MOMENTUM MAPS & CLASSICAL FIELDS

2. Covariant Field Theory

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Overview:

I develop some basic CFT from a covariant viewpoint, including:

- Geometry of the jet bundle and the Euler–Lagrange equations (analogous to that of the tangent bundle & the Lagrange equations in mechanics)
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- Geometry of the jet bundle and the Euler–Lagrange equations (analogous to that of the tangent bundle & the Lagrange equations in mechanics)

- Multisymplectic geometry (analogous to the geometry of the cotangent bundle)

- Conservation laws and Noether’s theorem using covariant momentum maps (generalizing the concept of momentum map familiar from mechanics)
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- **Covariant (or multisymplectic)** — dynamics described in terms of the finite-dimensional space of fields at a given event in spacetime.

Both are useful and have their own advantages.
Covariant Configuration Bundle $Y$

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- Compare $Y = \mathbb{R} \times Q \to \mathbb{R}$ in (time-dependent) mechanics

- Coordinates $(x^\mu, y^A) = (x^0, x^1, \ldots, x^n, y^1, \ldots, y^N)$ on $Y$.

- Conventions
Bosonic String:

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- Strings are maps \(\phi : X \to M\) (i.e., sections of \(X \to X \times M\))
- Think of \(\phi\) as being an \(M\)-valued scalar field on \(X\)
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The Jet Bundle $JY$

For first order theories:

$$J_x Y = \{[\phi] \mid \phi_1 \equiv \phi_2 \text{ at } x \text{ iff } \phi_1(x) = \phi_2(x) \text{ and } T_x \phi_1 = T_x \phi_2\}$$
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- $J_Y \to Y$ is an affine bundle, with fiber over $y \in Y_x$ being

$$J_y Y = \{ \gamma \in L(T_x X, T_y Y) \mid T\pi_{X,Y} \circ \gamma = \text{Id}_{T_x X} \}$$
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3. Underlying vector bundle has fiber

\[ L(T_x X, V_y Y) = \{\gamma \in L(T_x X, T_y Y) \mid T_{\pi_X,Y} \circ \gamma = 0\} \]
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- Coordinates on $J_Y$ are $(x^\mu, y^A, v^B_{\nu})$
The jet prolongation of $\phi : X \to Y$ is $j\phi : X \to JY$ given by $x \mapsto T_x\phi$
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In coordinates, $j\phi$ is

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Compare mechanics: $JY \approx \mathbb{R} \times TQ$
The Dual Jet Bundle

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- Its fiber over $y \in Y_x$ is

$$\{\text{affine maps } J_y Y \to \wedge^{n+1}_x X\}$$

Use affine maps as $JY$ is an affine bundle.
The Dual Jet Bundle

- $JY^*$ is the affine dual of $JY$

- Its fiber over $y \in Y_x$ is
  \[
  \{ \text{affine maps } J_y Y \to \Lambda_{x}^{n+1} X \} 
  \]

  Use affine maps as $JY$ is an affine bundle.

- Fiber coordinates on $JY^* \to Y$ are $(p, p_A^\mu)$, corresponding to the affine map
  \[
  v^A_\mu \mapsto (p + p_A^\mu v^A_\mu) \, d^{n+1}x
  \]

  where
  \[
  d^{n+1}x = dx^0 \wedge dx^1 \wedge \cdots \wedge dx^n
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  \[ d^{n+1}x = dx^0 \wedge dx^1 \wedge \cdots \wedge dx^n \]
- $J^* Y$ is a vector bundle.
- In mechanics, $J^* Y \approx T^* \mathbb{R} \times T^* Q$
Proposition: $JY^* \simeq Z$, where

$$Z_y = \{ z \in \Lambda_{y}^{n+1} Y \mid i_v i_w z = 0 \text{ for all } v, w \in V_y Y \}.$$
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- $z \in Z$ takes the form

$$z = pd^{n+1}x + pA^\mu dy^A \wedge d^n x_\mu$$

where $d^n x_\mu = \partial_\mu \perp d^{n+1}x$. 

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where $d'^n x_\mu = \partial_\mu \land d^{n+1} x$.

- Intrinsically, the isomorphism $\vartheta : Z \to JY^*$ is

$$\langle \vartheta(z), \gamma \rangle = \gamma^* z \in \Lambda_{x}^{n+1} X$$

where $z \in Z_y$, $\gamma \in J_y Y$ and $x = \pi_{XY}(y)$. 
Since $Z$ is a bundle of $(n+1)$-forms, it carries a tautological $(n+1)$-form $\Theta$ defined by

$$\Theta(z) = \pi^*_Y z$$

and $\Theta$ is the multi-Liouville form, $\Omega = -d\Theta$ is the multisymplectic form. $(Z, \Omega)$ is the covariant or multi-phase space.
Canonical forms:

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- In coordinates,

$$\Theta = p_A^\mu dy^A \wedge d^n x_\mu + pd^{n+1}x$$

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Remarks:

- The affine terms $pd^{n+1}x$ in $Z$ and $\Theta$ are crucial; they are responsible for
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- General multisymplectic geometry?

- Poisson brackets?
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Coordinates on $JY$: $(x^\mu, \phi^A, h_{\sigma\rho}, \phi^A_{\mu}, h_{\sigma\rho\mu})$
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- Coordinates on $JY$: \((x^\mu, \phi^A, h_{\sigma\rho}, \phi^A_{\mu}, h_{\sigma\rho\mu})\)
- Coordinates on $Z$: \((x^\mu, \phi^A, h_{\sigma\rho}, p, p^A_{\mu}, \rho^{\sigma\rho\mu})\)
Lagrangian Dynamics

The Lagrangian density:

\[ L: JY \rightarrow \Lambda^{n+1} X \]

In coordinates

\[ L = L(x^\mu, y^A, v^A_{\mu}) \, d^n + 1 \times x. \]

No regularity assumption on \( L \); it would fail in almost all examples.
Lagrangian Dynamics

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No regularity assumption on \( L \); it would fail in almost all examples.
The Legendre transformation:

\[ \mathcal{F} \mathcal{L} : JY \rightarrow JY^* \text{ defined by} \]

\[ \langle \mathcal{F} \mathcal{L}(\gamma), \gamma' \rangle = \mathcal{L}(\gamma) + \frac{d}{d\varepsilon} \mathcal{L}(\gamma + \varepsilon(\gamma' - \gamma)) \bigg|_{\varepsilon=0}. \]
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- In coordinates

$$p_A^\mu = \frac{\partial L}{\partial v_A^\mu} \quad \text{and} \quad p = L - \frac{\partial L}{\partial v_A^\mu} v_A^\mu.$$
The Cartan form:

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- $\Theta_L = (FL)^* \Theta$

- In coordinates

$$\Theta_L = \frac{\partial L}{\partial v^A_\mu} \, dy^A \wedge d^n x_\mu + \left( L - \frac{\partial L}{\partial v^A_\mu} \, v^A_\mu \right) d^{n+1} x.$$
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- Cool fact: $\mathcal{L}(j_\phi) = (j_\phi)^* \Theta_L$
The Euler–Lagrange equations:

The following are equivalent. For a section \( \phi : X \to Y \),

- \( \phi \) is a critical point of the action

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\mathcal{A}(\phi) = \int_X \mathcal{L}(j\phi)
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- For all vector fields \( \xi \) on \( JY \),
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  j\phi^*(\xi \downarrow \Theta\mathcal{L}) = 0
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The Euler–Lagrange equations:

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  \[ A(\phi) = \int_X L(j\phi) \]

- For all vector fields \( \xi \) on \( JY \),
  \[ j_{\phi}^*(\xi \downarrow d\Theta_L) = 0 \]

- In coordinates
  \[ \frac{\partial L}{\partial y^A}(j\phi) - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial v^A_{\mu}}(j\phi) \right) = 0. \]
Bosonic String:

The Lagrangian is the (negative of the) energy:

\[ \mathcal{L} = -\frac{1}{2} \sqrt{-hh} h^\rho{}_{\sigma} g_{AB} v^A{}_{\sigma} v^B{}_{\rho} d^2x \]
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- The Legendre transform is

\[ p^A_{\mu} = -\sqrt{-h} h^{\mu \nu} g_{A B} v^B_{\nu} \]

\[ \rho^{\sigma \rho \mu} = 0 \]

\[ p = \frac{1}{2} \sqrt{-h} h^{\mu \nu} g_{A B} v^A_{\mu} v^B_{\nu} \]
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\]

- So the Cartan form is
  \[
  \Theta_{\mathcal{L}} = \sqrt{-h} \left( -h^{\mu\nu} g_{AB} v^B_\nu d\phi^A \wedge dx^\mu + \frac{1}{2} \sqrt{-h} h^{\mu\nu} g_{AB} v^A_\mu v^B_\nu d^2x \right).
  \]
The E–L equations $\delta L/\delta \phi^A = 0$ and $\delta L/\delta h_{\alpha\beta} = 0$ are

\[
(h^{\mu\nu} g_{AB}(\phi) \phi^B_{,\nu};\mu)_{;\mu} = 0
\]  

(1)

\[
\left(\frac{1}{2} \sqrt{-h} h^{\mu\nu} g_{AB}(\phi) \phi^A_{,\mu} \phi^B_{,\nu}\right) h_{\alpha\beta} = g_{CD}(\phi) \phi^C_{,\alpha} \phi^D_{,\beta}
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(2)
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- (1) is the harmonic map equation for \( \phi \)
- (2) does two things:
  - it says \( h \) is conformally related to \( \phi^* g \): \( \Lambda^2 h_{\alpha\beta} = (\phi^* g)_{\alpha\beta} \),
  - and it determines the conformal factor: \( \Lambda^2 = \frac{1}{2} h^{\mu\nu} g_{AB}(\phi) \phi^A_{,\mu} \phi^B_{,\nu} \)
Covariant Momentum Maps & Noether’s Theorem

Suppose $\eta$ is an automorphism of $Y$, covering a diffeomorphism of $X$. We may lift $\eta$ to an automorphism of various bundles over $Y$.

Jet prolongations: $\eta_{JY} : JY \rightarrow JY$ defined by $\eta_{JY}(\gamma) = T\eta_{Y} \circ \gamma \circ T\eta_{X}^{-1}$

Canonical lifts: $\eta_{Z} : Z \rightarrow Z$ defined by $\eta_{Z}(z) = (\eta_{Y})^{\ast}(z)$

Proposition

Canonical lifts are special covariant canonical transformations: $\eta^{\ast}Z \Theta = \Theta$
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  \[ \eta_Z(Z) = (\eta_Y)_*(Z) \]
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Multimomentum Maps

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such that

$$dJ(\xi) = \xi_Z \lrcorner \Omega$$

Here $\xi_Z$ is the infinitesimal generator on $Z$ corresponding to $\xi \in g = \text{Lie}(G)$. 


Multimomentum Maps

Suppose $\mathcal{G}$ is a Lie group of automorphisms of $Y$ (not necessarily finite-dimensional). If $\mathcal{G}$ acts by covariant canonical transformations (or multisymplectomorphisms), a covariant momentum map (or multimomentum map) for this action is a map

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$J$ intertwines the group action with the multisymplectic structure via the above equation.
Proposition

If $\mathcal{G}$ acts by special covariant canonical transformations, then

$$J(\xi) = \xi_Z \downarrow \Theta$$

is a special covariant momentum map.
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Indeed, $dJ(\xi) = di_{\xi_Z} \Theta = (\mathbb{L}_{\xi_Z} - i_{\xi_Z} d')\Theta = i_{\xi_Z} \Omega.$
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is a special covariant momentum map.

- Indeed, $dJ(\xi) = di_{\xi_Z} \Theta = (\mathbb{L}_{\xi_Z} - i_{\xi_Z} d') \Theta = i_{\xi_Z} \Omega$.

- An alternate formula: $J(\xi)(z) = \pi^*_Y (\xi_Y \lrcorner z)$
Proposition

If $G$ acts by special covariant canonical transformations, then

$$J(\xi) = \xi_Z \hookrightarrow \Theta$$

is a special covariant momentum map.

- Indeed, $dJ(\xi) = di_{\xi_Z} \Theta = (\mathbb{L}_{\xi_Z} - i_{\xi} d')\Theta = i_{\xi_Z} \Omega$.

- An alternate formula: $J(\xi)(z) = \pi^*_{YZ}(\xi_Y \hookrightarrow z)$

- In coordinates: if we write $\xi_Y = \xi^\mu \frac{\partial}{\partial x^\mu} + \xi^A \frac{\partial}{\partial y^A}$,

  $$J(\xi) = (p_A^\mu \xi^A + p_{\xi^\mu}) d^n x_\mu - p_A^{\nu} \xi^\nu dy^A \wedge d^{n-1}x_{\mu\nu}$$
Bosonic String

The gauge group is $\mathcal{G} = \text{Diff}(X) \ltimes C^\infty(X, \mathbb{R}^+)$.
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- $\text{Diff}(X)$ — material relabelings
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- \( \text{Diff}(X) \) — material relabelings

- \( C^\infty(X, \mathbb{R}^+) \) conformal rescalings
The Lie algebra is $g \approx \mathcal{X}(X) \ltimes C^\infty(X)$.
The Lie algebra is \( g \approx \mathfrak{X}(X) \ltimes C^\infty(X) \)

For \((\xi, \lambda) \in g\), the infinitesimal generator is

\[
(\xi, \lambda)_Y = 2\lambda h_{\sigma\rho} \frac{\partial}{\partial h_{\sigma\rho}} - \left( h_{\sigma\mu} \xi^\mu_{,\rho} + h_{\rho\mu} \xi^\mu_{,\sigma} \right) \frac{\partial}{\partial h_{\sigma\rho}} + \xi^\mu \frac{\partial}{\partial x^\mu}
\]

Note: there is no \( \frac{\partial}{\partial \phi^A} \) component here, as \( \phi \) is a scalar field.
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Note: there is no $\frac{\partial}{\partial \phi^A}$ component here, as $\phi$ is a scalar field.

The multimomentum map is

$$J(\xi, \lambda) = \left[ \rho^{\sigma\rho\mu} \left( 2\lambda h_{\sigma\rho} - h_{\sigma\nu} \xi^\nu,_{\rho} - h_{\rho\nu} \xi^\nu,_{\sigma} \right) + p \xi^\mu \right] d^1 x^\mu$$

$$- \left( p_A^{\mu} \xi^\nu d\phi^A + \rho^{\sigma\rho\mu} \xi^\nu dh_{\sigma\rho} \right) \epsilon_{\mu\nu}$$

where $d^2 x_{\mu\nu} = \epsilon_{\mu\nu}$. 
Let $\mathcal{G}$ act on $Y$ by bundle automorphisms.
Symmetries

Let $G$ act on $Y$ by bundle automorphisms.

- $\mathcal{L}$ is **equivariant** (or $G$-covariant) if

$$\mathcal{L}(\eta_{J^1Y}(\gamma)) = (\eta_X)_* \mathcal{L}(\gamma)$$

for all $\gamma \in JY$. 

This will be a fundamental assumption in all that follows.

Infinitesimally, this is $\delta_{\xi} \mathcal{L} = 0$, where

$$\delta_{\xi} \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x^\mu} \xi^\mu + \frac{\partial \mathcal{L}}{\partial y^A} \xi^A + \frac{\partial \mathcal{L}}{\partial v^A_{\mu}} (\xi^A_{,\mu} - v^B_{\nu} \xi^B_{,\mu} + v^B_{\mu} \partial \xi^A y^B) + \mathcal{L} \xi^\mu_{,\mu}$$

is the variation of $\mathcal{L}$. 

MARK J. GOTAY (PIMS, UBC)  
MOMENTUM MAPS & CLASSICAL FIELDS  
Olomouc, August, 2009  
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- Infinitesimally, this is $\delta_\xi L = 0$, where
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  \delta_\xi L = \frac{\partial L}{\partial x^\mu} \xi^\mu + \frac{\partial L}{\partial y^A} \xi^A + \frac{\partial L}{\partial v^A_\mu} \left( \xi^A ,_\mu - v^A ,_\nu \xi^\nu ,_\mu + v^B_\mu \frac{\partial \xi^A}{\partial y^B} \right) + L \xi ,_\mu ,_\mu
  \]
  is the variation of $L$. 
Thm

Let $\mathcal{L}$ be $G$-equivariant. Then:

1. $F\mathcal{L}$ is also equivariant, i.e., $\eta_Z \circ F\mathcal{L} = F\mathcal{L} \circ \eta_{JY}$
Thm

Let $\mathcal{L}$ be $\mathcal{G}$-equivariant. Then:

- $F \mathcal{L}$ is also equivariant, i.e., $\eta_Z \circ F \mathcal{L} = F \mathcal{L} \circ \eta_{J\mathcal{Y}}$
- The Cartan form $\Theta_{\mathcal{L}}$ is invariant, i.e., $\eta^*_{J\mathcal{Y}} \Theta_{\mathcal{L}} = \Theta_{\mathcal{L}}$
Let $\mathcal{L}$ be $G$-equivariant. Then:

- $\mathcal{F}_{\mathcal{L}}$ is also equivariant, i.e., $\eta_Z \circ \mathcal{F}_{\mathcal{L}} = \mathcal{F}_{\mathcal{L}} \circ \eta_{JY}$

- The Cartan form $\Theta_{\mathcal{L}}$ is invariant, i.e., $\eta^*_{JY} \Theta_{\mathcal{L}} = \Theta_{\mathcal{L}}$

- The map $J^\mathcal{L}(\xi) := \mathcal{F}_{\mathcal{L}}^* J(\xi) : JY \to \Lambda^n(JY)$ is a momentum map for the prolonged action of $G$ on $JY$ relative to $\Omega_{\mathcal{L}} = -d\Theta_{\mathcal{L}}$. That is to say,

$$\xi_{JY} \downarrow \Omega_{\mathcal{L}} = dJ^\mathcal{L}(\xi).$$

Moreover,

$$J^\mathcal{L}(\xi) = \xi_{J^1Y} \downarrow \Theta_{\mathcal{L}}.$$
Divergence Form of Noether’s Thm

If $L$ is $G$-covariant, then for each $\xi \in g$, $d\left[ (j_\phi)^\ast J L(\xi) \right] = 0$ for any section $\phi$ of $\pi_{XY}$ satisfying the Euler–Lagrange equations.

The quantity $(j_\phi)^\ast J L(\xi)$ is called the Noether current, and this theorem states that the current is conserved.

Proof
If $\phi$ is a solution of the Euler–Lagrange equations, then $(j_\phi)^\ast (W_{\Omega L}) = 0$ for any vector field $W$ on $JY$. In particular, set $W = \xi_{JY}$ and simply apply $(j_\phi)^\ast$ to $\xi_{JY}_{\Omega L} = dJ L(\xi)$.

$\blacksquare$
Divergence Form of Noether’s Thm

If $\mathcal{L}$ is $\mathcal{G}$-covariant, then for each $\xi \in \mathfrak{g}$,

$$d \left[ (j\phi)^* J^\mathcal{L}(\xi) \right] = 0$$

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for any vector field $W$ on $JY$. In particular, set $W = \xi_{JY}$ and simply apply $(j\phi)^*$ to

$$\xi_{JY} \hookrightarrow \Omega_\mathcal{L} = dJ^\mathcal{L}(\xi).$$

■
Local Expressions

the "Lagrangian multimomentum map"

\[ J_L(\xi) = \left( \frac{\partial L}{\partial v_A^\mu} \xi_A^\mu + \left[ L - \frac{\partial L}{\partial v_A^\nu} v_A^\nu \right] \xi^\mu \right) d^n x^\mu - \frac{\partial L}{\partial v_A^\mu} \xi^\nu dy_A \wedge d^n x^{\mu-1} \]

the Noether current is

\[ (j^1_\phi)^* J_L(\xi) = \left[ -\frac{\partial L}{\partial v_A^\mu} (j^1_\phi)(L_{\xi \phi}) A^\mu + L(j^1_\phi) \xi^\mu \right] d^n x^\mu \]

where the "Lie derivative of \( \phi \) along \( \xi \) is

\[ L_{\xi \phi} = T_{\phi} \circ \xi - \xi \circ \phi \]; i.e.,

\[ (L_{\xi \phi})^A_{\mu, \nu} = \phi^A_{, \nu} \xi^\mu - \xi^A_{, \phi} \circ \phi \]
the “Lagrangian multimomentum map” is

\[ J^L(\xi) = \left( \frac{\partial L}{\partial v_A^\mu} \xi^A + \left[ L - \frac{\partial L}{\partial v_A^\nu} v^A_{\nu} \right] \xi^\mu \right) d^n x{\mu} - \frac{\partial L}{\partial v_A^\mu} \xi^\nu dy^A \wedge d^{n-1} x{\mu\nu} \]
the “Lagrangian multimomentum map” is

\[
J^L(\xi) = \left( \frac{\partial L}{\partial V^A_\mu} \xi^A + \left[ L - \frac{\partial L}{\partial V^A_\nu} \nu^A_\nu \right] \xi^\mu \right) d^nx_\mu - \frac{\partial L}{\partial V^A_\mu} \xi^\nu dy^A \wedge d^{n-1}x_{\mu\nu}
\]

the Noether current is

\[
(j^1 \phi)^* J^L (\xi) = \left[ - \frac{\partial L}{\partial V^A_\mu} (j^1 \phi)(\mathcal{L}_\xi \phi)^A + L(j^1 \phi)\xi^\mu \right] d^nx_\mu
\]

where the “Lie derivative of \( \phi \) along \( \xi \) is

\[
\mathcal{L}_\xi \phi = T_\phi \circ \xi_X - \xi_Y \circ \phi; \quad \text{i.e.,} \quad (\mathcal{L}_\xi \phi)^A = \phi^A,\nu \xi^\nu - \xi^A \circ \phi
\]
A computation gives the useful expression for the Noether divergence:

\[ d \left[ (j_\phi)^* J^L(\xi) \right] = \left\{ \frac{\delta L}{\delta \phi^A} (\mathbb{L}_\xi \phi)^A + \delta_\xi L \right\} (j^1_\phi) d^{n+1}x \]

from which again Noether’s theorem is immediate.
Bosonic String

The Noether current is:

\[ j(\phi, h)^* J^L(\xi, \lambda) = \sqrt{-h} g_{AB} \left( h^{\mu\nu} \phi^A_{,\rho} \phi^B_{,\nu} \xi^\rho - \frac{1}{2} h^{\sigma\rho} \phi^A_{,\sigma} \phi^B_{,\rho} \xi^\mu \right) d^1 x_\mu. \]  \hspace{1cm} (3)

Note again that \( \lambda \) does not appear on the RHS.