MOMENTUM MAPS & CLASSICAL FIELDS

4. Gauge Symmetries and IV Constraints

MARK J. GOTAY
Overview:

Here I discuss:

- gauge groups
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- gauge groups
- the Vanishing Theorem
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- gauge groups
- the Vanishing Theorem
- the relationship between the E-M map and (first class) secondary constraints
Gauge Groups

Let $G$ be a group of automorphisms of $Y$.

- Assume that $G$ is a covariance group, i.e., $L$ is $G$-covariant.
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**Proposition**

The induced action of $G$ on $Y$ stabilizes the space of solutions to the E–L equations.

- The proof relies upon: $L$ equivariant $\implies \Theta_L$ invariant, and the way the E–L equations are formulated in terms of the Cartan form.
Gauge Groups

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- Assume that $\mathcal{G}$ is a covariance group, i.e., $\mathcal{L}$ is $\mathcal{G}$-covariant.

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- The proof relies upon: $\mathcal{L}$ equivariant $\implies \Theta_{\mathcal{L}}$ invariant, and the way the E–L equations are formulated in terms of the Cartan form.

- Thus a covariance group is a symmetry group.
Chern–Simons shows that the converse fails.

What distinguishes a gauge group from a mere covariance group is:

Localizability

$G$ localizable provided that for each pair of disjoint hypersurfaces $\Sigma_1$ and $\Sigma_2$ in $X$ and each $\xi \in g$, there is a Lie algebra element $\chi \in g$ such that

$\chi Y |_{\pi^{-1}XY(\Sigma_1)} = \xi Y |_{\pi^{-1}XY(\Sigma_1)}$ and

$\chi Y |_{\pi^{-1}XY(\Sigma_2)} = 0$

For strings, $\text{Diff}(X) \ltimes C^\infty(X, \mathbb{R}^+)$ is localizable.

The Poincaré group is not.
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Definition

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- Some CFTs have no gauge symmetries: Proca on a frozen background. BUT: this is not generally covariant.

- Any **generally covariant** ("parametrized") CFT has nontrivial gauge symmetry, since for such theories the projection of $\mathcal{G} \subset \text{Aut}(Y)$ in $\text{Diff}(X)$ in “large”.
Definition

Instantaneous gauge transformations

These are diffeomorphisms of the final constraint set $C_{\tau}$ which preserve the induced presymplectic form $\varpi_{\tau}$. The instantaneous gauge algebra is $\ker \varpi_{\tau}$. Of course, $G$ usually does not act in the instantaneous formalism. Nonetheless, using the instantaneous energy-momentum map $E_{\tau}: P_{\tau} \to g^*$, naturally defines a Lie subalgebra $g_{C_{\tau}} \subset X(C_{\tau})$ which serves as an (infinitesimal) action. (Elements of $g_{C_{\tau}}$ are the Hamiltonian VFs of the $E_{\tau}(\xi)$.)
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- Of course, $\mathcal{G}$ usually does not act in the instantaneous formalism.

- Nonetheless, using the instantaneous energy-momentum map $\mathcal{E}_\tau : \mathcal{P}_\tau \to \mathfrak{g}^*$, $\mathfrak{g}$ naturally defines a Lie subalgebra $\mathfrak{g}_{C_\tau} \subset \mathfrak{X}(C_\tau)$ which serves as an (infinitesimal) action. (Elements of $\mathfrak{g}_{C_\tau}$ are the Hamiltonian VFs of the $\mathcal{E}_\tau(\xi)$.)
Key assumption here: all fields are variational.
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It follows that *if \( \mathcal{G} \) is a gauge group in the covariant sense and all fields are variational, then \( \mathcal{G} \) "acts" on \( C_T \) by gauge transformations in the instantaneous sense.*
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It follows that *if* $\mathcal{G}$ *is a gauge group in the covariant sense and all fields are variational*, then $\mathcal{G}$ "acts" on $C_\tau$ by gauge transformations in the instantaneous sense.

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Recall that $\mathcal{G}$ is assumed known at the outset.
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For strings, both \( \text{Diff}(X) \ltimes C^\infty(X, \mathbb{R}^+) \) and its subgroup \( \text{Diff}(X) \) are full.

A sufficient condition for fullness will come later.
The Vanishing Theorem

Here we begin the program of relating initial value constraints to zero levels of the energy-momentum map by proving the Vanishing Theorem

- Let $\Sigma_+$ and $\Sigma_-$ be two hypersurfaces in $X$ which form the boundary of a compact region; we call $\Sigma_+$ and $\Sigma_-$ an admissible pair of hypersurfaces.

(We have in mind the case of a spacetime $X$ with $\Sigma_+$ and $\Sigma_-$ deformations of a given hypersurface to the future and past, respectively)
Figure: An admissible pair of hypersurfaces
Lemma

For every compact oriented hypersurface $\Sigma$ there is a *disjoint* hypersurface $\Sigma'$ such that $(\Sigma, \Sigma')$ form an admissible pair.
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Now suppose that we have a field theory with gauge group $\mathcal{G}$ in which all fields are variational. Then from the first Noether theorem and Stokes’ theorem,

$$
\int_{\Sigma^+} \tau^*_+ (j^1 \phi)^* J^L (\xi) = \int_{\Sigma^-} \tau^*_- (j^1 \phi)^* J^L (\xi).
$$

for every solution $\phi$ of the Euler–Lagrange equations and admissible pair of hypersurfaces $\Sigma^+$ and $\Sigma^-$, where $\tau_{\pm} : \Sigma_{\pm} \rightarrow X$ are the inclusions.
By localizability, this integral vanishes.
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**Vanishing Thm**

Let $\mathcal{L}$ be the Lagrangian density for a field theory with gauge group $G$. Then for any solution $\phi$ of the Euler–Lagrange equations and hypersurface $\Sigma$, the energy-momentum map on $\Sigma$ in the Lagrangian representation vanishes:

$$\int_{\Sigma} \tau^* (j^1\phi)^* J^\mathcal{L}(\xi) = 0$$

for all $\xi \in g$, where $\tau : \Sigma \rightarrow X$ is the inclusion.
Bosonic String

Using the expression already given for the Lagrangian multimomentum map given previously, the VT forces:

$$g_{AB}(h^{00}\phi^A,0\phi^B,0) - h_{11}^{AB}\phi^A,1\phi^B,1) = 0$$

—superHamiltonian constraint in Lagrangian disguise

$$g_{AB}h^{0\mu}\phi^B,\mu\phi^A,1 = 0$$

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(First Class) Secondary Constraints

Goals are to show that:

- the vanishing of the instantaneous EM map over $\Sigma_{\tau}$ yields (first class) secondary IV constraints
- it yields all such constraints.

**Thm**

Suppose that the Euler–Lagrange equations are well-posed. Then

$$C_{\tau} \subset \mathcal{E}_{\tau}^{-1}(0)$$
Proof

Let \((\varphi, \pi) \in C_\tau\). Then there is a solution \(\phi \in \mathcal{Y}\) of the Euler–Lagrange equations with initial data \((\varphi, \pi)\). Set 
\[\sigma = FL \circ j \phi \circ \tau\] so that \(\sigma\) is a holonomic lift of \((\varphi, \pi)\). Now apply the Vanishing Theorem to \(\phi\), obtaining for each \(\xi \in \mathfrak{g}\),

\[
0 = \int_{\Sigma} \tau^* (j \phi)^* J^L(\xi) = \int_{\Sigma} \tau^* (j \phi)^* FL^* J(\xi)
\]

\[
= \int_{\Sigma} \sigma^* J(\xi) = \langle E_\tau(\sigma), \xi \rangle
\]

\[
= \langle E_\tau(\varphi, \pi), \xi \rangle
\]

as \(\sigma\) is a holonomic lift of \((\varphi, \pi)\). Thus \((\varphi, \pi) \in \mathcal{E}_\tau^{-1}(0)\). \(\blacksquare\)
This result shows that the conditions $\langle \mathcal{E}_\tau, \xi \rangle = 0$ are secondary initial value constraints.
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Recall the standing assumption that all secondary constraints are first class
Energy-Momentum Theorem, Version I

If $\mathcal{G}$ is full and $\mathcal{E}_{\tau}^{-1}(0)$ is connected, then

$$C_{\tau} = \mathcal{E}_{\tau}^{-1}(0)$$
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Energy-Momentum Theorem, Version II
If $\mathcal{E}_\tau^{-1}(0)$ is coisotropic, then

$$C_\tau = \mathcal{E}_\tau^{-1}(0)$$
Version II is more appealing than Version I, as:

- It requires no a priori knowledge of $C\tau$.
- Its hypothesis is straightforward (although not necessarily trivial) to verify in practice.
- It eliminates the necessity of having to worry about the connectedness of $E^{-1}\tau(0)$.

We also obtain a useful criterion for fullness:

$$Thm\ G_{full}^{+}C_{\tau}\ coisotropic\iff C_{\tau}=E^{-1}\tau(0)\iff E^{-1}\tau(0)\ coisotopic.$$
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- One has similar results for primary constraints vis-à-vis a certain momentum map.
The Adjoint Formalism

Dynamic and Atlas Fields

The arena for dynamics is the primary constraint set $P_\tau \subset T^* Y_\tau$. Our first task is to invariantly split the fields and their momenta on $(P_\tau, \omega_\tau)$ into the dynamic fields $\psi$ and their conjugate momenta $\rho$ and the nondynamic atlas fields $\alpha$.

Cotangent reduction gives $P_\tau / (\ker \omega_\tau)$ $\approx T^* D_\tau \ker \omega_\tau$ are kinematic directions $D_\tau$ comprise the dynamic fields
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The Adjoint Formalism

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- \( \ker \omega_\tau \) are kinematic directions
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Massive bundle-theoretic constructions lead to

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\mathcal{P}_\tau \approx T^*D_\tau \times A_\tau
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**Thm**

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- \( A_\tau \) comprise the atlas fields
- They are certain invariant combinations involving kinematic fields and elements of \( \mathfrak{g} \)
Massive bundle-theoretic constructions lead to

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- \( \mathcal{A}_\tau \) comprise the atlas fields
- They are certain invariant combinations involving kinematic fields and elements of \( \mathfrak{g} \)
- Instead of dynamics on the presymplectic space \( \mathcal{P}_\tau \), we have “parametric dynamics” on the symplectic space \( T^*\mathcal{D}_\tau \)
Massive bundle-theoretic constructions lead to

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- Instead of dynamics on the presymplectic space \( \mathcal{P}_\tau \), we have “parametric dynamics” on the symplectic space \( T^*\mathcal{D}_\tau \)

- The Thm does not hold if one replaces “atlas fields” by “kinematic fields” (subtle)
Then the atlas construction leads to

**Thm**

\[ H_{\tau,\zeta} \text{ is linear in the atlas fields } \alpha_j \]
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\textbf{Thm}

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- This devolves to the Hamiltonian being linear in \( \zeta \)
- BUT the Thm does not hold if one replaces “atlas fields” by “kinematic fields” \textit{(subtle)}

Combining these results with the Energy-Momentum theorem, we obtain item 4 of the introduction

\textbf{Thm}

\[ H_{\tau,\zeta} = \int_{\Sigma_{\tau}} \alpha_j \Phi^i(\Psi, \rho) d^n x_0 \]
The Adjoint Equations

\[ H = \langle \alpha, \Phi \rangle \] on \[ A \times T^* \mathcal{D} \]

Write \[ \Pi : T^* \mathcal{D} \to T \mathcal{D} \] for the "Poisson tensor"; i.e., the tensor such that the Hamiltonian vector field \[ X_f = \Pi \cdot df \]

So Hamilton's equations \[ X_H(\psi,\rho) = \omega = dh(\psi,\rho) = \Pi(\psi,\rho) \cdot \langle \alpha, D\Phi(\psi,\rho) \rangle. \]
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We simplify the discussion for clarity, dropping all subscripts.

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- So Hamilton’s equations $X_H(\psi, \rho) \cdot \omega = dH(\psi, \rho)$ are

$$\left( \begin{array}{c} \dot{\psi} \\ \dot{\rho} \end{array} \right) = \Pi(\psi, \rho) \cdot dh(\psi, \rho) = \Pi(\psi, \rho) \cdot \langle \alpha, D\Phi(\psi, \rho) \rangle.$$
Introduce an almost complex structure
\[ \mathcal{J}(\psi, \rho) : T_{(\psi, \rho)} \mathcal{D} \to T_{(\psi, \rho)} \mathcal{D} \]
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\[ \mathcal{J}(\psi, \rho) \cdot \mathcal{W} = \Pi(\psi, \rho) \cdot \langle \mathcal{W}, - \rangle \]
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Finally, define the adjoint of \( D\Phi(\psi, \rho) \) according to
\[ \langle \alpha, D\Phi(\psi, \rho) \cdot \mathcal{W} \rangle := \langle D\Phi(\psi, \rho)^* \alpha, \mathcal{W} \rangle. \]
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\]

Then we obtain Hamilton’s equations in adjoint form

\[
\begin{pmatrix}
\dot{\psi} \\
\dot{\rho}
\end{pmatrix} = J(\psi, \rho) \cdot D\Phi(\psi, \rho)^* \cdot \alpha
\]

which is item 5!
Introduce an almost complex structure
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