Jan. 14, 1999.

Daylight and Latitude.

Imagine the earth as a ball rotating around the z-axis, its illuminated half bounded by a great circle which lies on the plane $z = ty$, with $t > 0$. Let's say that $z > 0$ at the north pole. Then the latter lies in daylight — i.e., the sun is between Aries and Virgo. If this is not the case, just interchange north and south (day and night) ...

Question: How long is the night on a circle of latitude α ?

Obviously, if $a = \tan\alpha$ is greater than t, the entire circle lies on the sunny side of the dawn-dusk plane (i.e., in the Arctis). Let us therefore assume $a \leq t$. In that case, the plane cuts a nocturnal segment off the circle. How can we describe it?

The points of the circle satisfy $z^2 = a^2(x^2 + y^2)$. On the nocturnal side, we further have $z \leq ty$, hence

$$x^2 \leq \frac{t^2 - a^2}{a^2} \cdot y^2 \tag{1}$$

altogether. If ε is the angle described between dusk and midnight, this amounts to saying

$$\tan^2\varepsilon \geq \frac{\tan^2\alpha}{\tan^2\theta - \tan^2\alpha} \quad , \tag{2}$$

where $\theta = \arctan t$. The angle θ between the dawn-dusk plane and the ecliptic changes from $90°$ at the equinoxes to $(90 \pm 23)°$ at the solstices.

July 4, 1999.

Reading this again today, I found it hard to follow without a picture. Imagine the x, y-plane as the table-top, with y going off to the right. Part of the cone $z^2 = a^2(x^2 + y^2)$ is shown, balanced on its tip at the origin, the top rim being our circle of latitude. It is cut by the dawn-dusk plane $z = ty$, which slants more steeply (with angle $\theta \geq \alpha$) and is lit from the left. It cuts our disk of latitude (centre $= C$) in a segment AB parallel to the x-axis. The inequalities (1) and (2) are best understood within the plane of that disk. The angle ε is half of $\angle ACB$.