International Mathematical Olympiad 1998.

(July 15, Taipei, Taiwan.)

1. In the quadrilateral $ABCD$, the diagonals AC and BD are perpendicular, and the opposite sides AB and DC are not parallel. Suppose that the point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.

Solution. The triangles ABP and CDP are clearly isosceles with the common vertex P. Assuming cyclicity — i.e., that AP and CP have equal length — the areas will be equal if the vertex angles APB and CPD add to 180°, or equivalently that the base angles $\alpha = PAB$ and $\delta = PDC$ add to 90°. It does not matter whether we consider the triangles BPC and APD instead of the pair just named: if the vertex angles of one pair are supplementary, then so are those of the other.

With all that freedom of choice, we may suppose (*pour fixer les idées*) that P lies in the first quadrant of the cross formed by DB (left to right) and CA (top to bottom) with the "origin" O. Consider the angles $\xi = CAP$ and $\eta = DBP$. Then $\alpha + \xi = OAB$ and $\alpha - \eta = ABO$ add to 90° in the triangle ABO. Replacing AB by DC, we get $(\delta + \eta) + (\delta - \xi) = 90°$ as well — *d'où le résultat.*

So far we have only proved the "only if". Suppose now that AP is shorter that CP. The circle around P through A and B will then cut PC and PD in new points C' and D'. Since triangle PAB is the same as $PD'C'$ it must be smaller than PDC, areawise.

2. In a competition, there are a contestants and b judges, where $b \geq 3$ is odd. Each judge rates a contestant as either "pass" or "fail". Supose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that $k/a \geq (b-1)/2b$.

Solution.

3. For any positive integer n, let $d(n)$ denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that $k = d(n^2)/d(n)$ for some n.

Solution. Let Γ be the set of all rational numbers of the form $d(n^2)/d(n)$. We claim that its integral elements are precisely the positive odd integers. We have $1 \in \Gamma$ by taking $n = 1$.

It is easy to see (via prime power decomposition) that Γ consists of all products of numbers of the form $(2s-1)/s$, with $s > 1$. Since all numerators are odd, it is impossible to get an even integer in Γ. It is also impossible to get rid of any factor 2 in the denominators. Any integer element of Γ is therefore a product of quotients $(2s-1)/s$, with s odd.

Let $m = 2^\nu t - 1$ with t odd. To show that m lies in Γ, consider the map $L : s \to 2s - 1$ from the odd integers to themselves. Iterating it ν times gives $L^\nu(s) = 2^\nu s - (2^\nu - 1)$. For any s, the product

$$\frac{L(s)}{s} \cdot \frac{L^2(s)}{L(s)} \cdots \frac{L^\nu(s)}{L^{\nu-1}(s)} = \frac{L^\nu(s)}{s}$$

certainly lies in Γ. But if we set $s = (2^\nu - 1)t$, we get $L^\nu(s)/s = (2^\nu t - 1)/t = m/t$. Since $t \in \Gamma$ by induction, and Γ is closed under multiplication, we are done.