GENERALIZED CONSTRAINT ALGORITHM
AND
SPECIAL PRESYMPLECTIC MANIFOLDS

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Abstract

A generalized constraint algorithm is developed which provides necessary and sufficient conditions for the solvability of the canonical equations of motion associated to presymplectic classical systems. This constraint algorithm is combined with a presymplectic extension of Tulczjew's description of constrained dynamical systems in terms of special symplectic manifolds. The resultant theory provides a unified geometric description as well as a complete solution of the problems of constrained and a priori presymplectic classical systems in both the finite and infinite dimensional cases.

I. *Introduction*

Recently, Tulczyjew has given a description of constrained classical systems in terms of special symplectic manifolds [1-5]. This elegant theory adequately describes the dynamics of first-class systems in which (in the sense of Dirac[6]) no secondary constraints appear.

In a different approach [7-9], we have developed a geometric constraint algorithm which completely solves the problem of defining, obtaining and solving "consistent" canonical equations of motion for presymplectic dynamical systems. This algorithm is phrased in the context of global infinite-dimensional presymplectic geometry, and generalizes as well as improves upon the local Dirac-Bergmann theory of constraints [6]. The algorithm is applicable to the degenerate Hamiltonian and Lagrangian formulations of constrained systems [10] as well as to a priori presymplectic systems.

In this paper, we consolidate Tulczyjew's theory and our presymplectic techniques obtaining a complete unified geometric treatment of constrained and a priori presymplectic dynamical systems in terms of special presymplectic manifolds. This combined approach has several advantages over either method taken individually. The notion of special symplectic manifold, as Tulczyjew has pointed out, allows a uniform treatment of classical physics including relativistic and nonrelativistic dynamics as well as provides a basis for generalization to field theories, encompassing in particular the Poincaré-Cartan (multisymplectic) formalism [5, 11, 12]. Besides yielding geometrical insight into the mechanics of the presymplectic constraint algorithm, special symplectic techniques are indispensible in the consideration of singular dynamical systems, where, for instance, they may be used to "unfold" singular constraint submanifolds (cf. §VIII).

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On the other hand, our presymplectic methods are capable of treating completely general constrained and a priori presymplectic dynamical systems. Specifically, given a physical system described by a presymplectic phase space (M, ω) and a Hamiltonian H on M, the algorithm finds whether or not there exists a submanifold N of M along which the canonical equations of motion

$$i(X)\omega = - dH \tag{1.1}$$

hold; if such a submanifold exists, the algorithm provides a *constructive* method for finding it. Moreover, the "final constraint submanifold" N is *maximal* in the sense that it contains any other submanifold along which (1.1) is satisfied.

In contrast, Tulczyjew's program is not constructive, that is, Tulczyjew does not consider the "Dirac constraint problem" per se, but rather only describes the finished product. Except under very special conditions (viz., when no secondary constraints appear in the theory), one must be given the final constraint submanifold N before Tulczyjew's techniques can be applied. The presymplectic constraint algorithm therefore can be used to extend Tulczyjew's theory of constrained dynamical systems to those in which secondary constraints are present.

There is, however, one profound difference between the synthetic approach of this paper and that proposed by Menzio and Tulczyjew [4], centering on the role of the integrability conditions in the theory. In the formulation of Menzio and Tulczyjew, certain integrability conditions are imposed which effectively demand that the final constraint submanifold N be first class. The integrability conditions associated with the presymplectic constraint algorithm, however, place no restriction on the class of N.

It is our contention that the integrability conditions of Menzio and Tulczyjew are inappropriate for a description of the dynamics of constrained classical systems. In fact, it turns out that these conditions are sufficient but not necessary for solutions of (1.1) to exist. Consequently, the imposition of such integrability conditions will artificially eliminate from consideration a great many systems of genuine physical interest (e.g., the Proca field).

Menzio and Tulczyjew claim that discarding constrained classical systems which are not first class *a priori* is acceptable, since such systems can never be the classical limits of consistent quantum theories. While this latter remark is -- strictly speaking -- true, there seems to be no compelling reason to eliminate such systems from consideration on the classical level. Furthermore, a theorem of Śniatycki [13] shows that it is usually possible to reformulate the dynamics of constrained systems in a manner such that the resulting dynamics is first class. Failing this, one may of course quantize the reduced phasespace [9].

Therefore, we feel that Menzio and Tulczyjew's dictum that the dynamics of constrained classical systems be first class a priori is unnecessarily severe. It is our opinion that there is much to be gained, and little to be lost, by developing techniques which are capable of treating constrained systems of arbitrary class.

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The language used throughout this paper is that of infinite-dimensional presymplectic geometry. Notation and terminology are summarized in the Appendix.

II. *Presymplectic Geometry and Classical Mechanics*

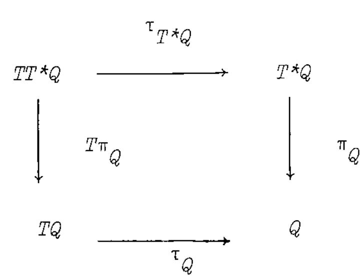
Let M be a Banach manifold, and suppose that ω is a closed 2-form on M. Then (M, ω) is said to be a *strong symplectic* manifold if the map $\flat : TM \to T^*M$ defined by $\flat(X) := i(X)\omega$ is a toplinear isomorphism. However, it may happen that $\flat$ is injective but not surjective, in which case (M, ω) is called a *weak symplectic* manifold, ω being *weakly* nondegenerate. Generically, $\flat$ will be neither injective nor surjective and ω is then *degenerate*. For brevity, weakly nondegenerate and degenerate manifolds will often be referred to simply as *presymplectic*. When M is finite-dimensional, there is of course no distinction between weak and strong symplectic forms.

Physically, M represents the phasespace of a classical system, while ω is a generalization of the Poisson (or Lagrange) bracket [14].

The standard example of a symplectic manifold is the cotangent bundle $\pi_Q : T^*Q \to Q$ of any Banach manifold Q. Indeed, on T^*Q there exists a canonical 1-form Θ_Q (the *Liouville form*) defined by the universal property

$$\alpha^*(\Theta_Q) = \alpha, \tag{2.1}$$

where α is any 1-form on Q. Alternatively, since the diagram



commutes, Θ_Q may be characterized as follows:

$$\langle v | \Theta_Q \rangle = \langle T\pi_Q(v) | \tau_{T^*Q}(v) \rangle, \tag{2.2}$$

where $v \in T(T^*Q)$. The Liouville form determines the exact symplectic structure

$$\Omega_Q = d\Theta_Q. \tag{2.3}$$

It is not difficult to show that Ω_Q so defined is weakly nondegenerate, and moreover that (T^*Q, Ω_Q) is strongly symplectic iff Q is reflexive [15].

The mechanics of the cotangent bundle case can be better understood by examining the local representatives of the above formulas. Let $U \subset F$ be a chart, where F is the model space for Q. The local representative of $m \in T^*Q$ is $(x, \sigma) \in U \times F^*$, and for $v \in T_m(T^*Q)$, one has $v = (x, \sigma) \oplus (a, \pi)$ in $(U \times F^*) \oplus (F \times F^*)$. It follows that

$$\tau_{T^*Q}(v) = (x, \sigma) \in U \times F^*$$

and

$$T\pi_Q(v) = (x, a) \in U \times F.$$

Therefore, (2.2) becomes, employing the shorthand notation $a \oplus \pi := (x, \sigma) \oplus (a, \pi)$,

$$\Theta_Q(x, \sigma)\cdot(a \oplus \pi) = \langle a|\sigma\rangle. \tag{2.4a}$$

Similarly, one calculates that

$$\Omega_Q(x, \sigma)\cdot(a \oplus \pi, b \oplus \tau) = \langle b|\pi\rangle - \langle a|\tau\rangle . \tag{2.4b}$$

In the finite-dimensional case, these formulas are not nearly so mysterious. If $(T^*U;q^i, p_i)$ is a natural bundle chart for T^*Q, (2.4a) and (2.4b) become

$$\Theta_Q|T^*U = p_i dq^i \tag{2.5a}$$

and

$$\Omega_Q|T^*U = dp_i \wedge dq^i. \tag{2.5b}$$

Physically, the weak and strong symplectic manifolds one almost always encounters are cotangent bundles. Indeed, *physics in the Hamiltonian formulation is none other than mechanics on cotangent bundles.* The manifold Q is the configuration space of the physical system, its cotangent bundle T^*Q is momentum phasespace and the canonical 1-form Θ_Q is the integrand in the Principle of Least Action.

There do, however, exist physically interesting systems whose phasespaces are not cotangent bundles and whose symplectic forms are not exact. An example of such a system was given by Souriau [16], who investigated the dynamics of a freely spinning massive particle in Minkowski spacetime from a symplectic viewpoint (here,

$M = \mathbb{R}^6 \times S^2$). Systems of this type do not possess configuration manifolds and consequently do not admit Hamiltonian or Lagrangian formulations (at least in the usual sense).

Furthermore, the geometry of classical systems need not be strongly symplectic. This phenomenon is characteristic of systems with an infinite number of degrees of freedom, where ω may be presymplectic even when there are no constraints (e.g., the Klein-Gordon field [9, 15]). An example of an a priori presymplectic dynamical system has been provided by Künzle [17], who obtained genuinely presymplectic phase spaces for spinning particles in curved spacetimes.

The most important application of presymplectic geometry is to the theory of constrained classical systems. Typically, (e.g., electromagnetism, gravity), the constraints take the form of internal consistency conditions on the dynamics of the system.

Such constraints appear when one transforms from the Lagrangian to the Hamiltonian formalism. A physical system, described by a configuration space Q and a Lagrangian L, is cast into canonical form by "changing variables" from $(q^i, \dot{q}^i)$ to (q^i, p_i) and replacing L by the Hamiltonian H through $H(q, p) = p_i\dot{q}^i - L(q, \dot{q})$. Mathematically, this transition is accomplished via the *Legendre transformation* $FL: TQ \to T^*Q$ defined by

$$\langle w | FL(z) \rangle : = \frac{d}{ds} L(z + sw) \Big|_{s=0} , \qquad (2.6)$$

where $z, w \in TQ$.

Presymplectic manifolds arise when FL is not a diffeomorphism [14], in which case the Legendre transformation defines a submanifold $FL(TQ)$ of T^*Q. This is the starting point of the Dirac-Bergmann constraint theory [6], in which $FL(TQ)$ is called the *primary constraint* submanifold. $FL(TQ)$ will inherit a presymplectic structure from T^*Q by pulling Ω_Q back to M via the inclusion $j: FL(TQ) \to T^*Q$). The degree of degeneracy of $w = j^*\Omega_Q$ depends entirely upon the behavior of FL. On $FL(TQ)$ Hamilton's equations take the form (1.1).

Another example of an a priori presymplectic system is provided by Lagrangian dynamics, where the fundamental dynamical arena is not *momentum* phasephase T^*Q, but rather *velocity* phasespace TQ. Whereas T^*Q carries a canonical exact symplectic structure, TQ does not. Nonetheless, it is always possible to transfer the exact symplectic structure Ω_Q on T^*Q to TQ by pull back via FL. Generically, however, this induced structure will not be symplectic, but merely presymplectic, depending upon the regularity properties of FL.

III. *Canonical Systems and their Classification*

It is useful to have a classification scheme for generalized submanifolds of presymplectic manifolds which is both mathematically convenient and physically meaningful. Dirac first developed a local classification of submanifolds of strongly symplectic manifolds by describing them in terms of certain types of constraint functions (see refs. [4], [6], [9] and [18] for details concerning this approach). Tulczyjew and Śniatycki [13] have found an intrinsic generalization of Dirac's classification scheme, which is extended here to the presymplectic case. This classification is of the utmost significance insofar as the physical interpretation of the constraint algorithm is concerned, and has important applications to both the gauge theory and the quantization of presymplectic dynamical systems [8, 9].

Let N be a g-submanifold of the presymplectic manifold (M, ω) with inclusion j. The manifold N is called a *constraint* g-submanifold, and the triple (M, ω, N) a *canonical system*. Define the *symplectic complement* $TN^{\perp}$ of $\underline{TN}$ in TM to be

$$TN^{\perp} = \{Z \in T_N M \text{ such that } \omega(X, Z) = 0 \text{ for all } X \in \underline{TN}\}.$$

The *annihilator* $TN^{\vdash}$ of $\underline{TN}$ in T^*M is

$$TN^{\vdash} = \{\alpha \in T^*_N M \text{ such that } \langle X|\alpha\rangle = 0 \text{ for all } X \in \underline{TN}\}.$$

The constraint g-submanifold N is said to be

(*i*) *isotropic* if $\underline{TN} \subset TN^{\perp}$,

(*ii*) *coisotropic* or *first class* if $TN^{\perp} \subset \underline{TN}$,

(*iii*) *weakly symplectic* or *second class* if $\underline{TN} \cap TN^{\perp} = \{0\}$, and

(*iv*) *Lagrangian* if $TN = TN^{\perp}$ [19].

If N does not happen to fall into any of these categories, then N is said to be *mixed* constraint g-submanifold.

From the point of view of the g-submanifold N, this classification reduces to a characterization of the naturally induced presymplectic structure ω_N on N. Indeed, $TN^{\perp} \cap \underline{TN} = \underline{\ker \omega_N}$, where $\omega_N := j^*\omega$. In particular, N is isotropic iff $j^*\omega = 0$.

As an illustration, let $C \subset Q$. Then T^*C is a second class submanifold of (T^*Q, Ω_Q). Furthermore, the constraint submanifold $\pi_Q^{-1}(C) \subset T^*Q$ is first class.

Let α: $Q \to T^*Q$ be a <u>closed</u> 1-form. By virtue of the definition (2.1) of Θ_Q, it follows that the image $\alpha(Q)$ of Q under α is an isotropic submanifold of (T^*Q, Ω_Q):

$$\alpha^*\Omega_Q = d\alpha^*\Theta_Q = d\alpha = 0.$$

In fact, $\alpha(Q)$ is <u>maximally</u> isotropic and hence Lagrangian. If α were only densely defined, however, then the image of α would be merely isotropic.

Thus the zero-section Q of T^*Q provides a natural example of a Lagrangian constraint submanifold. Also, for each $m \in Q$, the fiber $\pi_Q^{-1}(m)$ is a Lagrangian submanifold. A Lagrangian submanifold, as these examples indicate, generalizes the classical coordinate and momentum representations.

IV. *Canonical Dynamics of Presymplectic Systems*

The presymplectic form ω and the phasespace M have only kinematical significance -- the *dynamics* of the physical system (M, ω) is determined by specifying on M a closed 1-form α, the *Hamiltonian form*. One then solves the *generalized Hamilton equations*

$$i(X)\omega = \alpha \tag{4.1}$$

for the *evolution vectorfield* X. Once X has been determined, one appeals to the standard results of differential equation theory in order to integrate X, thereby obtaining the dynamical trajectories of the system in phasespace.

When (M, ω) is strongly symplectic, the induced map $\flat : TM \to T^*M$ is an isomorphism. Consequently, in this case (4.1) possesses a unique solution $X = \flat^{-1}(\alpha)$. Since X is every where defined and smooth, it gives rise to a unique local flow [20].

We now calculate the local representative of X in the strongly symplectic case. Let V be a (contractible) chart on M, and suppose for simplicity that M is Hilbertable. Then, Darboux's Theorem [21] asserts the existence of a reflexive Banach space F and a chart $U \subset F$ such that

$$(V, \omega|V) \approx (T^*U, \Omega_U).$$

Furthermore, since V is contractible, $\alpha|V = -dH$, where H is the ordinary Hamiltonian. If $m = (x, \sigma) \in T^*U$, and $Y = b \oplus \tau \in F \times F^*$ is a vector at (x, σ), then

$$\begin{aligned} i_Y\alpha(m) &= - DH(x, \sigma)\cdot(b \oplus \tau) \\ &= - \overline{D}H(x, \sigma)\cdot b - \dot{D}H(x, \sigma)\cdot \tau. \end{aligned}$$

Similarly, writing $X(x, \sigma) = a \oplus \pi \in F \times F^*$, (2.4b) yields

$$\begin{aligned} i_Y i_X\omega(m) &= \Omega_U(x, \sigma)\cdot(a \oplus \pi, b \oplus \tau) \\ &= \langle b|\pi\rangle - \langle a|\tau\rangle. \end{aligned}$$

Comparing this expression with the previous one, equation (4.1) implies that the local representative of X is

$$X(x, \sigma) = \dot{D}H(x, \sigma) \oplus -\overline{D}H(x, \sigma).$$

In the finite-dimensional case, this reduces to

$$X = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i},$$

the integral curves of which are found by solving Hamilton's equations:

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}.$$

Turning now to the presymplectic case, there are four major difficulties one encounters when trying to solve the generalized Hamilton equations associated with a presymplectic dynamical system (M, ω, α):

(*i*) These equations are typically inconsistent and consequently will not possess globally defined solutions; if an evolution vectorfield X exists at all, then in general it will be defined only on some g-submanifold N of M [22];

(*ii*) X does not necessarily define a differential equation on N, that is, $X \notin \underline{T_N \overline{N}}$ in general;

(*iii*) The solution X, if it exists, need not be unique; and

(*iv*) X will usually be discontinuous so that it may not possess even a locally defined flow.

Difficulty (*i*), the *existence problem*, is encountered even in well-behaved systems, e.g., the Klein-Gordon field, for which $M = H^1 \oplus L^{2*}$ and $N = H^2 \oplus H^{1*}$. Physically, N is to be regarded as a *constraint g-submanifold*, that is, $\overline{N} \subseteq M$ consists of those states of the system which are *physically realizable*. The implication is that states in M not contained in $\overline{N}$ are *dynamically inaccessible* to the system, since the equations of motion cannot be integrated at such points.

The *constraint problem* (*ii*) is of fundamental significance, and in the degenerate case presents the major obstacle to solving the equations (4.1). The generalized Hamilton equations are to be considered as evolution equations for the system, and hence must be *differential* equations. However, in order for the vectorfield X to be interpretable as a differential equation, it is necessary that X be "tangent" to $\overline{N}$ in the sense that $X \in \underline{T_N \overline{N}}$. In other words, if X is to describe the evolution of the system in phasespace, then it must generate a (local) flow. Since X is defined only along the g-submanifold N, it can (at best) give rise to a flow on N -- only if X is tangent to $\overline{N}$. Physically this has the interpretation that the motion of the system is constrained to lie in $\overline{N}$.

The existence and constraint problems will be the subjects of the next section, while (*iii*), the *uniqueness problem* -- which signals the presence of gauge degrees of freedom in the theory -- has been discussed elsewhere [8, 9].

The *integration problem* (*iv*) can be very severe for presymplectic systems as well. As discussed above, the interpretation of equations (4.1) as evolution equations requires that X be integrable, i.e., X must give rise to a well-defined (possibly local) flow. The demand that X be tangent to $\overline{N}$ gives a necessary, but certainly not sufficient, condition for X to be integrable. The difficulty is that X is not necessarily continuous (as it may not be defined globally; e.g., the Klein-Gordon field) so that the standard theorems on the existence and uniqueness of flows of vector fields are not applicable. Unlike difficulties (*i*) - (*iii*), the integration problem lies mainly outside the province of symplectic geometry and is better considered from the viewpoint of global analysis and the theory of partial differential equations [15]. Consequently, this problem will not be considered further here.

In this paper, techniques will be developed which will (eventually) enable one to "solve" problems (*i*) - (*iii*). In view of the first difficulty, however, the initial step in the "solution" must be to answer the question: "What does one mean by 'consistent equations of motion,' and how does one obtain and solve such equations?"

V. *The Presymplectic Constraint Algorithm*

In this section we present an improved version of our presymplectic constraint algorithm [7-9] which correctly handles the infinite dimensional case where the evolution may be defined only on a dense subset rather than globally. Given a presymplectic dynamical system $(M_1, \omega_1, \alpha_1)$, our procedure will be used to select a certain g-submanifold N of M_1 upon which one can define and solve "consistent equations of motion." More precisely, this technique will provide necessary and sufficient conditions for the existence of a g-submanifold N of M_1 such that the equations

$$i(X)\omega_1 = \alpha_1 \tag{5.1}$$

hold when restricted to N, i.e.,

$$[i(X)\omega_1 - \alpha_1]|N = 0,$$

with X *tangent* to $\overline{N}$.

Begin by noting that if α_1 is everywhere contained in the range of $\flat_1$, then the required solution (not necessarily unique) of the equations (5.1) is simply any smooth element of $\flat_1^{-1}\{\alpha_1\}$. In the generic case, however, this will not be so. But there may exist points of M_1 (such points being assumed to form a g-submanifold M_2 of M_1), for which $\alpha_1|M_2$ is in the range of $\flat_1|M_2$. One is thus led to try and solve equation (5.1) restricted to M_2, i.e.,

$$[i(X)\omega_1 - \alpha_1]|M_2 = 0. \tag{5.2}$$

Equation (5.2) evidently possesses solutions, but only in an *algebraic* sense. In accord with the discussion of the constraint problem in §IV, one must demand that X solve (5.2) in a *differential* sense, viz., that $X \in \underline{T}_{M_2}\overline{M}_2$, or else the equations of motion will try to evolve the system "off $\overline{M}_2$" into an unphysical domain.

This requirement will not necessarily be satisfied, forcing a further restriction of (5.1) to the g-submanifold M_3 of M_2 defined by

$$M_3 := \{m \in M_2 \text{ such that } \alpha_1(m) \in \underline{\overline{TM}_2}^{\flat}\},$$

with the shorthand notation $T_P\overline{P} =: \overline{TP}$. It must now be ensured that the solution to (5.1) restricted to M_3 is in fact tangent to $\overline{M}_3$; this will in general necessitate yet more restrictions.

It is now clear how the algorithm must proceed. A string of *constraint g-submanifolds* [22]

$$\ldots \to M_3 \xrightarrow{j_3} M_2 \xrightarrow{j_2} M_1$$

is generated, defined as follows:

$$M_{\ell+1} := \{m \in M_\ell \text{ such that } \alpha_1(m) \in \underline{\overline{TM}_\ell}^{\flat}\}.$$

Once the constraint algorithm so defined is set into motion, only one of four distinct possibilities may occur. They are:

Case 1: There exists a K such that $M_K = \phi$;

Case 2: Eventually, the algorithm produces a g-submanifold $M_K \neq \phi$ such that $\dim M_K = 0$;

Case 3: There exists a K such that $M_K = M_{K+1}$ with $\dim M_K \neq 0$; and

Case 4: The algorithm does not terminate.

In the first case, $M_K = \phi$ means that the generalized Hamilton equations (5.1) have no solutions at all in any sense. In principle, this means that $(M_1, \omega_1, \alpha_1)$ does not accurately describe the dynamics of any system.

The second possibility results in a constraint g-submanifold which consists of isolated points. The equations (5.1) are consistent, but the only possible solution is $X = 0$ and there is no dynamics.

For case three, one has a constraint g-submanifold M_K and completely consistent equations of motion on M_K of the form

$$[i(X)\omega_1 - \alpha_1]|M_K = 0, \qquad (5.3)$$

with X tangent to $\overline{M}_K$. It is this g-submanifold M_K (the *final* constraint

g-submanifold) which corresponds to the g-submanifold N discussed in §III.

The situation described in case four is only possible for systems with an infinite number of degrees of freedom. In this circumstance, the final constraint g-submanifold can be taken to be the intersection M_∞ of all the g-submanifolds M_ℓ. One then recovers cases (1) - (3) depending upon whether $M_\infty = \phi$, $dim\ M_\infty = 0$, or $0 < dim\ M_\infty \leq \infty$.

If the algorithm terminates with some final constraint g-submanifold M_K $(1 \leq K \leq \infty)$, then *by construction* one is assured that at least one solution X to the canonical equations exists and furthermore that this solution is tangent to $\overline{M}_K$. Note that X need not be unique, for one can add to it any element of $ker\ \omega_1 \cap \underline{T\overline{M}_K}$. In addition, it is obvious, again by construction, that the final constraint g-submanifold is *maximal* in the following sense: if N is any other submanifold along which the equations (5.1) are satisfied, then $N \subseteq M_K$.

We have shown [7] that this constraint algorithm generalizes the local Dirac-Bergmann theory of constraints [6]. Indeed, it is possible to characterize the closures of the constraint g-submanifolds M_ℓ, as follows:

$$\overline{M}_\ell = \{m \in \overline{M}_{\ell-1} \text{ such that } \langle Z|\alpha_1\rangle(m) = 0 \text{ for all } Z \in T\overline{M}^{\perp}_{\ell-1}\}.$$

The constraint functions $\langle T\overline{M}^{\perp}_{\ell-1}|\alpha_1\rangle = 0$ which define $\overline{M}_\ell$ in $\overline{M}_{\ell-1}$ are none other than Dirac's *ℓ-ary constraints*.

This presymplectic constraint algorithm provides a geometrically intuitive and conceptually simple method for defining and solving consistent equations of motion on a presymplectic manifold. It provides a *constructive* solution to the existence and constraint problems of §IV, and is of very general applicability, requiring only that the phasespaces involved be Banach manifolds.

VI. Special Presymplectic Manifolds

Here, we broaden Tulczjew's notion of "special symplectic manifold" [1, 23] so as to encompass the presymplectic formalism necessary for the description of completely general dynamical systems.

A *special symplectic manifold* is a quintuple (P, p, M, λ, μ), where $p: P \to M$ is a fiber bundle, λ is a 1-form on P, and μ is a fiber-preserving diffeomorphism $P \to T^*M$ such that $\mu^*\Theta_M = \lambda$.

Essentially, one is transferring the symplectic structure on T^*M to P via μ. The 2-form $d\lambda$ on P is weakly nondegenerate, and strongly nondegenerate iff M is reflexive.

A *special presymplectic manifold* is obtained by relaxing the requirement that μ be a diffeomorphism. A special presymplectic manifold is therefore a degenerate "copy" of a cotangent bundle.

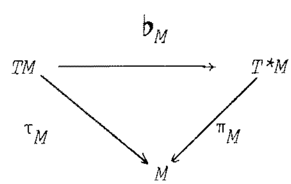
Example 1: If Q is the configuration space of a physical system, then momentum phasespace $(T^*Q,\ \pi_Q,\ Q,\ \Theta_Q,\ id_Q)$ is a special symplectic manifold.

Example 2: The Lagrangian system $(TQ, \tau_Q,\ Q,\ FL^*\Theta_Q,\ FL)$ is a special presymplectic manifold, where $L\colon TQ \to R$ is the Lagrangian. In a bundle chart $U \times F$ for TQ, one has

$$FL^*\Theta_Q(u,\ e) = \dot{D}L(u,\ e) \oplus 0.$$

[In finite-dimensions, $FL^*\Theta_Q = \frac{\partial L}{\partial \dot{q}^i}\, dq^i$]. The 2-form $dFL^*\Theta_Q$ is strongly (weakly) symplectic iff the velocity Hessian $\ddot{D}\dot{D}L(u,\ e)$, viewed as a linear map $F \to F^*$, is strongly (weakly) nondegenerate [14].

Example 3: Let $(M,\ \omega)$ be a presymplectic manifold. Then $(TM,\ \tau_M,\ M,\ \flat_M{}^*\Theta_M,\ \flat_M)$ is a special presymplectic manifold, where $\flat_M$ is the map of TM to T^*M induced by ω. The presymplectic structure $d\flat_M{}^*\Theta_M$ on TM is denoted $\dot{\omega}$.



Consider the special case $M = T^*Q$. The local representative

$$\flat_{T^*U}\colon\ U \times F^* \times F \times F^* \to U \times F^* \times F^* \times F^{**}$$

of $\flat_{T^*Q}\colon\ T(T^*Q) \to T^*(T^*Q)$ is

$$\flat_{T^*U}\,(x,\ \sigma,\ e,\ \pi) = (x,\ \sigma,\ \pi,\ -e).$$

Consequently, one has

$$\lambda(x,\ \sigma,\ e,\ \pi) = (\ \pi,\ -e) \oplus (0,\ 0).$$

In a finite-dimensional natural bundle chart $(T(T^*U);\ q^i,\ p_i,\ \dot{q}^i,\ \dot{p}_i)$, this expression becomes

$$\lambda = \dot{p}_i dq^i - \dot{q}^i dp_i\ .$$

Example 4: If $U \subset F$ is a chart for Q, then

$$T(T^*U) = U \times F^* \times F \times F^*,$$

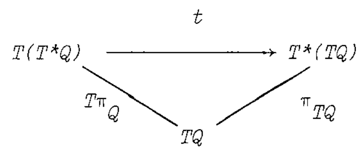
while

$$T^*(TU) = U \times F \times F^* \times F^*.$$

The map t: $U \times F^* \times F \times F^* \to U \times F \times F^* \times F^*$ given in charts by

$$t(x, \sigma, e, \pi) = (x, e, \pi, \sigma)$$

extends to a well-defined diffeomorphism t: $T(T^*Q) \to T^*(TQ)$ (for an intrinsic definition of t, see ref [2]). Since the diagram



commutes, it follows that $(T(T^*Q), T\pi_Q, TQ, t^*\Theta_{TQ}, t)$ is a special symplectic manifold. Here,

$$\lambda(x, \sigma, e, \pi) = (\pi, 0) \oplus (\sigma, 0)$$

or, in finite-dimensions,

$$\lambda = \dot{p}_i dq^i + p_i\, d\dot{q}^i.$$

Combining examples (3) and (4), one sees that $T(T^*Q)$ can be realized as a special presymplectic manifold in two completely different ways. This fact is of fundamental significance for mechanics, since it provides the geometric link between the Hamiltonian and Lagrangian formalisms in terms of which the Legendre transformation is defined (cf §VIII). Note, however, that both *special* presymplectic structures on $T(T^*Q)$ give rise to the same *symplectic* structure, since

$$d\flat_{T^*Q}{}^*\Theta_{T^*Q} = \dot{\omega} = dt^*\Theta_{TQ}.$$

*****

Of particular importance for dynamics are the isotropic g-submanifolds of special presymplectic manifolds. Generalizing the construction at the end of §III, one has the following interpretation of such g-submanifolds in terms of generating forms.

Theorem [Śniatycki and Tulczyjew]: Let (P, p, M, λ, μ) be a special presymplectic manifold, j_N: $N \to M$ a g-submanifold of M, and α a closed 1-form on N. Define $\mathcal{D}(\alpha) = \{y \in p^{-1}(j_N(N)) | \langle Z|\lambda\rangle = \langle u|\alpha\rangle$ for all $Z \in T_yP$ and $u \in \underline{TN}$ with $Tp(Z) = u\}$. Then $\mathcal{D}(\alpha)$ is an isotropic g-submanifold of $(P, d\lambda)$ with inclusion $j_{\mathcal{D}}$, the map $p_{\mathcal{D}}$ defined by the commutative diagram

$$\begin{array}{ccc} \mathcal{D}(\alpha) & \xrightarrow{\;j_{\mathcal{D}}\;} & P \\ {\scriptstyle p_{\mathcal{D}}}\downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow[\;j_N\;]{} & M \end{array}$$

is a submersion, the fibers of $p_{\mathcal{D}}$ are connected, and $j_{\mathcal{D}}{}^*\lambda = p_{\mathcal{D}}{}^*\alpha$.

Conversely, suppose that $\mathcal{D}$ is an isotropic g-submanifold of $(P, d\lambda)$ with inclusion $j_{\mathcal{D}}$ such that $N := p\circ j_{\mathcal{D}}(\mathcal{D})$ is a g-submanifold of M and the induced projection $p_{\mathcal{D}}$ defined by the commutative diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\;j_{\mathcal{D}}\;} & P \\ {\scriptstyle p_{\mathcal{D}}}\downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow[\;j_N\;]{} & M \end{array}$$

is a submersion with connected fibers. Then there exists a unique closed 1-form α on M such that $j_{\mathcal{D}}{}^*\lambda = p_{\mathcal{D}}{}^*\alpha$. Furthermore, $\mathcal{D} \subseteq \mathcal{D}(\alpha)$.

The 1-form α is the *generating form* of $\mathcal{D}(\alpha)$, and $\mathcal{D}(\alpha)$ is said to be *generated* by α. Symbolically, we write

$$\mathcal{D}(\alpha) = \mu^{-1}\{\alpha(N) + \underline{TN}^{\vdash}\}.$$

If N happens to be a Banach submanifold of M, then $\mathcal{D}(\alpha)$ is actually Lagrangian.

The proof of the above result, given in [23] for submanifolds of special symplectic manifolds, in fact holds for <u>generalized</u> submanifolds of special <u>pre</u>-symplectic manifolds.

<u>Example 5</u>: Let $L\colon\ TQ \to \mathbb{R}$ be a Lagrangian. According to the above theorem, the Lagrangian submanifold $\mathcal{D}(dL)$ of $(T(T^*Q),\ T\pi_Q,\ TQ,\ t^*\Theta_{TQ},\ t)$ generated by dL is defined by $t^*\Theta_{TQ} = T\pi_Q{}^*(dL)$. In a natural bundle chart $(T(T^*U);\ q^i,\ p_i,\ \dot{q}^i,\ \dot{p}_i)$, this becomes

$$\dot{p}_i dq^i + p_i d\dot{q}^i = dL,$$

or, more suggestively,

$$\dot{p}_i = \frac{\partial L}{\partial q^i}\ ,\ p_i = \frac{\partial L}{\partial \dot{q}^i}\ .$$

Example 6: Let N be a Banach submanifold of T^*Q, and $H: N \to \mathbb{R}$ a Hamiltonian. The exact 1-form $-dH$ generates an isotropic submanifold $\mathcal{D}(-dH)$ of $(T(T^*Q), \tau_{T^*Q}, T^*Q, \flat_{T^*Q}{}^*\Theta_{T^*Q}, \flat_{T^*Q})$. In the natural bundle chart of Example [5], N may be described by the vanishing of certain functions $\phi^\alpha(q, p)$, $\alpha = 1, \ldots,$ codim N. $\mathcal{D}(-dH)$ is then locally given by $\flat_{T^*Q}{}^*\Theta_{T^*Q} = -\tau_{T^*Q}{}^*dH$ subject to the constraints $\phi^\alpha = 0$. Using Lagrange multipliers, one has the local expressions

$$\dot{q}^i = \frac{\partial \bar{H}}{\partial p_i} + \lambda_\alpha \frac{\partial \phi^\alpha}{\partial p_i}$$

$$\dot{p}^i = -\frac{\partial \bar{H}}{\partial q^i} - \lambda_\alpha \frac{\partial \phi^\alpha}{\partial q^i} \quad ,$$

where $\bar{H}$ is any extension of H to T^*Q. Physically, $N = FL(TQ)$ is Dirac's primary constraint submanifold, the ϕ^α are primary constraints, and the above two equations are the Dirac-Hamilton equations of motion (cf. [6], [9] and §VIII).

VII. *Generalized Constraint Algorithm*

Let $(M_1, \omega_1, \alpha_1)$ be a presymplectic dynamical system, and consider the generalized Hamilton equations

$$i(X)\omega_1 = \alpha_1. \tag{7.1}$$

We now restate the presymplectic constraint algorithm of §V, which provides the necessary and sufficient conditions for the solvability of (7.1), in terms of special presymplectic manifolds.

Construct the special presymplectic manifold $(TM_1, \tau_1, M_1, \flat_1{}^*\Theta_1, \flat_1)$, where $\flat_1$: $TM_1 \to TM_1{}^*$ is the map induced by ω_1. The closed 1-form α_1 on M_1 generates, according to the theorem in the last section, an isotropic g-submanifold

$$\mathcal{D}_1 = \flat_1{}^{-1}\{\alpha_1(M_1)\}$$

of $(TM_1, \dot{\omega}_1)$. The secondary constraint g-submanifold

$$M_2 = \tau_1(\mathcal{D}_1)$$

consists of those points of M_1 along which there exist algebraic solutions X of (7.1), viewed as smooth sections of $\mathcal{D}_1$.

The g-submanifold $\mathcal{D}_1$ will be a *differential equation* with respect to M_2 -- that is, vector fields $X: M_2 \to \mathcal{D}_1$ will solve (7.1) in a differential sense -- iff the integrability conditions

$$\mathcal{D}_1 \subseteq \underline{TM_2}$$

are satisfied. If this is not the case, then one must restrict attention to the subset

$$\mathcal{D}_2 = \mathcal{D}_1 \cap \underline{\overline{TM}_2}$$

of TM_1.

The motion of the system is thereby constrained to lie in the closure of the tertiary constraint g-submanifold.

$$M_3 = \tau_1(\mathcal{D}_2)$$

of M_1. Demanding that $\mathcal{D}_2$ be tangent to $\overline{M}_3$ (i.e., $\mathcal{D}_2 \subsetneq \underline{\overline{TM}_3}$) may necessitate a further restriction to $\mathcal{D}_3 = \mathcal{D}_2 \cap \underline{\overline{TM}_3}$ etc.

Thus, the algorithm leads to a sequence of isotropic constraint g-submanifolds M_ℓ given by

$$M_\ell = \tau_1(\mathcal{D}_{\ell-1}), \tag{7.2}$$

where

$$\mathcal{D}_\ell = \mathcal{D}_{\ell-1} \cap \underline{\overline{TM}_\ell}\ , \tag{7.3}$$

and

$$\mathcal{D}_1 = \flat_1^{-1}\{\alpha_1(M_1)\}.$$

If the algorithm terminates with some non-empty final constraint g-submanifold M_K $(1 \leq K \leq \infty)$, then $\mathcal{D}_K = \mathcal{D}_{K+1} \subsetneq \underline{\overline{TM}_K}$. Consequently, there exists at least one solution $X \in \underline{\overline{TM}_K}$ such that

$$[i(X)\omega_1 - \alpha_1]|M_K = 0. \tag{7.4}$$

The fact that $\mathcal{D}_K$ is not usually transverse to the fibers of $\underline{\overline{TM}_K}$ is indicative of the generic non-uniqueness of the evolution vectorfield X. Specifically, X is unique iff the fiber dimension of $\mathcal{D}_K \cap \underline{\overline{TM}_K}$ is everywhere unity, in which case the canonical system (M_1, ω_1, M_K) is second class.

There are two regularity conditions that must be satisfied for the successful application of the algorithm: (*i*) Each set $\tau_1(\mathcal{D}_{\ell-1})$ must be a g-submanifold of M_1, and (*ii*) The fibers of $\mathcal{D}_{\ell-1}$ over $\tau_1(\mathcal{D}_{\ell-1})$ must be isomorphic [24]. If, at the ℓth step of the algorithm, either of these two conditions fails to hold, then one must judiciously choose a g-submanifold M_ℓ' of $\tau_1(\mathcal{D}_{\ell-1})$ such that the fibers of $\mathcal{D}_{\ell-1}|M_\ell'$

are isomorphic and then proceed with the algorithm applied to M_ℓ'. A proper treatment of such singularities, which is necessary for the correct physical interpretation of certain systems, will be given elsewhere [25] (see also [13]).

The above technique should be compared with that proposed by Menzio and Tulczyjew [4]. From the presymplectic standpoint, the integrability conditions

$$\mathcal{D}_\ell \subseteq \underline{\overline{T}[\tau_1(\mathcal{D}_\ell)]}.$$

are applied during the course of the algorithm and consequently are automatically satisfied on the final constraint g-submanifold M_K, i.e., if M_K exists, then by construction

$$\mathcal{D}_K \subseteq \underline{\overline{T}[\tau_1(\mathcal{D}_K)]}.$$

Therefore, integrability has no relation to the class of the canonical system (M_1, ω_1, M_K).

We note that this generalized constraint algorithm is applicable to any dynamical system determined by the specification of a submanifold $\mathcal{D}_1$ of "admissible" vector fields. Eqns. (7.2, 7.3) contain the essence of the Dirac constraint problem and are quite independent of the origin of $\mathcal{D}_1$.

VIII. Applications

(1) The Lagrangian Formulation of Mechanics

Let Q be the configuration space of a physical system, and TQ its velocity phase space. We want to include the case of field dynamics, where the Lagrangian may be only densely defined. Typically, one takes the domain of L to be the restriction T_CQ, where C is a manifold domain in Q.

For $w \in T_CQ$, we define the *energy* $E: T_CQ \to \mathbb{R}$ of L by

$$E(w) = \langle w|FL(w)\rangle - L(w),$$

where the Legendre transformation $FL: T_CQ \to T^*Q$ is given by (2.6). Pulling Ω_Q back to T_CQ, one obtains a generically presymplectic form $\Omega_L = FL^*\Omega_Q$. Our task is to define and solve consistent Lagrange equations of the form

$$i(X)\Omega_L = -dE. \qquad (8.1)$$

Consider the special presymplectic manifold $(T(T_CQ), \tau_{T_CQ}, T_CQ, \flat_L{}^*\Theta_{T_CQ}, \flat_L)$,

where $\flat_L: T(T_CQ) \to T^*(T_CQ)$ is induced by Ω_L. The 1-form $-dE$ on T_CQ generates an isotropic g-submanifold $\mathcal{D}_1 = \flat_L^{-1}\{-dE(T_CQ)\}$ of $(T(T_CQ), \dot{\Omega}_L)$. The constraint algorithm, applied to $\mathcal{D}_1$, then proceeds as in §VII, eventually (if the problem is solvable) producing a differential equation $\mathcal{D}_K \subseteq \mathcal{D}_1$ and a final constraint

g-submanifold $\tau_{T_C Q}(\mathcal{D}_K)$ of $T_C Q$. One is then assured of the existence of a section X: $\tau_{T_C Q}(\mathcal{D}_K) \to \mathcal{D}_K$ such that

$$\left[i(X)\Omega_L + dE\right] | \tau_{T_C Q}(\mathcal{D}_K) = 0. \tag{8.2}$$

(2) The Second-Order Equation Problem

The consistent Lagrange equations that follow from the constraint algorithm are typically a set of coupled first-order differential equations -- a feature of theories which are described mathematically by presymplectic geometries. Variational as well as physical considerations demand, however, that the Lagrange equations be a set of coupled second-order differential equations. Specifically, the equations of motion (8.1) will follow from a variational principle iff the *second-order equation* condition

$$\tau_{T_C Q}(X) = T(\tau_Q | T_C Q)(X) \tag{8.3}$$

holds at every point in the domain of X [26].

It is therefore necessary to find the conditions under which the Lagrange equations (8.2) admit solutions which are in fact second-order equations. Formally, special presymplectic techniques combined with the constraint algorithm allow us to easily solve this problem. Indeed, define

$$T_C^2 Q = \{X \in T(T_C Q) \,|\, (8.3) \text{ holds}\}.$$

The isotropic g-submanifold

$$\mathcal{D}_1' = \flat_L^{-1}\{-dE(T_C Q)\} \cap T_C^2 Q$$

consists of those vectors which satisfy both (8.1) and (8.3) along $\tau_{T_C Q}(\mathcal{D}_1')$. Applying the constraint algorithm to $\mathcal{D}_1'$, one obtains a second-order differential equation $\mathcal{D}_F'$ whose sections are solutions of the Lagrange equations along $\tau_{T_C Q}(\mathcal{D}_F')$.

Typically, however,

$$\tau_{T_C Q}(\mathcal{D}_F') \subset \tau_{T_C Q}(\mathcal{D}_K),$$

where $\mathcal{D}_K$ is as in example (1) above. Furthermore, it may happen that $\mathcal{D}_F' = \phi \neq \mathcal{D}_K$; and in the case $\mathcal{D}_F' \neq \phi$, there may not exist globally smooth sections $\tau_{T_C Q}(\mathcal{D}_F') \to \mathcal{D}_F'$ even though such sections of $\tau_{T_C Q}(\mathcal{D}_K) \to \mathcal{D}_K$ exist. Elsewhere [26] we have, subject to certain regularity conditions, proved the existence of, as well as classified, certain g-submanifolds of $\tau_{T_C Q}(\mathcal{D}_F')$ along which smooth sections exist.

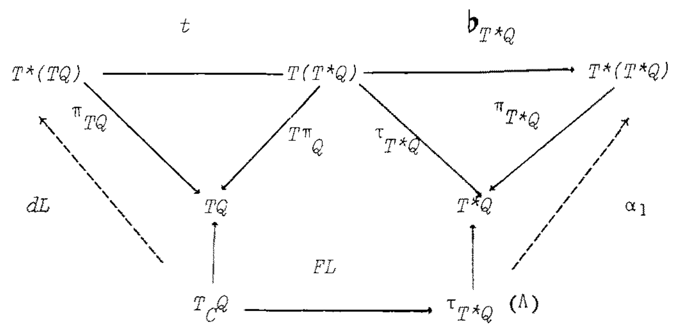
(3) The Hamiltonian Formulation of Mechanics [27]

Given a Lagrangian L on the restricted velocity phasespace $T_C Q$, one may Legendre transform to the Hamiltonian description as follows: the 1-form dL on $T_C Q$ generates an isotropic g-submanifold $\Lambda = t^{-1}\{dL(T_C Q)\}$ of the special symplectic manifold $(T(T^*Q), T\pi_Q, TQ, t^*\Theta_{TQ}, t)$. However, $T(T^*Q)$ may be viewed as a special presymplectic manifold $(T(T^*Q), \tau_{T^*Q}, T^*Q, \flat_{T^*Q}{}^*\Theta_{T^*Q}, \flat_{T^*Q})$. The g-submanifold $\tau_{T^*Q}(\Lambda)$ of T^*Q is the primary constraint g-submanifold M_1 of the Dirac-Bergmann theory. Indeed,

$$\tau_{T^*Q}(\Lambda) = \tau_{T^*Q} \circ t^{-1} \circ dL(T_C Q)$$

$$= FL(T_C Q)$$

as may be verified in charts. These constructions are summarized in the following diagram:



If the projection $\Lambda \to \tau_{T^*Q}(\Lambda)$ is a submersion whose fibers are connected [28], then Λ is generated by a unique closed 1-form α_1 on $M_1 = \tau_{T^*Q}(\Lambda)$ [29]. The form α_1 is the Hamiltonian 1-form for the presymplectic Hamiltonian system (M_1, ω_1), where $j_1: M_1 \to T^*Q$ is the inclusion and $\omega_1 = j_1{}^*\Omega_Q$.

There are two equivalent ways to proceed with a Hamiltonian analysis of the system. For example, one may apply the algorithm directly to Λ, effectively generating solutions of

$$i(X)\Omega_Q = \bar{\alpha}_1,$$

where $\bar{\alpha}_1$ is any extension of α_1 on M_1 to T^*Q. One thus obtains a symplectic version of the Dirac-Bergmann technique [6, 7]. Note, however, that this method only relies upon the existence of Λ, not the Hamiltonian 1-form $\bar{\alpha}_1$. Consequently, one has here a way to do Hamiltonian dynamics without ever mentioning Hamiltonians.

On the other hand, one may proceed more in the spirit of §VII by directly solving the Hamilton equations

$$i(X)\omega_1 = \alpha_1$$

associated to $(M_1, \omega_1, \alpha_1)$. In this case, the constraint algorithm is directly applied to the isotropic g-submanifold $\flat_1^{-1}\{\alpha_1(M_1)\}$ of the special presymplectic manifold $(TM_1, \tau_1, M_1, \flat_1{}^*\Theta_{M_1}, \flat)$.

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The Proca Field

As a concrete example of the generalized constraint algorithm applied to an infinite-dimensional second class system, we now work out the details for the Proca field in the Hamiltonian formulation.

The 3 + 1 decomposed Proca Lagrangian is

$$L(A,\dot{A}) = \tfrac{1}{2}\int_{\mathbb{R}^3} (\vec{\nabla}A_\perp)^2 - 2(\vec{\nabla}A_\perp)\cdot\dot{\vec{A}} + \dot{\vec{A}} - (\vec{\nabla}\times\vec{A})^2 + m^2A_\perp^2 - m^2\vec{A}^2\}d\mu,$$

where the vector field A is decomposed $A = (A_\perp, \vec{A})$, $\mathbb{R}^3$ denotes a constant-time Cauchy surface in Minkowski spacetime and μ is some measure on $\mathbb{R}^3$.

One must first decide on a choice for velocity phasespace. The configuration space should be some Hilbert space of all four-vectors $(A_\perp, \vec{A})$. As L contains at most first spatial derivatives of A, an appropriate choice for configuration space is the manifold domain

$$C = H_\perp^1 \times \vec{H}^1$$

of

$$Q = L_\perp^2 \times \vec{L}^2,$$

with the obvious notational shorthand, where H^1 is the first Sobolev space on $\mathbb{R}^3$. Velocity phasespace, that is, the manifold of all $(A,\dot{A})$ is then the restriction of TQ to C:

$$T_CQ = (H_\perp^1 \times \vec{H}^1) \oplus (L_\perp^2 \times \vec{L}^2),$$

as no spatial derivatives of $\dot{A}$ appear in L. The measure μ can then be taken to be the ordinary L^2 measure on $\mathbb{R}^3$.

To Legendre transform to the Hamiltonian description a la example (3), we must calculate dL. For $(A,\dot{A}) \in T_CQ$ and $\mathfrak{a}\oplus\mathfrak{b} \in T(T_CQ)$,

$$dL(A,\dot{A}):(a+b) = \int_{\mathbb{R}^3} \{(\vec{\nabla}A_\perp - \vec{\dot{A}})\cdot\vec{\nabla}a_\perp$$

$$+ (\vec{\dot{A}} - \vec{\nabla}A_\perp)\cdot\vec{b} - (\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\times\vec{a})$$

$$+ m^2 A_\perp a_\perp - m^2\vec{A}\cdot\vec{a}\}d\mu$$

Appealing to the theorem of §VI, one finds that the isotropic g-submanifold $dL(T_CQ) \subset T^*(TQ)$ consists of those points

$$(A,\dot{A}) \oplus (\sigma, \pi) \in (C\times Q) \oplus (Q^*\times Q^*)$$

such that

$$\langle a|\sigma\rangle = \int_{\mathbb{R}^3}\{(\vec{\nabla}A_\perp - \vec{\dot{A}})\cdot\vec{\nabla}a_\perp + m^2A_\perp a_\perp$$

$$- (\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\times\vec{a}) - m^2\vec{A}\cdot\vec{a}\}d\mu \tag{8.4a}$$

$$\langle b|\pi\rangle = \int_{\mathbb{R}^3} (\vec{\dot{A}} - \vec{\nabla}A_\perp)\cdot\vec{b}\; d\mu \tag{8.4b}$$

for arbitrary $a \in C$ and $b \in Q$. Here, the natural pairing $\langle\ |\ \rangle$: $TQ \times T^*Q \to \mathbb{R}$ is defined by

$$\langle(A,\dot{A})|(A,\pi)\rangle = \int_{\mathbb{R}^3}\{\vec{\dot{A}}\cdot\vec{\pi} + \dot{A}_\perp\pi_\perp\}d\mu. \tag{8.5}$$

According to (8.5), (8.4b) implies that

$$\pi_\perp = 0. \tag{8.6}$$

Applying t^{-1}, we have that

$$\Lambda = t^{-1}\{dL(T_CQ)\}\subset T(T^*Q)$$

consists of those points

$$(A,\pi) \oplus (\dot{A},\sigma) \in (C\times\vec{L}^{2*}) \oplus (Q\times Q^*)$$

for which (8.4a, b) hold with $\pi = (0,\vec{\pi})$.

Viewing Λ as an isotropic g-submanifold of the special symplectic manifold $(T(T^*Q),\ \tau_{T^*Q},\ T^*Q,\ \flat_{T^*Q}{}^*\Theta_{T^*Q},\ \flat_{T^*Q})$, one finds that the primary constraint g-submanifold $M_1 = \tau_{T^*Q}(\Lambda)$ of T^*Q is

$$M_1 = C \times \vec{L}^{2*}.$$

The condition (8.6) is therefore a primary constraint.

The induced projection $\Lambda \to M_1$ is clearly a submersion whose fibers are connected. Thus, Λ is generated by a closed 1-form α_1 on M_1. In fact, $\alpha_1 = -dH_1$, where the Hamiltonian H_1 on M_1 is

$$H_1(A,\pi) = \tfrac{1}{2}\int_{\mathbb{R}^3}\{\vec{\pi}^2 + 2(\vec{\nabla}A_\perp)\cdot\vec{\pi} + (\vec{\nabla}\times\vec{A})^2 - m^2A_\perp^2 + m^2\vec{A}^2\}d\mu$$

for $(A,\pi) \in M_1$ (cf. [29]).

We now apply the constraint algorithm to solve the field equations

$$i(X)\omega_1 = -dH_1 \tag{8.7}$$

of the presymplectic dynamical system (M_1, ω_1, dH_1), where j_1: $M_1 \to T^*Q$ is the inclusion and $\omega_1 = j_1{}^*\Omega_Q$.

The first step is to calculate the isotropic g-submanifold $\mathcal{D}_1 = \flat_1^{-1}\{-dH_1(M_1)\}$ of $(TM_1, \tau_1, M_1, \flat_1{}^*\Theta_{M_1^*}, \flat_1)$. If $(A,\pi) \in M_1$ and $b \oplus \tau \in TM_1$,

$$\begin{aligned} dH_1(A,\pi)\cdot(b\oplus\tau) = \int_{\mathbb{R}^3} \{&\vec{\pi}\cdot\vec{\tau} + \vec{\nabla}b_\perp \cdot\pi \\ &+ (\vec{\nabla}A_\perp)\cdot\vec{\tau} + (\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\times\vec{b}) \\ &- m^2A_\perp b_\perp + m^2\vec{A}\cdot\vec{b}\}d\mu. \end{aligned} \tag{8.8}$$

Writing $X(A,\pi) = a \oplus \sigma \in TM_1$, (8.7) becomes

$$\omega_1(a\oplus\sigma,\ b\oplus\tau)\,|\,(A,\pi) = -dH_1(A,\pi)\cdot(b\oplus\tau). \tag{8.9}$$

From the definition of ω_1, (2.4b) and (8.5),

$$\omega_1(a+\sigma,\ b+\tau)\,|\,(A,\pi) = \int_{\mathbb{R}^3}\{\vec{b}\cdot\vec{\sigma} - \vec{a}\cdot\vec{\tau}\}\,d\mu\ .$$

Substituting this expression into (8.9), and then comparing with (8.8), one calculates

$$X(A,\pi) = (a_\perp,\ \vec{\pi} + \vec{\nabla}A_\perp) \oplus (0,\ \Delta\vec{A} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - m^2\vec{A}) \tag{8.10}$$

iff

$$\vec{\nabla}\cdot\vec{\pi} + m^2 A_\perp = 0 \, . \tag{8.11}$$

Note that these formal expressions are well-defined iff $\vec{A} \in \vec{H}^2$ and $\vec{\pi} \in \vec{H}^{1*}$. Thus, we

$$\mathcal{D}_1 = \{(A,\pi) \oplus (a,\sigma) \;\varepsilon\; \left[(H^1_\perp \times \vec{H}^2)\times \vec{H}^{1*}\right] \oplus \left[(H^1_\perp \times \vec{H}^1)\times \vec{L}^{2*}\right]$$

such that

$$a = (a_\perp, \vec{\pi} + \vec{\nabla}A_\perp),$$

$$\sigma = (0, \Delta\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - m^2\vec{A}),$$

and

$$\vec{\nabla}\cdot\vec{\pi} + m^2 A_\perp = 0\}.$$

Proceeding with the algorithm, the secondary constraint g-submanifold $M_2 = \tau_1(\mathcal{D}_1)$ along which algebraic solutions X to (8.7) exist is

$$M_2 = \{(A,\pi) \;\varepsilon\; (H^1_\perp \times \vec{H}^2) \times \vec{H}^{1*} \,|\, \text{(8.11) holds}\}.$$

We now check the integrability conditions: is $\mathcal{D}_1 \subseteq \overline{\underline{TM_2}}$?

$$\overline{\underline{TM_2}} = \{(A,\pi) \oplus (a,\sigma) \;\varepsilon\; TM_1 \,|\, \text{(8.11) holds and } \vec{\nabla}\cdot\vec{\sigma} + m^2 a_\perp = 0\}.$$

From the definition of $\mathcal{D}_1$, however,

$$\vec{\nabla}\cdot\vec{\sigma} + m^2 a_\perp = \vec{\nabla}\cdot\{\Delta\vec{A} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - m^2\vec{A}\} + m^2 a_\perp$$

$$= m^2 \{a_\perp - \vec{\nabla}\cdot\vec{A}\} \neq 0,$$

so that $\mathcal{D}_1 \not\subseteq \overline{\underline{TM_2}}$. Thus, we consider

$$\mathcal{D}_2 = \mathcal{D}_1 \cap \overline{\underline{TM_2}}$$

$$= \{(A,\pi) \oplus (a,\sigma) \;\varepsilon\; \mathcal{D}_1 \,|\, a_\perp - \vec{\nabla}\cdot\vec{A} = 0\}.$$

Calculating the tertiary constraint g-submanifold M_3, we have

$$M_3 = \tau_1(\mathcal{D}_2) = M_2,$$

and M_2 is the final constraint g-submanifold.

Thus, the constraint algorithm terminates, and we are assured that at least one solution X to (8.7) restricted to M_2 exists. From (8.10) and the expression for $\mathcal{D}_2$, one finds for $(A,\pi) \in M_2$

$$X(A, \pi) = (\vec{\nabla}\cdot\vec{A}, \vec{\pi} + \vec{\nabla}A_\perp) \oplus (0, \Delta\vec{A} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - m^2\vec{A}).$$

These are just the Proca equations:

$$dA_\perp/dt \equiv a_\perp = \vec{\nabla}\cdot\vec{A}$$

$$d\vec{A}/dt \equiv \vec{a} = \vec{\pi} + \vec{\nabla}A_\perp$$

$$d\pi_\perp/dt \equiv b_\perp = 0$$

$$d\vec{\pi}/dt \equiv \vec{b} = \Delta\vec{A} - \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - m^2\vec{A}.$$

Clearly, X is unique. That the Proca equations give rise to a well-defined flow on $\overline{M_2}$ follows from the hyperbolic version of the Hille-Yoshida Theorem [15].

The Proca canonical system is thus (M_1, ω_1, M_2) and is second class. Indeed, let $(A, \pi) \in M_2$ and $b \oplus \tau \in TM_1$. Then $b \oplus \tau \in TM_2^\perp$ iff

$$0 = \omega_1(a\oplus\sigma, b\oplus\tau)|(A,\pi)$$

$$= \int_{\mathbb{R}^3}\{\vec{b}\cdot\vec{\sigma} - \vec{a}\cdot\vec{\tau}\}d\mu.$$

for arbitrary $a\oplus\sigma \in \underline{TM_2}$. Taking $a\oplus\sigma = (0,\vec{a}) \oplus 0$, the above expression will vanish iff $\vec{\tau} = 0$. On the other hand, if $a\oplus\sigma = (-m^{-2}\vec{\nabla}\cdot\vec{\sigma}, \vec{0}) \oplus (0,\vec{\sigma})$, then this expression is zero iff $\vec{b} = 0$. Consequently, $b\oplus\tau \in TM_2^\perp$ iff $b \oplus \tau = (b, \vec{0}) + 0$. But such a $b \oplus \tau$ is an element of $\underline{TM_2}$ iff $b_\perp = 0$. Thus, $\underline{TM_2} \cap TM_2^\perp = \{0\}$ and the Proca canonical system is weakly symplectic.

Appendix

List of Symbols

- $\alpha^*\Theta$ pullback of Θ by α
- d exterior derivative
- D Frechet derivative
- $\overline{D}/\dot{D}$ partial Frechet derivative along the base/fiber of a fiber bundle

FL	Legendre transformation $TQ \to T^*Q$ induced by L
H^n	nth Sobolev space on $\mathbb{R}^3$
Θ_Q	canonical 1-form on T^*Q
i	interior product
$\overline{N}$	topological closure of N
π_Q	cotangent bundle projection $T^*Q \to Q$
t	canonical diffeomorphism $T(T^*Q) \to T^*(TQ)$
T	tangent functor
Tf	pushforward, prolongation of f
T_CQ	restriction of TQ to $C \subsetneq Q$, $TQ\|C$
TM	tangent bundle of M; set of all smooth vectorfields on M
$\underline{TN}$	image $Tj(TN)$ of TN in TM, where $j: N \to M$ is an inclusion
$TN^\flat$	image of TN in T^*M under $\flat$
$TN^\perp$	symplectic complement of $\underline{TN}$ in TM
$TN^\vdash$	annihilator of $\underline{TN}$ in T^*M
$\overline{TN}$	$T_N\overline{N}$
T^2Q	the diagonal in $T(TQ)$
T^*Q	cotangent bundle of Q; set of all smooth 1-forms on Q
τ_Q	tangent bundle projection $TQ \to Q$
ω, ω_1	presymplectic forms
$\dot{\omega}$	"special" presymplectic form on the tangent bundle of a presymplectic manifold
Ω_Q	canonical symplectic form on T^*Q
Ω_L	symplectic form on TQ; $FL^*\Omega_Q$
$\flat_M$	map $TM \to T^*M$ induced by the presymplectic form ω on M
$\langle \| \rangle$	dualization $TM \times T^*M \to \mathbb{R}$
$\cdot$	dualization $E \times E^* \to \mathbb{R}$ for Banach spaces
$\|$	restriction (not pullback)

Terminology and Conventions

All manifolds and maps appearing in this paper are assumed to be C^∞.

The symbol TM (T^*M) denotes both the tangent (cotangent) bundle of M and the space of all smooth vectorfields (1-forms) on M. Usually, lower-case italic letters will refer to tangent vectors, while upper-case italics will denote vectorfields.

Let Q be a manifold, τ_Q: $TQ \to Q$ its tangent bundle, and $(U;q^i)$ a chart on Q. For $w \in T_mQ$, the chart $(TU;q^i,\dot{q}^i)$ on TQ defined by

$$q^i(w) = q^i \circ \tau_Q(w)$$

$$\dot{q}^i(w) = \langle w | dq^i \rangle$$

is said to be a *natural bundle chart*. One can similarly define natural bundle charts on cotangent bundles and higher-order bundles.

Let $j: N \to M$ be a map of a Banach manifold N into a Banach manifold M. The pair (N,j) is said to be a

(*i*) *Banach submanifold* of M if j is an injective immersion (i.e., both j and Tj are injective and $Tj(TN)$ splits in TM),

(*ii*) *manifold domain* of M if both j and Tj are injective and have dense range,

(*iii*) *submanifold domain* of M if (N, j) is a manifold domain of the injectively immersed submanifold N of M, and

(*iv*) *submersion* of N onto M if j and Tj are surjective and $\ker Tj$ splits in TN.

Throughout this paper, the term *generalized submanifold* ("g-submanifold") refers to any pair (N, j) which is a Banach submanifold, a manifold domain or a submanifold domain.

We now briefly explain how one calculates locally, following Refs. [15] and [20]. If $U \subset E$ is a chart on a manifold Q, then $T^*U = U \times E^*$ is a chart on T^*Q, and a point $m \in T^*Q$ has the local representation $m = (x,\sigma)$ where $x \in U$, $\sigma \in E^*$. A chart on $T(T^*Q)$ is $T(T^*U) = (U \times E^*) \oplus (E \times E^*)$. Thus a tangent vector X to T^*Q has the local representation $x(m) = (x,\sigma) \oplus (a, w)$ where $a \in E$ and $w \in E^*$. We will often suppress the base point (x,σ) and simply write this as $X =: a \oplus w$. Thus, for example, if α is a 1-form on T^*Q, the interior product $i(X)\alpha(m)$ is written locally as $\alpha(x,\sigma)\cdot(a \oplus \pi)$.

In general, we try to keep our notation and terminology consistent with that of references [14], [15] and [20].

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Notes and References

1. W. Tulczyjew, Symposia Mathematica 14, 247 (1974).
2. W. Tulczyjew, in Differential Geometric Methods in Mathematical Physics, Lecture Notes in Math., #570, 457, 464 (Springer-Verlag, Berlin, 1977).
3. W. Tulczyjew, Acta Phys. Polon., B8, 431 (1977).
4. M. Menzio and W. Tulczyjew, Ann. Inst. H. Poincaré, A28, 349 (1978).
5. J. Kijowski and W. Tulczyjew, A Symplectic Framework for Field Theories, to appear (Springer-Verlag)
6. P.A.M. Dirac, Lectures on Quantum Mechanics, Belfer Graduate School of Science Monograph Series #2 (1964).
7. M.J. Gotay, J.M. Nester and G. Hinds, J. Math. Phys. 19, 2388 (1978).
8. M.J. Gotay and J.M. Nester, Presymplectic Hamilton and Lagrange Systems, Gauge Transformations and the Dirac Theory of Constraints, to appear (Proc. of the VII Int'l. Colloq. on Group Theoretical Methods in Physics, Austin, 1978).
9. M.J. Gotay, Presymplectic Manifolds, Geometric Constraint Theory, and the Dirac-Bergmann Theory of Constraints, Dissertation, University of Maryland, 1979.

10. M.J. Gotay and J.M. Nester, Ann. Inst. H. Poincare, A30, 129 (1979).

11. J. Kijowski, Commun. Math. Phys., 30, 99 (1973).

12. J. Kijowski and W. Szczyrba, Commun. Math. Phys., 46, 183 (1976).

13. J. Sniatycki, Ann. Inst. H. Poincaré, A20, 365 (1974).

14. R. Abraham and J. Marsden, Foundations of Mechanics, second ed., (Benjamin-Cummings, New York, 1978).

15. P. Chernoff and J. Marsden, Properties of Infinite-Dimensional Hamiltonian Systems, Lecture Notes in Mathematics #425 (Springer-Verlag, Berlin, 1974).

16. J.-M. Souriau, Structures des Systemes Dynamiques, (Dunod, Paris, 1970).

17. H.P. Künzle, J. Math. Phys., 13, 739 (1972).

18. A. Lichnerowicz, C.R. Acad. Sci. Paris, A280, 523 (1975).

19. A Lagrangian subspace $\underline{TN}$ of TM is necessarily closed, so that if N is Lagrangian, then N must be a Banach submanifold of M.

20. S. Lang, Differential Manifolds, (Addison-Wesley, Reading, Mass., 1972).

21. A. Weinstein, Adv. Math., 6, 329 (1971).

22. It is assumed that all of the spaces appearing in this paper are in fact generalized submanifolds in the sense of the Appendix (cf. §VII).

23. J. Sniatycki and W. Tulczyjew, Indiana U. Math. J., 22, 267 (1972).

24. In finite-dimensions, condition (ii) is tantamount to requiring that $dim\,\{ker\ \omega_1 \cap TM_{\ell-1}\}$ be constant on M_ℓ.

25. J.M. Nester and M.J. Gotay, The Dynamics of Singular Presymplectic Systems, (work in progress).

26. J.M. Nester and M.J. Gotay, Presymplectic Lagrangian Systems II: The Second-Order Equation Problem, University of Maryland Preprint #79-141, (1979) (to be published).

27. See also Exercise 5.3L of [14].

28. This is equivalent to the almost regularity of L (cf. ref. [10]).

29. Equivalently, $FL^*\alpha_1 = -dE$.

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