Incidence in Affine 3-Space.

In the following, arguments will be given and heard about something called "space", or rather, the three kinds of objects contained therein, namely points, lines, and planes. For real sticklers, we'll have to add that, yes, there is at least one plane, but we refuse to stick pedantically to the word "incident" to describe the relations between our objects, using instead the usual language of position and meeting. We make the following assumptions.

1) Every line has at least two points; and any two points determine a unique line on which they lie.
2) Every plane has at least three points; and any three non-collinear points determine a unique plane on which they lie.
3) Any two planes intersect, if at all, in a unique line; and every line can be presented as the intersection of two or more planes.
4) Given a line and a point separate from it, the plane determined by them contains a unique line not meeting the given line but containing the given point.

Vocabulary: Three or more points are *collinear* if they lie on the same line. Three or more lines are *concurrent* if they meet in a single point. Lines and/or points are *coplanar* if they lie in the same plane. Two lines are *parallel* if they are coplanar but do not meet.

Notation: We shall use capital Roman block and script letters for points and lines, respectively, capital Greek letters for planes.

Example: *If a plane* Π *contains two points* P_1 *and* P_2*, it must contain the entire line* $\mathcal{L}$ *determined by them.* Indeed, $\mathcal{L}$ is the intersection of two planes, say, Π_1 and Π_2, hence P_1 and P_2 are in Π, Π_1, and Π_2. Assuming $\Pi \neq \Pi_1$ (otherwise switch subscripts), consider their line $\mathcal{L}'$ of intersection. Since it contains P_1 and P_2, it must equal $\mathcal{L}$.

Breathtaking, ain't it? At least it shows that a plane Π which does *not* contain a line $\mathcal{L}$ either does not meet it at all or contains only one point of it. In the latter case, Π is called *transversal* to $\mathcal{L}$.

Remark: *If a plane* Π *is transversal to a line* $\mathcal{L}$ *it is transversal to all lines parallel to* $\mathcal{L}$. Indeed, let $\mathcal{L}'$ be such a parallel, Π' the plane of $\mathcal{L}$ and $\mathcal{L}'$. Consider $P = \mathcal{L} \cap \Pi$ and $\mathcal{L}'' = \Pi \cap \Pi'$. If $\mathcal{L}''$ did not intersect $\mathcal{L}'$, it would be, like $\mathcal{L}$, a parallel to $\mathcal{L}'$ through P.

Lemma: *If* Π_i*, for* $i = 1, 2, 3$*, are planes with 3 distinct pairwise intersections* $\mathcal{L}_{ij}$*, these are either concurrent or mutually parallel.*

Proof. Suppose $\mathcal{L}_{12}$ and $\mathcal{L}_{13}$ intersect in a point P. Then they cannot both be parallel to $\mathcal{L}_{23}$. Suppose, therefore, that $\mathcal{L}_{12}$ and $\mathcal{L}_{23}$ intersect in a point P'. Then P' lies in $\Pi_1 \cap \Pi_2 \cap \Pi_3$ and hence equals P.

Corollary: *If* $\mathcal{L}$ *and* $\mathcal{L}'$ *as well as* $\mathcal{L}'$ *and* $\mathcal{L}''$ *are parallel, then so are* $\mathcal{L}$ *and* $\mathcal{L}''$.

Proof. Omit the case, where the three lines are coplanar. Let Π and Π' denote the planes of $\mathcal{L}, \mathcal{L}'$ and $\mathcal{L}', \mathcal{L}''$, respectively. Now pick a point P^* on $\mathcal{L}$ but not on $\mathcal{L}''$. Because of parallelity of $\mathcal{L}$ and $\mathcal{L}'$ this point is not on $\mathcal{L}'$ either. Consider the plane Π^* spanned by P^* and $\mathcal{L}''$, as well as the intersection $\Pi \cap \Pi^* = \mathcal{L}^*$. Since P^* is neither on $\mathcal{L}'$ nor on $\mathcal{L}''$, these lines are distinct from $\mathcal{L}^*$. Therefore the three are pairwise parallel. In particular, $\mathcal{L}^*$ is a parallel to $\mathcal{L}'$ through P^*, and hence must be equal to $\mathcal{L}$.

Theorem: *Let* Π' *and* Π^* *be distinct planes defined by the triangles* $A'B'C'$ *and* $A^*B^*C^*$, *respectively. Then, if* $A'B'$ *and* A^*B^* *as well as* $B'C'$ *and* B^*C^* *are parallel, so are* $A'C'$ *and* A^*C^*.

Proof. Since $A'A^*$ and $C'C^*$ are coplanar, so are $A'C'$ and A^*C^*. Therefore it suffices to show that Π' and Π^* do not intersect. If they did, say, in a line $\mathcal{L}$, the latter would have to be parallel to $A'B'$ and A^*B^* by (3) applied to the triple of planes Π', Π^* and $A'B'B^*$. By the same token, $\mathcal{L}$ would have to be parallel to $B'C'$ and B^*C^*, which is impossible since $A'B'$ is not parallel to $B'C'$ (definition of "triangle").