PRESYMPLECTIC HAMILTON AND LAGRANGE SYSTEMS, GAUGE TRANSFORMATIONS
AND THE DIRAC THEORY OF CONSTRAINTS

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1. INTRODUCTION

Differential equations of the form

\[ i_X \omega = \alpha \]  \hspace{1cm} (1)

where \( \alpha \) is a closed 1 form (Hamiltonian) and \( \omega \) is a closed 2 form (Lagrange bracket) frequently arise in physics. If \( \omega \) is symplectic such equations have unique solutions.\(^1\) However if \( \omega \) is often presymplectic (i.e. \( \beta : X \rightarrow i_X \omega \) is not an isomorphism). This occurs, e.g., (i) for many infinite dimensional systems, (ii) \textit{a priori} - as in Künzle’s\(^2\) treatment of a spinning particle in curved spacetime, (iii) for systems derived from degenerate Lagrangians (i.e., the fiber derivative\(^1\) \( FL : TQ \rightarrow T^*Q \) is not a diffeomorphism) as in gravity and electromagnetism.

If \( \omega \) is presymplectic neither existence nor uniqueness of solutions to the system (1) is guaranteed.

We present a geometric constraint algorithm which provides the necessary and sufficient conditions for the existence of solutions to the generalized Hamilton equations (1) on a (finite or infinite dimensional) presymplectic manifold \( (M, \omega) \). We discuss the application of our algorithm to Lagrangian and Hamiltonian systems and indicate its relationship to the Dirac theory of constraints. We also discuss the non-uniqueness of the solutions of (1) and present an algorithm for determining the associated “space of true degrees of freedom”.

2. THE CONSTRAINT ALGORITHM\(^3\) Our constraint algorithm finds the unique maximal submanifold \( N \rightarrow M \) along which (1) possesses solutions \textit{tangent} to \( N \). This final

constraint submanifold is the limit $N = \bigcap M_\lambda$ of a string of sequentially constructed constraint submanifolds

$$M_{\lambda+1} = \{ m \in M_\lambda | \alpha_m \in \beta(TmM_\lambda) \}$$  \hspace{1cm} (2)

(with $M_1 = M$) which follow from applying consistency conditions to (1).

Theorem: The presymplectic Hamiltonian system (1) has solutions if $N$ is non-empty, in which case we have consistent equations of motion along the final constraint submanifold $N$

$$0 = (i_\omega \alpha - \alpha)_{|N} \in T^*_NX$$  \hspace{1cm} (3)

with solutions $X \in TN$.

If $\beta$ is closed and $TM$ is reflexive Banach (which we henceforth assume) an equivalent but much more convenient characterization of the constraint submanifolds is

$$M_{\lambda+1} = \{ m \in M_\lambda | \omega_{\lambda+1} = 0 \}$$  \hspace{1cm} (4)

where $T^*N = \{ (Z \in TM | \omega(Y,Z) = 0 \forall Y \in TW) \}$ is the presymplectic orthogonal complement.

3. DIRAC CONSTRAINT THEORY: When applied to systems described by degenerate Lagrangians our algorithm significantly generalizes as well as globalizes the local Dirac-Bergmann (DB) theory of constraints. In this case $M = FL(TQ) \hookrightarrow T^*Q$ is Dirac's primary constraint submanifold, $\omega = j^*\Omega$ with $\Omega$ being the canonical symplectic structure on $T^*Q$, and $\alpha = dH$ where the Hamiltonian $H$ satisfies

$$H \circ FL = E_L$$  \hspace{1cm} (5)

with $E_L(v) = \langle v | FL(v) \rangle - L(v)$.

The DB constraint theory works on all of phasespace ($T^*Q$) and proceeds to find (analytically) an extension of $H$ to $T^*Q$ and secondary constraints which will guarantee preservation of all constraints. Our (geometric) algorithm works directly on the primary constraint submanifold $M$. The constraint functions $\omega_{\lambda+1}$ in (4) correspond to Dirac's secondary constraints.

In recent years there has been some nice work done in giving a symplectic geometric description of the results of the DB constraint theory. (See the works
of Śniatycki, Tulczjew, Lichnerowicz, Marsden and Weinstein and ref. 3.) Our work (following the presymplectic approach of Hinds) geometrizes the constraint algorithm itself and thereby complements and extends this description.

In contrast to the Dirac method, our algorithm does not a priori require $M$ to be a submanifold of some given symplectic manifold (e.g. $T^*Q$). This fact is responsible for the wide range of applicability of the algorithm.

4. **LAGRANGIAN SYSTEMS** Our constraint algorithm can be applied directly to the inherently presymplectic Lagrange equations

$$i_{X_L} \omega_L = dE_L$$

on velocity phase space $TQ$. The presymplectic form $\omega_L$ is $J^\ast \omega$ or equivalently $-dJ^\ast dL$ where $J^\ast$ is the adjoint of Klein's almost tangent structure $J : TTQ \rightarrow TTQ$

which satisfies

$$\ker J = \text{Im} J = V(TQ) = \ker \pi_T.$$

The ability of our algorithm to deal directly with Lagrangian systems on $TQ$ is of particular importance because the fiber derivative $FL$ may be too pathological for a Hamiltonian formalism to exist in $T^*Q$. (For example (5) may not define a single valued function $H$.) There are two special cases of interest.

**Definition** $L$ is *almost regular* if $FL$ is a submersion onto its image and the fibers of $FL$ are connected.

**Definition** $L$ is *quasi-regular* if the leaf space $TQ/(\ker FL_\ast)$ of the foliation of $TQ$ generated by the involutive distribution $\ker FL_\ast$ is a manifold such that the canonical projection is a submersion. (Note almost regular implies quasi-regular.)

**Theorem** If $L$ is almost regular there exists a corresponding Hamiltonian formulation of the dynamics on a submanifold of $T^*Q$ and the Lagrangian and Hamiltonian formalisms are equivalent (i.e. each constraint submanifold of the Lagrangian system maps onto a corresponding constraint submanifold for the Hamiltonian system).

**Theorem** If $L$ is quasi-regular there is a corresponding equivalent generalized Hamiltonian system; however it is not given on a submanifold of $T^*Q$ but rather
on the leaf space $TQ/(\ker FL_*)$.

4A. **Second Order Equation Problem** Lagrangian systems differ from Hamiltonian systems in that from both physical\(^1\) and variational\(^{16}\) viewpoints the Lagrange equations (6) must be supplemented by the 2nd order equation condition

$$TQX = \tau_Q X \quad (7)$$

This condition was first investigated for homogeneous degenerate Lagrangians by Künzle\(^{18}\) who considered only degeneracies which arise from the homogeneity. He developed an algorithm which in principle solves the problem, but he gave no specific results in the general case. We have taken a different approach and considered the general degenerate Lagrangian, finding that (7) is compatible with our constraint algorithm.

**Theorem**\(^{19}\) If $L$ is quasi-regular there exists a 1 to 1 correspondence between solutions to (6,7) and solutions to the corresponding generalized Hamiltonian system on $TQ/(\ker FL_*)$.

5. **NON-UNIQUENESS OF SOLUTIONS** Presymplectic Hamiltonian systems in general do not have unique solutions. To any vector field $X \in TN$ satisfying (3) along the final constraint submanifold $N \xrightarrow{M_1} M_1$ one can add any element $Y$ in

$$K_1 = TN \cap TM_1^* \quad (8)$$

(These vector fields are generated by Dirac's primary 1st class constraints.)

The presence of these "gauge" vector fields indicates that $N$ contains redundant information. For certain purposes (e.g. quantization) one would like to eliminate this redundant information by constructing a *reduced phase space* (RPS) i.e. a "space of true degrees of freedom". One would like the RPS to be a symplectic manifold with a unique evolution determined by a Hamiltonian system.

5A. **Standard RPS** A reduced phase space can be constructed as follows: pull back (3) to $N$ obtaining

$$i_X^\alpha_N = \alpha_N \quad (9)$$

where $\alpha_N = i_{\omega_N}^\ast$ and $\omega_N = i_{\psi_0}^\ast$. The solutions to (9) are unique up to elements in
the characteristic vector fields of $\omega_N$. These vector fields form an involutive distribution which determines a foliation of $N$. On the leaf space of this foliation $\overline{N} = N/(K_N)$ there is a symplectic (hence unique solution) Hamiltonian system

$$i_X\omega = \alpha_N$$

induced by (9) - i.e. our constraint algorithm guarantees that $\alpha$ is admissible for $N$ in the sense of Lichnerowicz. We call $\overline{N}$ the standard RPS. Regarding $\overline{N}$ as the "space of true degrees of freedom" of the system is equivalent to regarding all of $K_N$ as gauge vector fields which corresponds to Dirac's extended Hamiltonian using all 1st class constraints (primary and secondary) as gauge generators. Dirac himself was uncertain as to whether this was correct - "I think it may be that all the first-class secondary constraints should be included among the transformations which don't change the physical state, but I haven't been able to prove it."20

5B. Gauge Vector Field Algorithm (GVA) RPS21 In order to see how the manifest $K_1$ gauge invariance of (3) could imply $K_N$ gauge invariance we have developed a gauge vector field algorithm (GVA) for extracting the "hidden" gauge transformations in (3). The associated GVA RPS is obtained by "moding out" the "gauge directions" in $N$: i.e. one wants to identify points connected by "gauge vector fields" (GVF's) since such points can be reached from the same initial data. The set of GVF's can be considerably larger than $K_1$ for if $Z$ is a GVF then so is $[X,Z]$ where $X$ is any solution to (3). This follows from the diagram

A similar construction shows that $[Z,Y]$ is a GVF whenever $Z$ and $Y$ are GVF's. Starting from $G_1 \equiv K_1$ (the manifestly obvious GVF's) and iterating one obtains the implicit GVF's
\begin{equation}
  G_{x+1} = G_x + [G_x, G_x] + [X, G_x] \tag{12}
\end{equation}

whose limit is an involutive distribution \( G \) - the Lie module generated by \( (\mathfrak{g}_X)^0 \).

The associated GVA RPS is the leaf space \( \tilde{N} = N/G \) of the foliation generated by \( G \). Is the GVA RPS the same as the standard RPS?

**Theorem** \( G = K_N \) iff the first class constraints \( [K_x, \alpha] \) are "good" constraints (i.e. their differentials do not vanish along \( N \)) where \( K_x \equiv TN \cap TM^A \).

In general \( G \) is a proper Lie submodule of \( K_N \) (i.e. some 1st class secondary constraints do not generate gauge transformations of (3)) and consequently \( \tilde{N} \) is larger than \( N \). In general \( \tilde{N} \) is inherently presymplectic. (In fact \( \tilde{N} \) can be odd dimensional!) Whenever \( G \neq K_N \) there is no natural symplectic Hamiltonian system on \( \tilde{N} \) for the unique evolution \( \tilde{x} \in T\tilde{N} \). Consequently regarding \( \tilde{N} \) as the space of true degrees of freedom has some undesirable features (e.g. How would one quantize such a system?). On the other hand \( N \) has the virtue of literally respecting the equations of motion (4).

Which RPS is the "space of true degrees of freedom"?

5C. A Possible Resolution If the Hamiltonian contains a built in gauge condition then \( G \) will respect it while \( K_N \) will not. So it may be possible to interpret \( G \neq K_N \) as a consequence of built in gauge conditions. Two examples may clarify this point. Consider the Lagrangian densities

\begin{equation}
  \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\lambda} (\partial_\mu A^\mu)^2 \tag{13}
\end{equation}

\begin{equation}
  \mathcal{L} = -\frac{1}{2} (\partial_\mu A^\nu) (\partial^\mu A^\nu) - A^\mu \partial_\mu \phi - \frac{\lambda}{2} \phi^2 \tag{14}
\end{equation}

where \( \lambda \) is a Lagrange multiplier and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). From (13) one obtains Maxwell's equations and the Lorentz condition \( \partial_\mu A^\mu = 0 \). If we interpret this system as electromagnetism with the Lorentz gauge condition imposed then it is easy to understand why \( G \neq K_N \), for \( G \) respects the Lorentz condition while \( K_N \) does not. The Lagrangian density (14) in fact yields the same effective equations as (13), though now one may be tempted to interpret it as what it appears to be - not E&M but an entirely different (massless, spin 1, divergence free) field. This
is the GWA interpretation of both (13) and (14). Thus whether or not Dirac's conjecture is correct (and which RPS is correct) depends on one's physical interpretation of the system. In general our algorithm permits a whole class of interpretations bounded by one (G) which extracts the gauge invariance of a given Lagrangian (and yields an inherently presymplectic RPS) and the standard one (KN) which extracts the maximum possible invariance of the given system (and has a symplectic Hamiltonian system for the unique evolution on its RPS). The former violates Dirac's conjecture while the latter satisfies it by interpreting G ≠ KN to be entirely a consequence of built-in gauge conditions.

4. We assume here and throughout that quotients and constraints yield sufficiently nice manifolds. For many interesting systems this is not true - one must then cut the system up into nice pieces. See refs. 8 and 12.
5. Actually this is a particular application of a general integrability algorithm that we have developed for systems of equations.
14. This generalizes Śniatycki's definition. 8
17. If L is degenerate (7) is not a consequence of (6).
20. See ref. 5, pp. 23,24.