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Newton versus Pell.

Let a and b be natural numbers, and think of a/b as derived from the algebraic integer $\alpha = a + b\sqrt{2}$. If a/b is an approximation of $\sqrt{2}$, the next step by Newton's Method would yield

$$\frac{a^2 + 2b^2}{2ab},$$

which corresponds to $\alpha^2 = (a + b\sqrt{2})^2$. Let us say that a/b is a *good* approximation if $a^2 - 2b^2 = \pm 1$, i.e., if the norm of α is a unit — so (a, b) is a solution of Pell's equation. Obviously all further steps in Newton's Method then have the same property.

Since all units in $\mathbf{Z}[\sqrt{2}]$ are powers of $\varepsilon = 1 + \sqrt{2}$, the terms of the Newton sequence would occur among those of $\alpha\varepsilon^n$. It is interesting to note the shape of the "next step" $\alpha \mapsto \alpha\varepsilon$ in this scenario (call it *Pell's Method* for short). Since $\alpha\varepsilon = (a + 2b) + (a + b)\sqrt{2}$, we would get

$$\frac{a + 2b}{a + b} = 1 + \frac{b}{a + b} = 1 + \frac{1}{1 + a/b},$$

in other words, the continued fraction development. Though much slower than Newton (enumerating versus successive doubling), it has well-known charms of its own.