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Catalan.

Let C_n be the number of bracketings of the non-associative $(n+1)$-tuple $\{A_0, \ldots, A_n\}$. Since each such bracketing contains one of the patterns $(A_0 \cdots A_k)(A_{k+1} \cdots A_n)$, we have the recursion relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \,. \tag{1}$$

For the formal power series $f(X) = \sum_k C_k X^k$, this means $f(X) = 1 + Xf(X)^2$, whence

$$f(X) = \frac{1 - \sqrt{1-4X}}{2X} \,. \tag{2}$$

Substituting $-4X$ for T in the square root series

$$g(T) = \sum_{k=0}^{\infty} \binom{1/2}{k} T^k \,, \tag{3}$$

which is obtained — remember? — by repeated formal differentiation of $(1+T)^{1/2}$, we get

$$f(X) = -\sum_{k=1}^{\infty} (-2^k) \binom{1/2}{k} (2X)^{k-1} \,. \tag{4}$$

Here the numerator of the "binomial" coefficient is the product of $(1/2 - \mu)$, with $\mu = 0, \ldots, k-1$. Multiplying each of its k factors with -2 yields $(2\mu - 1)$, which is positive except for $\mu = 0$. Hence it is not hard to compute

$$C_n = 2^n \frac{(2n-1)(2n-3)\cdots 1}{(n+1)!} = \frac{(2n)!}{(n+1)!\,n!} = \frac{1}{n+1}\binom{2n}{n} \,. \tag{5}$$