July 4, 1999.

Napoléon généralisé

ou: La défaite de la géométrie.

Every triple (α, β, γ) of complex numbers defines a map from triangles $\Delta = ABC$ to triangles $\Delta' = A'B'C'$ according to the prescription

$$A' = A + \alpha u\,, \quad B' = B + \beta v\,, \quad C' = C + \gamma w\,, \tag{$*$}$$

where $u = \vec{AB}\,, \quad v = \vec{BC}\,, \quad w = \vec{CA}\,.$

Theorem: *The similarity class of $A'B'C'$ is independent of the original ABC, whenever the triple (α, β, γ) satisfies the quadratic relation*

$$(\alpha - 1)(\beta - 1)(\gamma - 1) = \alpha\beta\gamma\,. \tag{$\dagger$}$$

Proof. A straightforward calculation shows that the vectors $A'B'$ and $B'C'$ are given by

$$u' = (1 - \alpha)u + \beta v \quad \text{and} \quad v' = (1 - \beta)v + \gamma w\,,$$

respectively. Using $w = -u - v$, the second equation becomes $v' = -\gamma u + (1 - \beta - \gamma)v$.

The similarity class of $A'B'C'$ can be identified with the complex number θ which turns one of these vectors into the other. Since (†) does not allow our triple to be trivial, we may assume $\beta \neq 0$ and define $\theta = (1 - \beta - \gamma)/\beta$. Finally we apply (†) in the disguised form $(1 - \alpha)(1 - \beta - \gamma) = -\gamma\beta$ to see that $\theta(1 - \alpha) = -\gamma$. Hence $\theta u' = v'$ for any u and v, as desired.

Remark: In terms of similarity classes, the map $\Delta \mapsto \Delta'$ defines a Möbius transformation $u/v \mapsto u'/v'$ whose degeneracy is equivalent to (†).

Scholium: The triangle $AA'B$ is isosceles with vertex A' if and only if $\alpha = (1 - ai)/2$ for some real a (its "slope"). Suppose all components of the triple are like this, with $\beta = (1 - bi)/2$ and $\gamma = (1 - ci)/2$ — in other words, $A'B'C'$ is obtained from ABC by attaching an isosceles triangle to each side, pointing "outward" if the "slope" is positive (remember that $\mathbf{C}$ is oriented). Then (†) turns into

$$bc + ac + ab = 1\,. \tag{$\ddagger$}$$

Napoleon's Theorem is just the case $a = b = c$: since the similarity class of $A'B'C'$ is independent of ABC, we might as well take the latter to be equilateral. It follows that $A'B'C'$ is *always* equilateral.