1. Semi-simplicity. If \( G \) is a group and \( F \) is a field, an \( FG\)-module \( M \) is a vector space over \( F \) with a linear \( G \)-action \( G \times M \to M \), i.e., one that satisfies
\[
\sigma(ax + by) = a\sigma(x) + b\sigma(y), \quad \sigma \in G, x, y \in M, a, b \in F.
\]
Thus, every \( \sigma \in G \) acts on \( M \) as a linear transformation \( \Delta(\sigma) \). Instead of considering the \( FG\)-module \( M \), we could equivalently speak of the representation \( \Delta \), which is a homomorphism \( G \to \text{Aut}_F(M) \).

CONVENTION: In this course, the groups \( G \) to be represented are tacitly assumed to have finite order, and all \( FG\)-modules, to have finite dimension over \( F \).

THEOREM (Maschke): Suppose that the order \( |G| \) is prime to the characteristic of \( F \), and let \( N \subseteq M \) be \( FG\)-modules. Then \( M = N \oplus N' \), where \( N' \subseteq M \) is an \( FG\)-module.

Proof. We need a left inverse ("retraction") \( p \in \text{Hom}_{FG}(M, N) \) for the inclusion \( N \hookrightarrow M \). It is easy to produce such a thing, say \( f \), in \( \text{Hom}_F(M, N) \). Take it and define
\[
p(x) = \frac{1}{|G|} \sum_G \sigma f(\sigma^{-1}x).
\]
For \( x \in N \), we have \( f(x) = x \) and hence \( p(x) = x \), so \( p \) is still a left inverse. Moreover, \( p(\tau x) = \tau p(x) \) for all \( \tau \in G \), q.e.d.

By induction, every \( FG\)-module \( M \) splits into a direct sum \( L_1 \oplus \cdots \oplus L_t \) of \( FG\)-modules which are irreducible in the sense that they have no \( FG\)-submodules. The uniqueness of the irreducible components \( L_i \), up to isomorphism, follows at once from general theorems (e.g. Jordan-Hölder or Krull-Schmidt), but we shall soon see it more directly.

Henceforth assume that the characteristic of \( F \) is prime to \( n = |G| \).

The group algebra \( FG \) consists of all "polynomials" \( \alpha = \sum_G a_\sigma \sigma \), with products \( \alpha \beta \) defined by the distributive law and the multiplication in \( G \).

LEMMA 1.1: Let \( L \) and \( N \) be irreducible \( FG\)-modules with \( L \subseteq FG \) a left ideal. Then \( LN = 0 \) unless \( L \) and \( N \) are isomorphic.

Proof. For every \( x \in N \), the product \( Lx \) is either \( O \) or \( N \), since it is an \( FG\)-submodule of \( N \). In case \( Lx = N \), the surjection \( L \to N \) by \( \lambda \mapsto \lambda x \) is an isomorphism, q.e.d.

COROLLARY: Let \( FG = L_1 \oplus \cdots \oplus L_t \) be a complete decomposition of \( FG \) as a (left) \( FG\)-module. Then every irreducible \( FG\)-module \( N \) is isomorphic to a suitable \( L_j \).

Proof. Since \((FG)N = N\), we must have \( L_j N \neq 0 \) for some \( j \). Then \( L_j \cong N \) by the Lemma; q.e.d.

Now collect the aforementioned \( L_j \) into isomorphism classes, say \( \mathcal{T}_1, \ldots, \mathcal{T}_s \), and let \( M_i \) be the direct sum of all the \( L_j \) which belong to \( \mathcal{T}_i \). Thus
\[
FG = M_1 \oplus \cdots \oplus M_s,
\]
with each \( M_i \) a sum of (say \( m_i \)) irreducible components of the same isomorphism type. \( \text{Note:} \) by the Lemma, \( i \neq j \) implies \( M_i M_j = 0 \).

LEMMA 1.2: If \( \pi_1 : FG \to M_i \subseteq FG \) is the obvious projection, \( \pi_i(\alpha) = \pi_i(1) \cdot \alpha \), for all \( \alpha \in FG \).

Proof.
\[
\pi_i(1)\alpha = \sum_j \pi_i(1)\pi_j(\alpha) = \pi_i(1)\pi_i(\alpha) = \sum_j \pi_j(1)\pi_i(\alpha) = 1 \cdot \pi_i(\alpha), \quad \text{q.e.d.}
\]

It is customary to abbreviate \( \pi_i(1) = e_i \). Since \( M_i \) is a left \( FG\)-module, it is obvious that \( \pi_i(\alpha) = \alpha e_i \), that \( e_i e_j = \delta_{ij} e_i \) (Kroneckers’s \( \delta \)), and that \( 1 = e_1 + \cdots + e_s \). The force of the lemma is that the \( e_i \) are central. In particular, \((\alpha e_i)(\beta e_i) = \alpha \beta e_i\) shows each \( M_i \) to be a ring with unity \( e_i \).
2. Characters. In this section, we keep the notation of the preceding one. In particular, \( I_1, \ldots, I_s \) are the available isomorphism classes of irreducible \( FG \)-modules, and \( e_1, \ldots, e_s \) are the corresponding central idempotents of \( FG \). An irreducible \( L \) in \( I_i \) is annihilated by all \( e_j \), except for \( e_i \), which acts on it as the identity. Let us once and for all choose a set \( L_1, \ldots, L_s \) of representatives for \( I_1, \ldots, I_s \).

Let \( \Delta : G \to \text{Aut}_F(M) \) be the representation associated with an \( FG \)-module \( M \). If \( \chi(\sigma) \) denotes the trace of \( \Delta(\sigma) \), the map \( \chi : G \to F \) is called the character of \( \Delta \) or of \( M \). We use the same name and notation for the associated linear functional \( FG \to F \).

**Lemma 2.1:** The \( FG \)-modules \( M \) and \( M' \) have the same character if they are isomorphic. The converse holds for \( F \) of characteristic 0.

**Proof.** Any isomorphism \( T : M \to M' \) must satisfy \( T\Delta(\sigma) = \Delta'(\sigma)T \), i.e., \( \Delta'(\sigma) = T\Delta(\sigma)T^{-1} \), for all \( \sigma \in G \); hence \( \chi = \chi' \). In particular, each isomorphism class \( I_i \) corresponds to a single character \( \chi_i \).

Now imagine a decomposition \( M = N_1 \oplus \cdots \oplus N_s \) into irreducibles. Then \( \chi(e_i) = \mu_i \delta_i \), where \( \mu_i \) is the multiplicity with which members of \( I_i \) occur among these components. If \( F \) has characteristic 0, this makes \( \chi = \chi' \) imply \( \mu_i = \mu'_i \), for all \( i \), so that \( M \) is isomorphic to \( M' \), q.e.d.

The next lemma shows how to compute the central idempotents \( e_1, \ldots, e_s \). Remember that \( M_i = (FG)e_i \) splits into \( m_i \) irreducible components, so that its character equals \( m_i \chi_i \).

**Lemma 2.2:**

\[
e_i = \frac{m_i}{n} \sum_{\sigma \in G} \chi_i(\sigma^{-1})\sigma.
\]

**Proof.** Let \( \chi_\infty \) be the character of \( FG \) itself. Then \( \chi_\infty(1) = n \), and \( \chi_\infty(\sigma) = 0 \) whenever \( 1 \neq \sigma \in G \). For \( \alpha = \sum_\sigma a_\sigma \tau \), this means that \( \chi_\infty(\sigma^{-1}\alpha) = na_\sigma \). On the other hand, \( \chi_\infty = \sum_j m_j \chi_j \) by (3), whence \( na_\sigma = \sum_j m_j \chi_j(\sigma^{-1}\alpha) \).

For \( \alpha = e_i \), this becomes \( na_\sigma = \sum_j m_j \chi_j(\sigma^{-1}e_i) = m_i \chi_i(\sigma^{-1}) \). The last equality is due to the fact that left multiplication by \( \sigma^{-1}e_i \) annihilates \( M_j \), for \( i \neq j \), and acts on \( M_i \) like \( \sigma^{-1} \); q.e.d.

Evaluating \( \chi_j(e_i) \) in (4), while keeping in mind that \( e_i \) acts like \( I_d \) on \( L_i \) and like 0 on all the other \( L_j \), we obtain the “first orthogonality relations”:

\[
\frac{1}{n} \sum_{\sigma \in G} \chi_i(\sigma^{-1})\chi_j(\sigma) = \begin{cases} 0 & \text{if } i \neq j \\ d_i/m_i & \text{if } i = j \end{cases}.
\]

To get a handle on the total number \( s \) of irreducible characters, we study the centre \( Z(FG) \). Breaking up \( G \) into its conjugacy classes \( C_1, \ldots, C_r \), we let \( \gamma_\nu \in FG \) stand for the sum over all \( \sigma \in C_\nu \). Clearly, \( \alpha = \sum_\nu a_\sigma \gamma_\nu \) in \( FG \) is central if and only if \( \alpha \) is fixed under conjugation by all \( \tau \in G \). This means that its coefficients \( a_\sigma \) must be constant on each \( C_\nu \), and hence \( \alpha \) is linear combination of the \( \gamma_\nu \). It follows that \( \gamma_1, \ldots, \gamma_r \) is a basis of \( Z(FG) \). On the other hand, \( e_1, \ldots, e_s \) is a linearly independent set in \( Z(FG) \); any relation \( e_1 e_1 + \cdots + e_s e_s = 0 \) would yield \( c_i e_i = 0 \) when multiplied by \( e_i \). Hence

\[
s \leq r \quad \text{and} \quad n = m_1 d_1 + \cdots + m_s d_s,
\]

where \( d_i \) denotes the \( F \)-dimension of a typical \( L \) in \( I_i \). Together, these relations are helpful for tracking down the possible types of irreducible \( FG \)-modules.

**Exercise:** If \( F \) has characteristic 0, show that \( n \mid m_i \). (Hint: The eigenvalues of any \( \Delta(\sigma) \) are of the form \( \zeta^\nu \), where \( \zeta \) is a primitive \( n \)-th root of 1. Let \( V_i \) be the \( Z \)-module generated by the elements \( \zeta^\nu e_i \), for \( \sigma \in G \) and \( \nu = 1, \ldots, n \). Multiplying (4) by \( (n/m_i)e_i \), obtain a relation which says that \( (n/m_i)V_i \subset V_i \). Using determinants, conclude that \( n/m_i \) is the root of a certain monic polynomial over \( Z \).)

2
3. Matrices. This section is in three parts: an abstract examination of matrices, some consequences for the components $M_i$ of $FG$ in the sum (3), and finally another version of the orthogonality relations (5).

I. Let $R$ be any ring, $V$ a (left) $R$-module, $E = \text{End}_R(V)$ its ring of endomorphisms. Then the ring $\mathcal{M}_m(E)$ of $m \times m$ matrices over $E$ is naturally isomorphic to $\text{End}_R(W)$, where $W = V \oplus \cdots \oplus V$ has $m$ components. Indeed, a matrix $(a_{ij})$ with entries in $E$ defines the endomorphism $\alpha : W \to W$ mapping $w = (v_1, \ldots, v_m)$ to the $m$-tuple whose $i$-th component is $\sum_j a_{ij}v_j$. Conversely, the matrix entry $a_{ij} \in E$ can be retrieved by following the obvious $j$-th injection $V \to W$ by a given $\alpha : W \to W$ and the $i$-th projection $W \to V$. Finally, it can be checked that the composite $\beta \alpha$ of two endomorphisms corresponds to the matrix product obtained by summing the composites $b_{ik}a_{kj}$ over $k$.

Let $R^o$ the “opposite” of $R$, i.e., a ring with the same elements as $R$ but with the order of multiplication reversed. $R^o$ is naturally identifiable with the ring of right multiplications in $R$, and this, in turn, is canonically isomorphic to $\text{End}_R(R)$, the endomorphisms of $R$ as a left $R$-module.

**Lemma 3.1:** Suppose that $R$, as a left $R$-module, is isomorphic to the direct sum $V \oplus \cdots \oplus V$ of $m$ copies of some $R$-module $V$. Then $R$ is isomorphic to the ring $\mathcal{M}_m(E^o)$, where $E = \text{End}_R(V)$.

*Proof.* By the preceding discussion, $R^o \simeq \mathcal{M}_m(E)$. To finish the proof, we note that the matrix transpose yields an isomorphism $\mathcal{M}_m(E^o) \to \mathcal{M}_m(E^o)$, a verification we leave to the reader, q.e.d.

II. Recall from Section 1, that $FG$ is the Cartesian product of rings $M_1, \ldots, M_s$, each $M_i$ being the direct sum of $m_i$ left ideals all isomorphic to a single $L_i$. Since $L_i$ is irreducible, all its non-zero endomorphisms are invertible (their kernels must be trivial, their images all of $L_i$), and hence $\text{End}_{FG}(L_i)^o$ is a division algebra $D_i$ over $F$. Applying Lemma 3.1 to $R = M_i$, we therefore have a ring isomorphism

$$M_i \simeq \mathcal{M}_{m_i}(D_i)$$

for every $i = 1, \ldots, s$. Letting $\delta_i$ denote the $F$-dimension of $D_i$, we conclude moreover that

$$\dim_F M_i = m_i^2\delta_i, \quad d_i = \dim_F L_i = m_i\delta_i, \quad \text{and} \quad n = m_1^2\delta_1 + \cdots + m_s^2\delta_s. \tag{8}$$

Since every element of any $D_i$ generates a finite field extension of $F$, we have $D_i = F$ and $\delta_i = 1$ for all $i$, whenever $F$ is algebraically closed.

III. The quotient $d_i/m_i$ appearing in the orthogonality relations (5) has thus been unmasked as $\delta_i$. Since characters are obviously constant on conjugacy classes we can rewrite these relations in the form

$$\frac{1}{n\delta_i} \sum_{\nu=1}^r x_i(C_{\nu})x_j(C_{\nu}^*)h_{\nu} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}, \tag{9}$$

with $h_{\nu} = |C_{\nu}|$ and $C_{\nu}^* = \{ \sigma \mid \sigma^{-1} \in C_{\nu} \}$. This can be interpreted as saying that the identity matrix $I_s$ is the product of the $s \times r$ matrix $(x_i(C_{\nu}))$ and the $r \times s$ matrix $(x_j(C_{\nu}^*)h_{\nu}/n\delta_j)$, where $i$ and $j$ always denote the row and column indices, respectively. If these matrices are square, the product of the factors can be reversed. In other words,

$$\frac{h_{\nu}}{n} \sum_{\nu=1}^r x_\nu(C_{\nu}^*)x_\nu(C_{\nu})/\delta_\nu = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}, \tag{10}$$

provided that $s = r$. These are the “second orthogonality relations”.

The proviso $r = s$ is always satisfied if $F$ is algebraically closed: in that case, each $M_i$ is a full matrix ring over $F$, and $Z(FG) = Z(M_1) \oplus \cdots \oplus Z(M_s)$ is the Cartesian product of $s$ copies of $F$. As a further bonus, the $\delta_i$ then disappear from the formula.