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FORMAL QUANTIZATION OF QUADRATIC MOMENTUM OBSERVABLES

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The problem of quantizing classical observables is a venerable one. It is well known that it is impossible to fully quantize all classical observables [1-3]; in fact, it is not even possible to consistently quantize the algebra of polynomials on a Euclidean phase space [1,3-6]. Consequently, research has centered on both quantizing restricted classes of observables [1,7,8] and weakening the notion of a "full" quantization [2,8-10]. To these ends, numerous quantization schemes have been developed (see, e.g., [7,10-15]).

In this paper, I study the quantization of quadratic momentum observables using the geometric quantization procedure of Kostant and Souriau [12-14] in the Schrödinger representation.

Consider a classical system with an oriented n-dimensional configuration space M. It is convenient (but certainly *not* necessary) in what follows to assume that M comes equipped with a semi-Riemannian metric g. Let $V = \ker T\pi$ be the vertical polarization on phase space T^*M, where $\pi: T^*M \to M$ is the projection.

I take the prequantization line bundle to be trivial and use the orientation on M to induce a metaplectic structure for V on T^*M. Relative to this data, the quantum Hilbert space is $L^2(M, |\det g|^{\frac{1}{2}})$.

Turning now to the classical observables, denote by $C_N(V)$ the subspace of $C^\infty(T^*M)$ consisting of functions which are at most N^{th}-degree polynomials along the fibers of π. Then $C_0(V) = \pi^* C^\infty(M)$ is the set of *configuration observables*. The space $C_1(V)$ consists of *linear momentum observables* $P_X + f$, where $f \in C_0(V)$ and P_X is the momentum in the direction of the vector field X on M.

The linear momentum observables play a distinguished role in quantization theory since they form a Schrödinger subalgebra [2,16] of the Poisson algebra $C^\infty(T^*M)$. In particular, this means that elements of $C_1(V)$ can be unambiguously and equivalently quantized in all viable quantization schemes [2,8]. This is so in geometric quantization because the flow of (the Hamiltonian vector field of) each such observable preserves the vertical polarization. Specifically, $P_X + f \in C_1(V)$ is quantized as the operator

$$Q(P_X + f) = -i(X + \tfrac{1}{2}\operatorname{div} X) \tag{1}$$

on $L^2(M,|det\ g|^{\frac{1}{2}})$, where $\hbar = 1$.

However, trouble arises when quantizing momentum observables of degree two or higher. Even when applicable, different quantization procedures will typically yield ambiguous and/or inequivalent results. Geroch [17] gives a succinct discussion of the problems that can occur. For example, due to factor-ordering ambiguities, the canonical quantization of an element of $C_N(V)$ is only determined up to the addition of a differential operator of order $N-2$; this quantization is therefore well defined only in the semiclassical limit (see [15] for details and [8] for an illustration). In geometric quantization, on the other hand, such observables present difficulties because their flows do not preserve the vertical polarization. By suitably pairing V with some transverse polarization $V^{\perp}$, Kostant [12] is able to avoid this problem and can actually quantize all polynomial momentum observables. But $V^{\perp}$ may not exist and, when it does, the quantizations resulting from different choices of $V^{\perp}$ agree only in the semiclassical limit. For this reason, I work solely with the vertical polarization, i.e., in the Schrödinger representation. Quantization then requires "moving" V and employs the Blattner-Kostant-Sternberg kernels [13,14].

Regarding $C_2(V)$, the main result is

Theorem: *Suppose that* $\{X_1,\ldots,X_K\}$ *is a collection of linearly independent commuting vector fields on* M. *Let* γ *and* ζ *be any homogeneous quadratic and linear polynomials, respectively. Then*

$$Q\left[\gamma(P_{X_1},\ldots,P_{X_K}) + \zeta(C_1(V))\right]$$
$$= \gamma(QP_{X_1},\ldots,QP_{X_K}) + \zeta(Q[C_1(V)]). \qquad (2)$$

Remark: There are several strong analytic and geometric conditions, not made explicit here, which must be satisfied before (2) rigorously follows. If any of these conditions fail, as is commonly the case, then (2) must be interpreted in a formal sense. These technical matters are discussed in [13,14,18].

Sketch of Proof: Write

$$\gamma(P_{X_1},\ldots,P_{X_K}) = \gamma^{ij}P_{X_i}P_{X_j}$$

where the γ^{ij} are constants. Without loss of generality, it may be assumed that γ^{ij} has rank K (for otherwise it would be possible to eliminate some constant linear combinations of the P_{X_i}'s). Then a constant linear transformation $X_i \to Y_i$ brings γ into the form

$$\gamma = \sum_{i=1}^{K} \varepsilon_i \left(P_{Y_i}\right)^2 ,$$

where $\varepsilon_i = \pm 1$. Furthermore, by the assumptions on the X_i and the construction of the Y_i, there exists a coordinate system $\{q^a\}$, $a = 1,\ldots,n$, such that $Y_i = \partial_i$, $i = 1,\ldots,K$. Finally, since any element of $C_1(V)$ can be written as $P_Z + f$ for some Z and f, the observable to be quantized becomes

$$\sum_{i=1}^{K} \varepsilon_i p_i^{\,2} + Z^a(q)p_a + f(q),$$

where $p_a = P_{\partial_a}$.

With this normal form, it is now straightforward to compute the LHS of (2) via the BKS transform (see §5-7 of [13] for the details of such calculations). The desired result then follows from a comparison of this with the RHS of (2), which is easily computed in these coordinates using (1). ∎

I now examine some of the implications of this result.

First of all, note that (2) yields

$$QF^2 = (QF)^2 \tag{3}$$

for all $F \in C_1(V)$. This may or may not be surprising, but in any case is at variance with the predictions of most other quantization schemes (e.g., [9-12,15]). On the other hand, (3) *is* consistent with von Neumann's rule (I) [19].

Secondly, simple examples show that (2) does not respect the "Poisson bracket → commutator" rule. This is unfortunate, but not entirely unexpected [1-7].

Thirdly, the conditions on the X_i's indicate that the Theorem is capable of quantizing only a limited subset of $C_2(V)$. But these requirements -- involutivity in particular -- are essential; the Theorem will fail without them.

For instance, it is not difficult to see that quantization cannot be multiplicative over noncommuting observables in general [7,17]. Indeed, suppose that X is nonvanishing and that g is a positive function on M chosen in such a way that $[gX, g^{-1}X] \neq 0$. Then

$$(QP_X)^2 = QP_X^{\,2} = Q(P_{gX}P_{g^{-1}X})$$

is not equal to the symmetrized RHS of (2), which in this case is just the anti-commutator of QP_{gX} with $QP_{g^{-1}X}$. Thus the Theorem is also incompatible with the "product → anti-commutator" rule.

More significant, however, is the fact that quantization is not necessarily *additive* over noncommuting observables. In particular, it follows from (1) and (3) that $QP_X^{\,2}$ depends only upon $X \otimes X$. But $QP_X^{\,2} + QP_Y^{\,2}$ will not usually depend only upon $X \otimes X + Y \otimes Y$ unless $[X,Y] = 0$. A simple example is provided by the free particle Hamiltonian on $M = \mathbf{R}^2$ in polar coordinates.

Thus geometric quantization violates von Neumann's additivity axiom (II) [19] and again this is at odds with most other quantization techniques.

The lack of additivity is really what prevents one from quantizing general elements of $C_2(V)$. On the other hand, the Theorem shows that one need only consider observables which are homogeneous quadratic in the momenta. Such an observable may be written

$$\gamma = \gamma^{ab}(q)p_a p_b.$$

Only in special cases can γ be easily quantized.

For example, if $rk\ \gamma = 1$ everywhere, then up to sign there exists a unique globally defined vector field X on M such that $\gamma^{ab} = X^a X^b$. Then $\gamma = P_X^2$ can be directly quantized.

When $rk\ \gamma > 1$ and the hypotheses of the Theorem are not satisfied, one must quantize γ by brute force. There is no problem with this in principle, but the BKS computations are intractable. To simplify them, it is necessary to find a suitable normal form for γ (note that the conditions on the X_i's in the Theorem also serve to guarantee the existence of such a normal form).

Other than the cases covered by the Theorem, apparently the only instance in which a suitable normal form exists is when $rk\ \gamma = n$. In particular, consider the Hamiltonian

$$H = \tfrac{1}{2}g^{ab}p_a p_b + V(q)$$

which can be formally quantized in Riemann normal coordinates [13,14]. The result is

$$QH = -\frac{1}{2}(\Delta - \frac{R}{6}) + V(q)$$

where Δ is the Laplace-Beltrami operator and R is the scalar curvature of g.

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