropiate approach to the subject.

INTRODUCTION

§1. As is well known, there are two directions in teaching geometry. The first, rather old one, is based on the assumption that the fundamental notions and axioms of geometry are innate to human beings, and that it therefore suffices – starting there – to unfold the whole system of theorems and proofs for the students, in the spirit of Euclid.

The other direction – which has lately gained much ground – begins with the insight, that students often show a far greater understanding of what the theorems say if these are illustrated by concrete examples in real situations, and are more likely to believe in their correctness if they are corroborated by inspection or even measurement – rather than being demonstrated by a rigorous discursive proof.

For the representatives of both directions, the "geometry" which the student must know is the contents of the same *theorems* – they differ only in what they see as valuable in these theorems. For those of the first direction – I shall take the liberty to call "logicians" hereafter – the essence lies in the educational value of the flawless argument; for those of the second direction – I might perhaps be allowed to call them "technicians" — it lies in the agreement of these theorems with physical experience, and in the possibility of using them for practical purposes and for the investigation of other scientific fields.

These two uses of geometry are respectively held in contempt by the most extreme representatives of the two directions, and – by the less extreme – acknowledged only sparingly and even more sparingly employed.

It seems to me that the inability of both parties, to convince each other of the value of what each of them holds dear in geometry, is due to their too narrow understanding of geometry – too narrow in the same way. For: they both forget that the systematic geometry course represents only an excerpt of the inexhaustible complex of spatial problems.

What impact this situation has on teaching, I shall presently try to explain.

§2. What is the "educational value" of a school subject? – Probably this: that the method of treating problems in this subject becomes so much the student's own that he also transfers it to other areas of his thinking.

We shall therefore focus on the following questions:

1) What is most typical in the method by which geometry problems are handled?

2) Why is it desirable – and is it desirable – to transfer it to other areas?

3) Is it possible to transfer it to other areas?

4) Do the logicians do everything necessary to ingrain it in the students?

Of course, we shall here only comment on those aspects which are apt to shed some light on our collection of exercises.

1) The geometric method is adorned by the words “logical thinking”, and it is thereby suggested that geometric truths are to be sought and found by certain recipes which are available a priori.

However, by a prescribed recipe one cannot find anything essentially new, since the new is always unknown, one does not know where it is, why it is needed, and often not even *that* it exists. With all our skill in proving the theorems of Euclid, we possess, for instance, no recipe for the direction to choose toward new research projects, in order to discover further relations of importance to us. For such really new things, even geometry contains no search engine.

One cannot speak of “logical seeking” and should not speak of “logical thinking”: this combination of words is misleading and has caused actual damage in education, because it has diverted the teachers’ thinking from an attentive examination of the thought-process!

If the word “logical” is taken to mean that which is free from contradiction, well arranged, and transparent, it may be used only in connection with the Euclidean *style of presentation*, which is so characteristic for geometry.

If, however, it is asked whether there is not something typical to emphasize in the process by which the geometric system arose, it can certainly be said that there is such a feature: it is *the relentless striving for utmost clarity.*

The “geometric method” would thus be: not to let up until a given problem has become entirely transparent, to formulate the result in the most vivid manner, and to arrange all obtained results so that they form a harmonious, well laid out system.

2) That harmony and order are the ideals of any research need probably not to be discussed [does it?] — at any rate, we shall herewith regard our second question as dealt with.

3) Can we achieve that our pupils enhance their knowledge in various other areas by striving for the utmost clarity? Personally I am convinced that better results are attainable in this regard, if only the pupils are given every opportunity, in their geometry lessons, to experience that geometric

method.

4) The question, however, of whether this is being accomplished by the logicians, must decidedly be answered in the negative. Indeed, they do not see the most responsible task hidden in the geometric system — because they think only of the finished theorems already in the books.

§3. Building a system means creating order out of chaos. Before one can spread out the system in front of the admiring public, one has — darting about in a somewhat unsystematic way — brought together the following things:

a. One has taken *cognisance* of the *existence* of the research area in question — by having isolated the features belonging to it in diverse phenomena, and joined them in a common category.

b. One has formulated the first *problems with sufficient clarity.*

c. Gingerly brought out those *concepts*, to which the other concepts of the area can be reduced (let us recall the hopeless state of mechanics, before the concept of "mass" was introduced by Newton.)

d. Determined those elementary relations between these concepts — the *axioms* — from which all others can be deduced as logically necessary consequences (in many areas of natural science they are still being sought today.)

e. Invented the most appropriate *style* for the *presentation* of the system.

For every science, items (a) and (b) seem to be brought about in a rather long semi-conscious progress, by many people, under the influence of life itself. In geometry (as opposed to mechanics, for instance), this seems to be the case for (c) as well. On the other hand, (d) and (e) might well be considered as the wonderful and immensely valuable achievement of one man — Euclid. (In this context, the imperfections later discovered in his system pale, of course, into insignificance!)

What do the logicians think of all this? — They believe that the geometric axioms are incarnated in the newborn ready-made, and therefore the most important task of the Ancient Greeks was this: to doubt everything, even the most evident, as much as possible, in order subsequently — with the help of these axioms — to remove the doubts; and that precisely this constitutes the "scientific" aspect of the geometric method.

And they place this in the foreground so persuasively, that the technicians too, even when they are becoming teachers of geometry, remain under the spell of this notion of what science is about. Since, however, they abhor proving the obvious, they maintain a hostile attitude toward the whole

Euclidean style of presentation.

The logicians do not show the *progress from chaos to system*, which comes with the *systematic treatment* of the subject, and thus "geometry" appears through them to be a mental game cut loose from anything material, and instead of operating with *concepts* — which are, after all, obtainable only by a personal act of abstracting from one's own live experience — pupils deal with *names* and *diagrams* which often fail to remind them of anything they know.

Nevertheless: there are always those rare pupils who, without the teacher's help or knowledge, somehow fill some gaps in the instruction and come to appreciate the geometric method. But even among these are very few who show the courage to create order in some other area; for *this* they certainly did not learn from the logicians!

However, it is not true that an untrained person has the geometric axioms — or even the elementary concepts — at his or her disposal. It is not even necessary to look into pre-euclidean history in order to see that space — just like any other area of exploration — shows itself to naive humanity in a rather chaotic form: evidence for this can be found every day by observing how helpless many people are when, in their activities, they must take into account the properties of space (packing, moving furniture, carrying it through a door, cutting cloth, making something with their own hands, etc.). It certainly will not occur to them to reduce the relevant spatial relations to planar problems, and to consider the appropriate lengths and angles in the plane. Even tradespeople, who through the kind of their trade have acquired some facility in the practical treatment of certain geometric problems, are only partially conscious of spatial laws and concepts, or not even that. This does not prevent the axioms of Euclidean geometry from appearing obvious to almost every person, as soon as he or she has occasion to begin thinking about them at all. But the *capacity of seeing* them as evident is not the same as *having them at one's disposal!*

§4. In order for a formative energy to emanate from a method, it is necessary that its success in mastering a given area be experienced personally, but more than that: this mastery must be seen as desirable.

How do logicians and technicians demonstrate the importance of geometry to their pupils? For the logicians, the question does not really exist: once there were these Greeks who had geometry growing out of their heads, and this was so scientific that now every pupil must know geometry. — Such a position is, of course, hardly apt to make the majority of pupils undergo the logico-geometric treatment with confidence.

In many pupils, enough vivid interest in geometry could be aroused, if they were shown how *everywhere densely* spatial problems penetrate everythything we do or know.

If, for the logicians, the agreement between geometric theorems and experience is only an accidental affair, which they cannot place in the centre of attention, one would expect a fundamentally different attitude from the technicians: for them geometry is, after all, the study of things in the physical world.

But — unhappily — the entire geometric aspect of this world is, for them, inscribed within the few dozen theorems of the geometry-book; all they do to establish the connection between the book and the world is this: to show practical applications for a part of *these* theorems.

Unfortunately these applications are not among the most interesting things one can find in Nature. Let us remember that the system which makes up the contents of the textbook aims mainly at the derivation of certain *formulas*, by means of which a *quantitative* mastery of space might be reached. — The Theorem of Pythagoras and the volume of the sphere are certainly noteworthy, and no other part of the exact sciences can do without them. But do we often have occasion to apply them in practical life? what is so particularly pleasing about them? why should we, come what may, teach exactly *these* formulas to pupils at an early stage, when they cannot yet critically judge their correctness? — The interest which these theorems can arouse in us does not so much lie in each of them individually, as in the possibility of fitting them into such a straightforward system, as well as in the use we can make of them in such diverse investigations of importance to us in so many ways. The Theorem of Pythagoras is *fascinating* only for those who find joy in *mental effort*. But this sort of joy is nipped in the bud by heuristic proofs, because the pupil can no longer come to the subsequent systematic exposition with fresh curiosity.

Considering the narrowness of the geometric material in propaedeutic geometry-books, one cannot avoid the impression that for the technicians, too, the whole plethora of spatial problems surrounding us remains closed off.

§5. In spite of this, it cannot be denied that "practical" teaching has more pupils reacting to the taught material with a certain liveliness: at any rate, they see more clearly what it is *about*, and they are better at picturing spatial objects. The development of the imaginative capacity is also often mentioned in connection with the practical method of instruction.

However, this development is much weaker and more one-sided than it could have been. The following is also overlooked: in order to internalize a pictorial proof completely, one already needs a sufficiently nimble imagination; without this, even such a proof is taken over rather mechanically; and by itself contributes little to that development, as it is quickly followed by an application of what was proved — and that usually has a *computational* rather than pictorial character.

There is yet another matter: technicians often do not distinguish what the *visual* investigation of a theorem yields for the development of the imagination and what the manipulation of *measurement* contributes to the verification of the theorem; indeed the measurement can often proceed without stimulating the appropriate visualization in any way. [1]

§6. We come to the conclusion that neither logicians nor technicians can, by their own methods, make what is important to them prevail, and that the cause of this is: the neglect of that preliminary work which must precede the study of the geometric system.

Our task would therefore be: to stimulate this preliminary work in the pupil's mind, and to guide it purposefully.

In whatever we do, our actions encounter spatial problems; even our stud-

[1] As an example, take the experimental method for finding the ratio of perimeter to diameter of a circle: a string is wrapped around an object with a circular cross-section (say, a tin can), then it is straightened and its length measured with a ruler. Likewise the diameter is measured. This is repeated, as means permit, with several round objects of various diameters. Then the respective ratios (which include some smaller than 3) are recorded, and their average is determined. Thereupon the teacher declares that, with more accurate measurements, this average would be roughly 3.14 (which it seldom is when the pupils do the measuring!).

Such a procedure has little to do with visualizing the lines involved here: a ratio is determined after the rounded string is pulled *straight*, and *in the picture itself no compelling reason is found* that the ratio must be this and no other. — How much more do the means of the discursive proof speak to the imagination! After a regular hexagon is inscribed in the circle, the pupils cannot only prove but also *see*, that its perimeter is equal to six radii, but smaller than the circular perimeter. From that moment on, it will simply be impossible for them to forget that π is greater than 3, but not by much!

On this point, as on various others, I have spoken more extensively in my brochure *"Wat kan en moet het Meetkundeonderwijs an een Niet-wiskundige geven"* (1924, J.B. Wolters, Groningen) ies in various other school subjects, such as geography, mechanics, physics, etc., involve questions which simply cannot be grasped without a sufficiently developed sense of space. For some of us, these very problems are sufficient motivation to work on our geometric insight, and they help us master the geometry course with ease. For many others, this is usually too complicated; on their own, they do not have the energy to analyse the geometric aspects (this is often also the reason for their lacking manual dexterity and seeing certain school subjects as unsurmountable obstacles), and thus they come to their geometry lesson completely unprepared.

It is the teacher's role to sort out the most appropriate examples from all this material, and to present them to the pupil in the most effective way — then the following results become attainable:

1. A sufficiently diverse reservoir of geometric images is acquired, along with the ability of mentally varying sizes and positions of the figures.

2. The elementary concepts of the geometry course will grow from concrete experience, and the corresponding terminology will be tied in with live images.

3. The diagrams in the geometry book will be seen as schematic, generalizing pictures of much that has been observed on real things, and not as objects of study sufficient onto themselves.

4. Pupils will learn how strongly the geometric properties of things shape the aspect of physical phenomena, and how important geometry therefore is for the natural sciences and technology.

5. It will be possible to confront the pupils with problems whose geometric content is perfectly accessible to them, but where they notice that a sure solution requires critical investigation: they will be impelled to *prove theorems!* And they will thereby experience how many things fall into place, when they formulate with utmost clarity everything they can regard as proved: they will be led in a natural way to the *geometric manner of presentation.*

Our collection should convince teachers that there is an abundance of problems for whose comprehension no theorems of the systematic course are required — but which, inversely, are apt to prepare a live understanding of those theorems.

Admittedly — in whatever question — it will rarely be avoidable to make tacit use of one rather sophisticated theorem or the other (e.g., Jordan Curve Theorem) and many a teacher might be inclined to accuse this method of the same misuse of images which taint that of the technicians. There is, however, an essential difference between the two methods: I emphasize that these preparatory exercises do not uncover, formulate, or prove *general theorems*, that the relations encountered in any particular case remain considered episodical, and that only the *the naming of concepts* be retained at this stage. However, the subconscious intake of impressions, which in the course of these exercises will lead to the recognition of theorems, is truly a necessary stage for the future command of the latter: this is the way the human mind works in building a science, there is no other way; even in number theory, a great many special examples usually precede the discovery of a general theorem.

USER'S GUIDE

§7. The questions are ordered by geometric categories and not in the sequence they must necessarily be put to the pupils. I thought this way would be the easiest for teachers to find appropriate exercise material; the judgment of what may be easier or harder for a given group of pupils depends anyway so much on the whole prehistory of their development that it cannot be fixed in a book once and for all.

It is not intended that each exercise should be taken as an "exercise for the geometry lesson." Many of them are meant to be tied into other kinds of the pupils' activities: drawing, sculpting, fashioning school-models, toys, tools, clothes, classroom decorations, etc. from all sorts of materials; visiting factories, doing nature excursions, learning physics, etc. The mathematics teacher should thus be attentive to whatever his pupils are busy at and also be in close contact with other teachers, in order to strive together for the common goal!

These problems differ greatly in their level of difficulty: some of them could and should already be put forward at the pre-school age; since, however, this was not done for most of the pupils we usually get in geometry class, it is — as experience shows — not entirely superfluous to go through them in middle school, too.

The collection is, of course, too large to be exhausted in a particular preparatory period. Many of them can, by the way, be usefully employed in the systematic course — as illustrations or to motivate the formulation of various theorems — where they would, of course, be treated in style different from what which is possible in a propaedeutic course.

The questions should be arranged in such a way that, at first, the very simplest concepts are teased out. However, it should not be forgotten that eventually there will be a transition to the systematic course; at the end of the preparatory period, questions should appear which can suggest the formulation of theorems. For this purpse, one may, of course, use problems which have been previously solved by intuitive *action*, and now draw the pupils' attention to relations which they had used as something obvious though they are not particularly self-evident (e.g., the similarity of triangles and quadrilaterals used in a cardboard construction; the drawing of a triangle with three given sides, where nobody had worried whether certain circles would always intersect).

§8. In discussing these problems with the pupils, it is advisable not to rush into showing them visual support material, but to let them have the opportunity of using their own ability to picture things: even if they do not get very far in this, the effort alone is useful, if only to heighten their awareneness as they finally look at the object. On the other hand — especially at the beginning — such visual support should not be skipped, even in the case of immediate right answers: the correct answer could turn out to be a pretty illusion, when you watch the pupil subsequently handling the object; certainly not all pupils will have answered; and finally, the significance of repeated sensual exposure for the vividness of our mental imagery can hardly be exaggerated!

The visual support material in the pupils' hands is an unequalled means for offering a glimpse into the life of their imaginations. Let the teacher allow them to perform as much as possible with their own hands — he will be witness to revelations, and only then will he know exactly what the pupils are missing in order to follow the seemingly simplest arguments with lucidity.

It is very valuable to ask for schematic drawings of whatever is being dicussed: in those very moments, the act of abstraction is accomplished by many a pupil.

One should never try to save time on the necessary manipulations: one will eventually be able to proceed much faster in the systematic course, and only then with real success!

§9. If, after all this, you proceed to the systematic course, taking it literally from the usual textbooks, you will still experience a serious disappointment. In fact, almost all pupils will still fail to see the point of "logical" proofs: for, as you know, the usual textbooks begin with the proofs of totally obvious theorems!

At this stage it is customary to say: "the simplest things are the most difficult". It is really high time to go beyond that platitude and to ask onseslf *why* "simplicity" is so difficult. One will come to the conclusion that it is not simplicity which makes those theorems difficult, but something so unsimple that even most teachers have no inkling of it, namely the *purpose* of these proofs!

In the sweat of their faces, many teachers try to act as if they could really find something to doubt in those theorems. Of course, this is a lie. No one has proved the equality of cross angles because he had previously doubted it. In reality *not the truth of theorem is being proved* but rather *its logical dependence on other theorems*, which are assumed already established and are no more evident than this theorem itself.

Of course, a beginner cannot grasp this kind of a problem: he is satisfied to know a theorem and to know he can rely on it, and not linger on the question of whether this theorem is held to be true as an independent axiom or as the logical consequence of other axioms.

Let us remark, that most people — *whatever their age* — are indifferent to this distinction (*unreasonably*, it would seem) — even if they have learnt their geometry from logicians.

It should be decided to skip the proofs of ovbious theorems for the first little while.

This will not damage the *clarity* of the presentation, as long as these theorems are not smuggled in tacitly, but are well formulated and clearly labelled as being assumed without proof.

Determining which theorems are true axioms (i.e., a set of statements which is logically independent and permits the deduction of all further theorems describing a given subject) — is a serious task, whose importance for science is very great, though not even acknowledged by all researchers. — If one wishes to acquaint some, and be it only the most talented, pupils with this question, postponing it till the end of the systematic course will achieve a greater success.

If both of these measures are taken:

1. the pupils are prepared for the systematic study of space, perhaps in the way sketched here,

2. the proofs of self-evident theorems are eliminated,

it will be seen that the pupils not only understand geometry, but will be able, under the teacher's guidance, to take an *active* part in building the system of geometric theorems — and only this will lead to a real mastery of the geometric method, so dear to the logicians, and also of the factual contents of geometry, so important to the technicians.

I. LENGTHS

For the pupils these exercises may be allowed to appear as a sport-like practice of visual sizing up. The teacher, however, should keep in mind that, apart from the — really very valuable ability of estimation — these exercises will oblige the pupils to take in a series of geometric forms, become aware of their existence, and to learn the terms they will find at the beginning of their systematic course, such as: being larger or smaller, setting off, comparing, measuring, including, being proportional to, etc., as abstractions from living experience.

1. Take a good look at this table, and tell me how many such tables could be put, close together, along that wall. (Record the different answers of the pupils as well as the number of votes received by each one.)

How could we test which of these answers is the closest? Do we really need that many tables and put them along the wall in order to check the answers?

a. Mark the length of the table on a string, and set off off the latter on the wall (the *integer* part of the desired number results *immediately*. In the beginning, deal only with whole numbers "we don't want it more exactly").

b. With strides or — less ambiguously — with footprints, determine the lengths of table and wall, and divide one of these numbers by the other. (The correct integer can be different this time — if the footprints on the table do not yield a whole number, and we round up or down. In the further course of the exercises, this must be discussed in detail; the first time around, it should only by noticed). "In the preceding case, we had measured the wall with the table: *the table* was our unit of length".

c. Measure table and wall in decimeters and point out the imprecision of the measuring process; if the decimeter does not clearly fit a whole number of times into the table or the wall, discuss which pair of whole numbers one would need to take in order to obtain a result which is surely too large or too small.

d. Measure with objects which do fit the table-length a whole number of times. What are the advantages of this method over the previous one?

e. Look for the common measure of both lengths. (Of course, during these *first exercises* one should not introduce all methods and all scruples, they are touched on here in order to be gradually developed by the pupils in the course of the exercises).

Compare the various estimated answers with the final result of measurement. Compute the average answer and its deviation from that result.

2. By what part of its own length (more than 1/2? than 1/3? than 2/3?) is this wall longer than that one? From where can you most easily see this? Estimate and then measure. Where to measure — on this wall or the one opposite? If there are protrusions, how do we get around them? Where to lay the yardstick: along the floor or at some convenient height — where do we have less to worry about?

As in the previous case, determine the average error of the answers obtained.

By what part is one wall longer than the other? How many parts — and which ones — does this wall contain of the other? How many parts does the other contain of this one? "Ratio" as a fraction.

3. Estimate diverse lengths variously oriented in space, and then measure them. Compare the average errors of estimation for horizontal and vertical lengths.

4. The same outside the school building for larger distances.

5. The same for very small objects.

6. The same for the diameters of round objects: table-tops, plates, glasses, coins; treetrunks, pencils, balls, etc. With which type of coin can you entirely cover the six-place number on a paper money bill?

7. Compare the average errors of estimation on small objects versus big ones.

8. Relative errors. Influence of absolute size on relative error. Effective arrangement length-measurements of small objects (with the help of instruments): measure the the total length of a series of small objects touching one another, and divide by their number.

9. Estimate and later measure certain lengths on the human body: distance between the pupils of both eyes; length of the middle finger; span of the spread hand, of both arms extended; stride, footprint. — Fix measurements on your own body — keeping in mind that it is growing! — in order to have measures with you at all times.

10. What part of the height of a medium-tall man would be the average height of a 3-, 6-, 10-year old child? Check it on various examples.

11. When do you say "narrow", when do yo say "thin"? — "thick" or "broad"? — a "short wall" or a "low wall"? Which is the "length" of brick,

and which are its “breadth” and “thickness”? When do you say “height”, when do you say “depth”?

12. Which animals have a long tail, which have short one? Which has a longer tail: a mouse or an elephant?

II. ANGLES

13. Can you tell the time by looking at a clock-dial where the numbers are illegible? What features would you use? What is *equal* on the dial of a watch and that of a clock tower, when they show the same time?

14. Draw, free-hand, the angle by which the minute hand turns in *ten minutes.* Cut this angle out of your paper and put it on the clock-dial for comparison. Correct it, and save the corrected “ten minute angle” as an angle measure.

15. Use two pens to form the angle through which the minute hand runs in twenty minutes. Put them on paper and draw that angle with a pencil. Check the result with the paper angle saved before.

!6. Using the same paper angle, draw the angles which the minute hand makes in 10, 20, 30, 40, 50, 60, 70 minutes.

17. Naming things: vertex, leg; straight, convex, concave angle. Which of the angles drawn are concave, convex, straight — how can you characterize them in words? With what symbols shall we distinguish the drawing of a straight angle from that of a straight line? Make paper models for concave angles.

18. Cut out four angles of 20 minutes each and glue them together — model of an angle which is greater than that of a full rotation.

19. Using the ten minute angle, make a five minute angle; then a 15 minute angle.

20. Draw a big triangle on the blackboard, then copy it on paper — “smaller” — but as similar as possible to the original. By how much have the sides become smaller? What about the angles?

21. On all kinds of objects, try to find angles which are equal to the 15 minute one — “right angles”. Make a right angle, without using the clock dial, by simply folding paper.

22. Draw two angles which have only the vertex in common. How many non-overlapping angles have been created? — Let us use letters as labels [for points], so that we talk about these angles more easily. Naming them

carefully: of the three letters [naming an angle] always put the one belonging to the vertex *between* the other two.

23. Draw two angles which have one leg and the vertex in common.

The pupils are to try and define the difference between any such pair of angles and a "supplementary" pair drawn by the teacher. The teacher draws supplementary and non-suppementary pairs in various positions, and points out the ones he calls "supplementary". The definition is to result from an analysis of what is seen.

How many angles supplementary to given one can you draw? Compare their sizes,

24. Define the concept "right angle" on the basis of its creation through paper-folding.

25. Namings: acute, right, obtuse angle. If two supplementary angles are not right angles, what can be said about each of them?

26. Assign numbers to the previously drawn angles, using the right angle as a unit. How big is a straight angle?

27. How big is our previous unit [10 minutes on the clock] in terms right angles?

"Degree" as a new unit. How many degrees are in the angle by which the minute hand turns in one minute?

"Degree" — "minute" — "second". How does this new minute [of angle] relate to the angle by which the minute hand turns in one minute of time?

28. A pupil is to stand in front of the class facing the window, then turn to the blackboard. By what angle has he turned? Let him turn about one, two, four, five right angles; about -1, -2, $+3$, -4, ... right angles. By what symbol should we distinguish $+a$ and $-a$ on a drawing?

29. Find a general formula for all angles whose legs are positioned the same way as those of a given angle ABC (taking into account the direction of rotation). Make models of such angles. The expression "congruent modulo 4 right angles" may appropriately be used.

30. By what angle has a man turned, from the moment he stood at the foot of the stairs, ready to climb them, to the moment he has just reached the next floor? ... or the following one? (Careful!)

31. Someone walks along the edge of a square-shaped field, starting in the middle of one of its sides. About what angle has he turned by the time he arrives at the starting point? What if the shape of the field is triangular, pentagonal, or round? What if he does a figure eight through the middle of the field?

32. On a piece of cardboard, the pupil is to sketch successively all angles he turns through, especially at street corners, on his way to school. From this, try to figure out the angle between the front walls of his and the school. Check it on a city map. If some parts of his path are not straight, think about possible errors. As a preliminary exercise, do a similar experiment inside the school building.

33. The distance between two successive windings of a screw is 4 mm. It has been driven 20 cm deep into a board. Through what angle has it been turned?

34. Which angles — formed by which straight lines — were we actually dealing with in the preceding exercises? Let the pupil again turn from the window to the blackboard: once with his right arm stretched out horizontally, again with that arm lifted a little, finally with it completely vertical. In all three cases, determine the angle between starting and finishing position of the arm. Which one of these three angles correponds to what we had — unanalysed — called the "angle of rotation"?

35. How do you have to attach a straight line to a screw, in order that the angle of rotation of this straight line is also the angle of rotation of the screw? — *Simply show it.*

36. How much faster than the hour-hand does the minute hand rotate on the clock-dial? What has a greater angular velocity: the earth turning on its axis or the hour-hand?

37. What arc-lengths (meant naively: measured with a string) are described by various points on a minute-hand in ten minutes? Can the lengths of the minute and hour hands be chosen in such a way that their end-points describe the same arc-lengths in the same time?

38. By how many right angles per second does a wheel turn, if its circumference is one meter and it is rolling down a straight path at the speed of three meters per second? Does it turn slower or faster than the second-hand on a clock?

39. Draw a tilted straight line on the blackboard. The pupil is to draw a "perpendicular" to it, which goes through a given point on the original line. Through the same point he is to draw a "vertical". (Can he do this if the blackboard itself is tilted?)

Two pupils are to stretch a string in the room somehow. Two others are to stretch a second string so that it goes through a certain point on the first one and is perpendicular to it. In how many ways can the second string be drawn?

III. THE STRAIGHT LINE AS AXIS OF ROTATION

40. Study the structure of a simple machine. Which parts rotate against each other? Does every rotating part turn about a rod going through it?

41. Fasten a solid between two immobile spikes. How can its various points still move after that? Are there immobile points other than the tips of the spikes? *"Axis of rotation."* What happens if we fix *one more* point? (On or off the axis).

42. Choose a solid which is such that part of the axis of rotation can be observed if the spikes are suitably positioned. For easy visibility, paint various places on its surface with lively colors.

43. Discuss questions (34) and (35) using the concept of "axis of rotation" and note at what angle to the latter the straight line [or arm] mentioned there should be oriented.

44. What kind of surface is cut out by the rotation of a straight line (knitting needle, knife-edge, ...) in sand, butter, etc., if this line goes through the axis of rotation at various angles; or does not even go through the axis?

45. Make string models for surfaces of rotation (cylinder, cone, one-sheet hyperboloid, plane). Watch them rotate on a potter's wheel.

46. As it is being screwed into [the wall], does the screw perform a simple rotation? Does it have points which remain immobile? How is its axis characterized?

47. On which principle are lathe and potter's wheel constructed? Forming clay on the potter's wheel. Contrast with oval vessels.

48. Checking whether a knife blade is straight.

49. Testing a ruler: a. Slide it along two fixed points on the paper; b. Turn it by 180° in the plane of the paper; c. Turn it by 180° in space around those two fixed points. — Critique of each of those methods. (a — can also be [an arc of] a circle; b — can be any centrally symmetric figure; c — the ruler has thickness, and turning it over brings the second edge onto the paper). Combine the first two methods.

50. Analogy between a straight line on the plane and a great circle on a sphere: the "straightest" line! Distinguish experimentally, by means of methods analogous to (49), between a great circle and any other line on the sphere. (Very desirable: a black globe for chalk drawings. Diameter about 50 cm).

IV. THE SHORTEST DISTANCE BETWEEN TWO POINTS

51. How long is the way from a point A at the foot of a mountain, dune, or hill, known to the pupil, to the station B situated high above? How should you walk along the slope to make it as short as you can? Would this be the very shortest distance between A and B?

52. Find the distance between Berlin and Moscow according to the railway tables. Estimate the distance by air between the same cities according to the globe. Would this be the very shortest spatial distance?

53. In which direction by compass should an airplane leave Berlin in order to reach Moscow by the shortest route? The same question for Berlin and Java. — Determine it on the globe by means of a taut string. — Is the arc of a circle of latitude the shortest line on the sphere between its two ends?

54. A worm crawls from a point A on one wall to the point B on the adjacent wall. Find its shortest path.

55. Draw the shortest path between two arbitrary points on (1) a cylinder, (2) a cone. Make these surfaces out of paper.

56. Which form does a stetched thread take? What about a thick rope? Observe this on various uses of ropes by workers. How must the two ends of the rope lie so that it can be streched straight?

V. STRAIGHT LINES FROM LIGHT RAYS

57. Three pupils are asked to stand along a straight line in front of the class — without using instruments of any kind; a fourth pupil is to check the line-up, again without aids. — On what basis is this task feasible?

58. Make an opening in each of two pieces of cardboard, and put up these screens one in front of the other, in such a way that a certain point in the classroom can be seen by looking through both of them. Check that from this point a string can be pulled straight through the two openings.

59. A pupil is to hold his finger in front of his eye at some distance so as to cover up a certain point; other pupils should convince themselves that the finger, the eye and that point lie on a straight line.

60. In a mirror, a pupil observes a point reflected in it; he then puts his finger on the mirror so as to cover the point. The positions of his eye, his finger, and that point are somehow marked. Let a thread be strung between these three positions, in order to probe the law of reflection. Vary the position of the point, keeping that of the eye the same. The direction of

a perpendicular [to the mirror] can be indicated by fixing a pencil, say, with wax, so that it forms a straight line with its own reflection.

61. Let schematic drawings be made for the last three experiments.

62. Determine at what distance from the eye an object (say, a plate, a coin) must be held, in order to cover a more distant object just barely. Make a schematic drawing.

63. Why do objects appear smaller to us, when we move away from them? *What* is really getting smaller in the process?

Make a schematic drawing — say, of a tree being observed by a person from various distances. What should be included in the drawing, which features of the objects are relevant for *our question*?

64. Estimate the angles under which the blackbord appears from various points in the classroom. Check with a home-made tool for measuring such angles: a tube attached to a piece of cardboard and turnable parallel to it; mark the two positions of the tube by making lines on the cardboard, then find the angle between them.

Later on, a finer instrument will be very desirable! — At the present stage of acquaintance with angles, it is very important to focus on their *legs*. Only later should this be supplanted by measuring arcs.

65. Estimate and then measure the angular distance between different points in space — between two stars when they are high in the sky and when they are near the horizon.

66. When the sun or the moon, near the horizon, is about to disappear behind an object (say, a tree-trunk or a chimney), estimate whether it will be hidden completely. Wait to see what happens; explain which quantity you were really estimating.

67. Which one do we see under a larger angle: the sun or the moon?

68. Into the picture of a landscape, draw the moon as big as it would appear to us in nature. Compare this with similar photograhic images of the moon. (This experiment was carried out by the magazine “Nature”).

69. Why does the moon run after us? Why do near objects pass us more rapidly than distant ones when we ride, say, in a train? — Make a schematic drawing.

70. Stereoscopic vision.

VI THE PLANE AND RULED SURFACES.

71. Using a taut string, check whether the table-top is *plane*.

72. You have two flat plates [slabs, tiles] and no other aids. Can you check whether they are *plane*? What kind of misunderstandig could arise? How many plates are necessary to remove any doubt? (Compare with no.49). In making flat glass surfaces by grinding them against one another, why are three of them required? Why would two not be sufficient?

73. Is the plane the only surface which allows drawing a straight line through every point? — Diverse surfaces — with or without names — which can be obtained by bending paper. In particular: cylinder, cone, helix [2] (circular or other), Möbius Band [3]. Discover such surfaces on nearby objects; build models.

74. What chracterises the *plane* among all *"ruled surfaces"*?.

75. Define cylinder and cone.

76. Ruled surfaces which cannot be made from paper. Models: a twisted pile of books or deck of cards; spiral staircase; one-sheeted hyberboloid; hyperbolic paraboloid. Make them from string.

VII. THE ENDLESS STRAIGHT LINE. PARALLEL LINES.

77. Through two points fixed somewhere in the middle of the classroom, mentally draw a straight line, and imagine in which points it penetrates the boundary of the room, and where it goes from there.

78. Fix a third point, and through it draw a second line meeting the first one. Demonstrate with two taut strings. Determine the angle between the two lines. Make a schematic drawing on the board. Now turn the second line around the third point in such a way that its point of intersection with the first line moves always in the same direction. Watch how the angle between the two lines changes. Mark a number of positions of the second line on the blackboard drawing. Imagine to what limit position the second line can be turned without stopping to intersect the first. — Where is the point of intersection of the two lines *in* the limit position? — Now the same thing while turning in the opposite direction. Again imagine the limit position and draw it on the board. How many limit positions are there? *Without influencing the pupils* let each of them decide, and make a statistic of the answers.

[2] A piece of this surface is obtained by radially cutting the region between two concentric circles, and lifting one side of the cut above the other until the surface is part of a spiral.

[3] See footnote for no.193

79. What kind of surface does the second line describe in the preceding example?

Look for *"parallel"* and non-parallel lines in the classroom, the street. — How should one define "parallel lines"?

80. Make a perspective drawing — even if only a sketch — of the front wall of the classroom, the two adjacent walls, the ceiling, and the floor.

81. Through a rectangular opening in a vertical screen, look at two rods perpendicular to it. Keeping the eye near the centre of the opening, fix the pair of rods successively at different heights: at the lower edge, at 1/3 of the height, in the middle, at the upper edge. For each position make a perspective drawing. The *"vanishing point"*.

VIII RELATIVE POSITIONS OF LINES AND PLANES.

82. Measure the angle between a slanted stick fixed to the floor and some straight line running along the floor from its base. Find another line running along the floor from the base of the stick and making the same angle. Is there a third such line? How does the angle between line and stick change as one varies the direction of the line on the floor? Which one of all these lines makes the largest, and which one the smallest, angle? How are they positioned to each other?

83. How do these two extremal angles change when the slant of the stick is varied?

84. Can you always draw a line on the floor which is perpendicular to the stick? Where is the second perpendicular? Can you place the stick in such a way that yet another line on the floor is perpendicular to it? What are the two extremal angles in this case? The *"line perpendicular to the plane."* (Preparation for the exact theory.)

85. In your surroundings, find examples of lines and planes which are perpendicular to each other. Examine a pencil held at a slant against a mirror, and its image.

86. Draw a straight line in some plane. Fix a point outside that plane and draw through it a straight line which intersects the first one. Turn the second line around the fixed point until it is parallel to the first one. Will it then still have points in common with the plane?

87. Draw parallel lines for planes in various positions. Find examples in the classroom.

88. How many straight lines *parallel* to a certain plane go through the same point? How many planes *parallel* to a certain straight line go through the same point?

89. The same question for *perpendicular*.

90. Through a straight line parallel to a certain plane, draw a second plane. Rotate the latter around the straight line, and watch how the line of intersection of the two planes is displaced. *"Parallel planes"* — which do not intersect. How is the line of intersection placed with respect to the first straight line (when the planes do intersect)?

IX. POSITIONS OF LINES AND PLANES WITH RESPECT TO A HORIZON.

91. A *"vertical"* line — the direction of a string with a weight hanging on it. *"Horizontal"* (line or plane) — as perpendicular to that. Draw the surface of water in a glass held at a slant (cross-section). The water-surface on the ocean: look at a globe, draw a cross-section.

92. The line of the horizon and a horizontal line. What is their relation?! What form does the line of the horizon have (say, for an observer on the high sea) and why?

93. *"Vertical plane".* — The pupils, who, by feeling and habit, are able to distinguish a vertical plane from a slanted one, are now to determine the characteristics of a vertical plane.

94. How many vertical planes can be drawn through a given point? Examples: (door, window, etc.) — look for them yourself!

95. Can a vertical plane be drawn through any straight line? How many? Can a horizontal plane be drawn through any straight line?

96. Does every plane contain a horizontal (a vertical) line?

97. Examples of objects which rotate around a horizontal; a vertical; a slanted line. Look for them yourself!

X. PLANE SECTIONS OF SURFACES. SHADOWS. PERSPECTIVE PICTURES.

98. What form does a surface of water have in a cylindrical glass held at different angles? — Imagine it, draw it, look at it.

99. The same for a funnel.

100. The same for containers of various shapes.

101. Hold a glass in such a way that its rim appears as a straight line (look with *one* eye!) Then move it downwards gradually. How does the rim appear now? How should it be drawn? — Put up a glass-pane between the eye and the glass and draw on it the rim just as you see it.

102. Is there a solid which can be drawn as a circle no matter how it is oriented?

103. What part of a spherical surface do we see when we look at it from various distances? From which distance would we see exactly half of it? — Experiment with the black globe. — Make schematic drawings.

104. What forms can the shadow of a sphere on a wall take — when the source of light is point-like? — Imagine, then experiment. Compare this with the water-surface in the funnel (which cases are lost there?).

105. Let there be a red and a green point-like source of light. Imagine the shadows when only the red or only the green are lit; when both sources are on. Check by experiment. Many point-like sources of light. Discuss the penumbras [partial shadows], draw them schematically, check eperimentally.

106. What breadth can the shadow of a narrow cylinder have (ignoring any penumbra)?

107. Why does a shadow in sunlight become sharper [less fuzzy] as the object moves closer to the wall?

108. A point-like source of light produces a circular shadow on a wall. What is the form of the object that makes the shadow?

109. Imagine the shadow-space of a given object — in diverse orientations toward a point-like source of light. Vary the distance of the object to the light source. Vary the distance and orientation of the object to the wall.

XI. ANGLES BETWEEN TWO AND THREE PLANES.

110. Open two books in such a way that the front and back covers respectively have the same inclination to each other — equal *"two-flat angles"*. What two-flat angles could be called "straight", "right", "acute", "obtuse", "concave"? — Look for such angles on nearby objects. One *"degree"*. Planes which are *"perpendicular"* with respect to each other.

111. What angles are formed by the edges of the book-covers when these form a right two-flat angle? — "Line angles" of the two-flat angle — how should they be defined? Convince yourself: the equality of two-flat angles corresponds to the equality of certain line angles — the line angle as a measure of the two-flat angle.

112. How can three planes be situated with respect to one another? — Imagine all cases, construct them from cardboard. — Surrounding examples.

113. In the classroom look for *“three-flat angles”* which can be see from their “inside” or from their “outside”.

114. From three pairs of respectively equal planar angles, make two three-flat angles neither of which can be made to fit into the other. (Use cardboard). A pair of gloves. An object and its mirror-image. Two spherical triangles whose respective sides are equal but arranged in different order. — Compare with planar triangles.

115. How many rooms can be adjacent to that [inner] corner of the ceiling. How many for the second floor, how many for the top floor. — How many non-overlapping three-flat angles are formed by three interecting planes?

116. Mentally vary the mutual inclinations of the planes. Construct models: from three pieces of cardboard, which are mutually slanted to one another, from three wire circles — as intersections of just those planes with the sphere; draw these circles on the black globe.

117. Are there, among those eight three-flat angles, pairs of equal ones? — “Congruent”. — “Symmetric”. (“Equal” should be reserved for *magnitudes*).

XII. SYMMETRY.

118. Out of folded paper, cut figures having some symmetry. What do you obtain if the paper is folded just once, what if it is folded twice or more?

119. Anticipate the results of the cutting and draw them. Reflect a given figure at a given straight line. *“Line of symmetry” (not axis* of symmetry).

120. Draw lines of symmetry on a figure so as to make it lose (resp. keep) its symmetry with respect to colour.

121. In the classroom, find surfaces with lines of symmetry. On the fronts of houses, or parts of them, look for lines of symmetry, or determine which features destroy a symmetry.

122. Formulate what kind of figure has a line of symmetry.

123. An object and its mirror image. Define “plane of symmetry”. Look for one in diverse objects. Symmetry of the human body, of animals. Are all fish symmetric as seen from the outside?

124. *“Centre of symmetry”* of a planar figure — define it for the pupils and ask them for examples.

125. Define regular polygons and ask the pupils to decide what kinds of symmetry they have.

126. Complete a given figure to a centrally symmetric one, with a given centre of symmetry.

127. Figures with (1) a line of symmetry, (2) a centre of symmetry, on the surface of the globe. In which cases can one half be made to cover the other?

128. "Second order axis of symmetry" as an extension of the concept of "centre of symmetry" of planar figures. Does every parallelepiped have second order axes of symmetry? — Find various nearby objects with second order axes of symmetry.

129. Third, fourth, $\ldots, n$-th order axes of symmetry. Find all second, third, fourth order symmetry axes of a cube. Symmetrical features of regular pyramids.

130. Does a skewed parallelepiped have any symmetries? *"Centre of symmetry"*. Remember centres of symmetry in planar (or spherical) figures — and compare. Can a skewed parallepiped be divided into two halves which can be made to coincide with each other? Does a regular tetrahedron have a centre of symmetry?

XIII. THE GEOMETRY OF MECHANISMS.

131. Make a schematic drawing of the treadle [crank] for a lathe or sewing machine; sketch, free hand, the paths taken by the end-points of the connecting-rod.

132. How do the various other points of the connecting-rod move? From strips of cardboard, make a model of this mechanism; lay it on white paper, and through holes in the strip, draw paths of various points of the rod. Imagine a continuous deformation of one path into the other as the observed point slides down the rod.

133. Vary the length relationships of the strips, and study the influence of this variation on the paths taken by end-points of the connecting-rod. The transformation of circular into rectilinear motion. Where is the fourth point of the corresponding quadrilateral? — Observe the various shapes of such quadrilaterals on machines.

134. Gears — cylindrical and conical. What is the purpose of the cone-shape? Find it in mechanisms.

135. What kind of paths can be described by various points of a plane which is fixed in one point and can move only in itself? — a sheet of paper tacked to the table. Similar question for a spherical shell fixed to a globe in one point. Are there further fixed points in that case?

136. Can a piece of paper move on a table in such a way that all its points describe circles of *equal size*? Are there immobile points in that case? Can a similar thing be done with the shperical shell? Centrifugal sieves. Why does one not rather allow a sieve to rotate around a fixed axis?

137. What paths can be described by the different points of a solid, if two of its points are held fixed? — what if only one point is fixed?

138. Give a general characterisation of translating motion and find examples of it.

139. What kind of path is described by a point on a screw which is being screwed in — if it lies on the axis, off the axis?

XIV. DEGREES OF FREEDOM.

140. How many data does one need to know in order to find a point on a surface? (where a treasure is buried on a field, where a tree is to be planted, where on the surface of the earth a city is located).

141. How many data determine the position of a point in space? (Exact position of an airplane, of a bird in the air, of an object in the room).

142. Different kinds of data. Geometric loci. The pupils are to reveal the data for the location of an object chosen by them, and the other pupils should then decide which object it is.

143. How many data does one need to know in order to determine the location and spatial orientation of a rod? The same for a strip of cardboard on the surface of the table.

144. How many data determine the location of a point on the surface of globe? — the orientation of a globe whose centre is fixed?

145. How many data (*"degrees of freedom"*) can be chosen arbitrarily to determine completely the location and spatial orientation of the following objects: a point in space; a point on the plane, on the surface of the globe, on the inside of the globe; a strip of cardboard on a table; the same if it is fixed in one point; and with the additional constraint of two extremal directions forming a given angle; a globe whose equator can move within a given plane; a pair of rods, one of which is fixed in one point while the other is attached to the first by a fixed joint allowing arbitrary turns; a wheel whose axle is fixed

in space, a wheel which can roll along a given rail, a wheel on a stationary car; a sewing machine.

146. Degrees of freedom of various parts of our bodies.

XV. DETERMINATE, INDETERMINATE, AND OVERDETERMIDED PROBLEMS.

The exercises of the following three parts are apt to suggest how a systematic discussion can help to find exact answers, and thus could serve as an introduction to the systematic course, and also as illustrations of it.

147. Three pupils are to stand in such a way that every pair of them stands at the same distance to each other. Use string and chalk as aids. Make a schematic drawing on the board.

148. Can this exercise be extended to four pupils? To four birds in the air? to five birds?

149. With ruler and compass, draw a triangle with three given sides. Let the pupils themselves prescribe the three lengths — are they completely arbitrary? Same exercise on the black globe.

150. Construct a quadrilateral with four given sides. Compare this exercise with the preceding one. How many degrees of freedom did the triangle have once its sides were fixed? How many does our quadrilateral have? Find additional data to make this exercise determinate (so that the quadrilateral, too, will no longer have any degrees of freedom).

151. Construct a triangle given one side and an adjacent angle. The geometric locus of the third vertex. Is the triangle completely determined by the choice of the third vertex? By what kind of data can this be accompished? — Similar things with spherical triangles. Is the triangle then still completely determined by *every* choice of the third vertex? — Similar things on a cylindrical surface (say, a circular one).

152. In several ways, presribe two data for a triangle and discuss the geometric locus of the third vertex. Analogously for spherical triangles.

153. On the sphere, prescribe two angles and, varying the length of the side between them, examine the tird angle; prescribe a certain value for the third angle, and — *groping* with your hands — determine the corresponding triangle; convince yourself that this is a *determinate* problem.

154. Analogous things with planar triangles. Why is the case of the planar triangle inderminate?

XVI. THE PARALLEL AXIOM AND ITS CONSEQUENCES.

155. A large floor is to be paved (ignoring its edges) with tiles of a single shape and size. What forms can the tiles have? Can they be triangles, quadrilaterals, pentagons? Can they have arbitrary angles?

156. Can the whole surface of the globe be divided into an arbitrary number of equilateral triangles of the same size?

157. Which regular polyhedra are possible?

158. What do we mean by an "enlarged picture" of a planar figure, or a "scale model" of a spatially extended object? Examples of figures whose respective angles are equal, but without proportional sides; with proportional sides, but without equal angles.

159. Can two triangles, quadrilaterals, pentagons of different sizes on the surface of the globe be similar?

160. In the plane and on the sphere, obtain a figure from a triangle by "multiplying" [all three sides by the same factor]. Can the figure thus obtained on the sphere be "rectilinear" (i.e., bounded by arcs of great circles)?

161. Determine the distance to an unreachable point (e.g, a tree on the other side of the river; a corner of the ceiling; the top of the tree).

162. Schematic drawings for a method of determining the radius of the earth; the distance to the moon, knowing the radius of the earth; the distance to the sun, knowing that to the moon; the distance to Sirius, knowing that to the sun.

163. All sides of a triangle are enlarged in the ratio 1:n. In what ratio is its area enlarged?

164. All edges of a parallelepiped are enlarged 2,3,4,... times. By how much is its total surface area, its volume, enlarged?

165. Why is it thought more economical to build *one* house with 8 apartments of equal size than to build 8 houses with one apartment each?

166. Eggs of two different sizes are for sale. One kind has a diameter one and a half time as long as the other, but is twice as expensive. Which one is a better deal (from whose point of view)?

167. A box is to be filled with as much as possible of a certain substance available in the form of balls of two different sizes: smaller ones and larger ones. Which kind would you choose?

168. Somebody has derived the formula $S = 3ab + bc$ for the "size" of a certain kind of solid, and it is known that a, b, c — are certain *linear*

dimensions of such a solid, while S — is either its surface or its volume. Can you tell from the formula, which of these two quanities is really given by S?

XVII. THE CIRCLE

169. A string is to be tied around a parcel having the form of a circular cylinder. How long should it be? — judged by sight! One side of a cylindrical box is to be covered with paper — what form and length should the piece of paper have?

170. You are to sew a skirt and a pair of trouseres having the same length. Which one requires more material? — make a rough estimate by schematically thinking of the skirt as one, the trousers as two, cylinders.

171. You have three cylindrical boxes of equal height, one being twice as wide as each of the two others. The side of the wide one is to be covered with red paper, the sides of the thin ones with green paper. Estimate of which paper you need more.

172. Around the equator of a ball as big as the earth, imagine a string. Now imagine this string lengthened by one metre, but still slung around the ball in a circle concentric with the equator. Estimate how far removed it will be from the surface of the ball.

173. What kind of buttons work better: flat ones or thick ones? — looking more like coins or more like marbles? For which kind would you have to make bigger button-holes, and bigger in which ratio, for buttons of the same diameter?

174. Through a circular opening in a piece of *paper* pass a coin which is larger than the opening. Estimate how small an opening would still work.

XVIII. SURFACES AND SECTIONS

175. From a piece of cardboard, cut out the shell of a cube.

176. Cut away pieces from the corners of a rectangular piece of cardboard, so as to make it possible to make an open box from what is left over. What will be the form of the cut-off pieces?

177. Draw a paper pattern for a conical lampshade whose base is an octagon, an n-gon, a circle. Invent various models whose shells can be made from a single piece of cardboard.

178. Given the plan for the exterior walls of a house, draw the horizontal projection of the roof, assuming that all parts of the roof make the same angle with the horizontal plane. How does the projection of the edge between

two "faces" of the roof change, when the latter make different angles with the horizontal plane? What forms do the different faces have? (Plan of the house: a square; a rectangle; a rectangular figure with some protruding walls; a regular octagon).

179. An eaves-trough with semicircular cross section consists of two equal halves which are joined together under some angle. What is the form of their common edge?

180. A cube is cut at right angles to its space diagonal. What is made by the cut? How does it depend on the distance of the cut from a corner?

XIX. TOPOLOGY

181. Schematic drawing of a knot.

182. Composition of an ordinary string [twine] — of an electric cable (a great number of threads are wound around one another so as to give each of them the same exposure to the surface).

183. Chains which can be taken apart.

184. Drawing a lasso.

185. On paper, n points are pairwise connected by lines. Can this configuration be drawn in a single stroke without repetition? — when n=4, 5, 6, ...? — Can all edges of a tetrahedron, an octahedron, a cube be traced in the way described above? — Try to invent more such "graphs" for which this is possible or impossible, respectively.

186. Colour a disc [pizza] divided into 2, 4, 6 sectors in such a way that adjacent sectors are coloured differently. What is the minimal number of colours required to do this?

187. The same for the case of three, five, seven sectors.

188. The same for two concentric circular discs, when the outer ring (respectively: the inner disc) is divided into two, three, four, five parts by radial lines.

189. Divide a square into such regions that two, three, four, five, ... colours are needed in order to colour any pair of adjacent regions differently.

190. Colour a tetrahedron, a cube, an octahedron so as to give different colours to any two adjacent faces.

191. Attach coloured beads (or use wax, clay) to the corners of a tetrahedron, a cube, an octahedron in such a way that any two beads on the same edge have different colours. How many colours are needed?— Compare with the preceding problem.

192. Divide the surface of a torus [dough-nut] into five, six, seven parts so as to require five, six, seven colours, respectively, to distinguish them as above.

193. From a strip of paper, make a cylinder and paint each of its surfaces (inner and outer) with a different colour. From a similar strip, make a Möbius Band [4] and try to colour it the same way.

194. On the surface of the cylinder, apply a thin streak of paint to each of its two rims. Try the same on the Möbius Band.

195. Cut through the Möbius Band along a line parallel to the rim. Repeat the process on the result of the first cut. How many rims are created by each cut?

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[4] If A, B, C, D are the successive corners of a rectangular strip of paper, so that a cylinder is obtained by gluing A to B and D to C, a (ruled) Möbius Band is produced by gluing the side AB to CD in such a way that A coincides with C and D with B.