COMPACT PARALLELIZABLE FOUR DIMENSIONAL SYMPLECTIC AND COMPLEX MANIFOLDS
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(Communicated by David G. Ebin)

ABSTRACT. Examples of compact symplectic manifolds with no complex
and/or Kähler structures are presented.

1. Introduction. Many examples of compact symplectic manifolds that carry
no positive definite Kähler metric are now known. Here we present some com-
 pact 4-dimensional manifolds that have symplectic structures but carry no complex
structures. More generally we prove

THEOREM 1.1. Let $E^4$ be a principal circle bundle over $E^3$, which in turn is a
principal circle bundle over a torus $T^2$, so that the first Betti number of $E^4$ satisfies
$2 \leq b_1(E^4) \leq 4$. Then
(i) if $b_1(E^4) = 2$ then $E^4$ has symplectic but no complex structures;
(ii) if $b_1(E^4) = 3$ then $E^4$ has both symplectic and complex structures but no
positive definite Kähler metrics; however $E^4$ carries indefinite Kähler metrics;
(iii) $b_1(E^4) = 4$ if and only if $E^4$ is a 4-torus $T^4$.

REMARKS. (1) Apparently the manifolds that occur in part (i) of Theorem 1.1
are the first examples of compact symplectic manifolds with no complex structures.
Van de Ven [VdV], Yau [Ya] and Brotherton [Br] have given examples of compact
4-dimensional almost complex manifolds with no complex structures. Brotherton
used Massey products to prove the nonexistence of complex structures on certain
parallelizable 4-dimensional manifolds.
(2) Thurston [Th] has given an example of a compact symplectic manifold
with no positive definite Kähler metric. (See also [Ab, CFG, CFL, We1].) In §3
we shall see that it is covered under part (ii) of Theorem 1.1. It is interesting to
note that this example already occurs in the work of Kodaira [Kod, Theorem 19].
An explicit description of the Kodaira-Thurston example as a complex manifold is
given in §3.
(3) The spaces $E^4$ are all real parallelizable (but only $T^4$ is complex paralleliz-
able in the sense of Wang [Wa]). By a blowing up procedure one can construct
nonparallelizable symplectic manifolds with no complex structure and/or positive
definite Kähler metric [Go].
(4) Most of the manifolds considered in Theorem 1.1 have explicit matrix real-
izations as nilmanifolds [CM]. See also [PS], where it is proved that a compact
manifold is a principal torus bundle over a torus if and only if it is a 2-step nilman-
ifold. The paper [BG] is also relevant.

Received by the editors July 10, 1987.
1980 Mathematics Subject Classification (1985 Revision). Primary 53C15; Secondary 53C55.
(5) As a corollary to part (ii) of Theorem 1.1, we observe that none of the complex structures mentioned there can be calibrated (in the sense of [Gro]) by a symplectic form, since otherwise the corresponding $E^4$ would admit a positive definite Kähler metric.

(6) For a general discussion of symplectic manifolds constructed as fiber bundles see [We2].

We wish to thank Mike Hoffman, Dusa McDuff, Jonathan Rosenberg, David Simms and Alan Weinstein for several very useful discussions.

2. The topology of a principal circle bundle over a principal circle bundle over a torus. The classification of principal circle bundles is well known:

**Theorem 2.1** [Kob, p. 35, Kos, p. 133]. There is a one-to-one correspondence between equivalence classes of principal circle bundles over a manifold $M$ and the cohomology class $H^2(M, \mathbb{Z})$. Furthermore, given an integral closed 2-form $\Phi$ on $M$, there is a principal circle bundle $\pi: E \to M$ with connection form $\eta$ such that $\Phi$ is the curvature of $\eta$ (that is, $\pi^*(\Phi) = d\eta$).

Now let $\alpha$ and $\beta$ be integral closed 1-forms on $T^2$ such that $\alpha$ and $\beta$ are everywhere linearly independent and the cohomology class $[\alpha \wedge \beta]$ generates $H^2(T^2, \mathbb{Z})$. Theorem 2.1 implies that for every integer $n$ there is a principal circle bundle $E^3 \to T^2$ corresponding to $n[\alpha \wedge \beta]$ and a connection form $\gamma$ on $E^3$ chosen so that the curvature of $\gamma$ is $n\alpha \wedge \beta$. The real minimal model of $E^3$ is thus

$$M(E^3) = \{ \alpha, \beta, \gamma \mid d\alpha = d\beta = 0, d\gamma = n\alpha \wedge \beta \}.$$ 

(We use the same notation for differential forms on base spaces and their pullbacks to total spaces.) Then $H^1(E^3, \mathbb{R}) = \{ [\alpha], [\beta] \}$ and $H^2(E^3, \mathbb{R}) = \{ [\alpha \wedge \gamma], [\beta \wedge \gamma] \}$ when $n \neq 0$.

If $n = 0$, $E^3$ is a 3-torus; otherwise $E^3$ can be realized as the compact quotient $\Gamma_n \backslash H_n$ where $H_n$ is the Lie group of matrices of the form

$$\begin{pmatrix}
1 & a & -c/n \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix}$$

and $\Gamma_n$ is the subgroup of $H_n$ consisting of those matrices for which $a, b$ and $c$ are integers.

Principal circle bundles $E^4 \to E^3$ are classified by $H^2(E^3, \mathbb{Z})$. When $n \neq 0$ the Gysin sequence yields $H^2(E^3, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{[n]}$; thus for each pair of integers $(p, q)$ there is a principal circle bundle corresponding to the class $p[\alpha \wedge \gamma] + q[\beta \wedge \gamma]$. Again we use Kobayashi’s Theorem 2.1 to conclude that the connection form $\eta$ of $E^4 \to E^3$ can be chosen so that its curvature form is precisely $p\alpha \wedge \gamma + q\beta \wedge \gamma$. It follows that when $n \neq 0$ the (real) minimal models of the $E^4$ are given as follows:

$$M(E^4) = \{ \alpha, \beta, \gamma, \eta \mid d\alpha = d\beta = 0, d\gamma = n\alpha \wedge \beta, d\eta = p\alpha \wedge \gamma + q\beta \wedge \gamma \}.$$ 

Clearly, $b_1(E^4) = 3$ if $p = q = 0$ and $b_1(E^4) = 2$ otherwise. The case $n = 0$ is similar: for each triple of integers $(p, q, r)$ there is an $E^4$ with

$$M(E^4) = \{ \alpha, \beta, \gamma, \eta \mid d\alpha = d\beta = d\gamma = 0, d\eta = r\alpha \wedge \beta + p\alpha \wedge \gamma + q\beta \wedge \gamma \}.$$ 

Then $b_1(E^4) = 4$ if $r = p = q = 0$ and $b_1(E^4) = 3$ otherwise.
LEMMA 2.2. The minimal model $M(E^4)$ is not formal if one of $n,p,q,r$ is different from zero.

PROOF. It suffices to find nonzero Massey products. Suppose $n \neq 0$ (the proof when $n = 0$ is simpler). Then the cohomology classes $[\alpha \wedge \alpha]$ and $[\alpha \wedge n\eta]$ are both zero, so that the Massey product $\langle [\alpha], [\alpha], [n\eta] \rangle$ is well defined. By definition it is represented by $\alpha \wedge \gamma$. Now $[\alpha \wedge \gamma] \neq 0$ for $E^3$; it is also nonzero in the cohomology of $E^4$ except when $p \neq 0$, $q = 0$. But in this case the Massey product $\langle [\beta], [\beta], [n\alpha] \rangle$ is nonzero.

3. **Proof of Theorem 1.1.** First let us note that in all cases $E^4$ has many symplectic forms. For example

$$\Omega = (a\alpha + b\beta) \wedge \gamma + (e\alpha + f\beta) \wedge \eta$$

is closed if $a, b, e, f$ are constants such that $fp - eq = 0$, and has maximal rank if $af - be \neq 0$.

PROOF OF (i). We use [Kod, Theorem 25]: A [complex] surface is a deformation of an algebraic surface if and only if its first Betti number is even. Suppose $E^4$ with $b_1(E^4) = 2$ had a complex structure. Then [Kod, Theorem 25] would imply that $E^4$ would have a positive definite Kähler metric. But now a result of [DGMS] would imply that $M(E^4)$ is formal, and this is impossible by Lemma 2.2.

REMARK. It is amusing to compare an $E^4$ with $b_1(E^4) = 2$ with the Kähler manifold $S^2 \times T^2$. Both are parallelizable and have the same Betti numbers. But $E^4$ has nonzero Massey products while $S^2 \times T^2$ does not.

PROOF OF (ii). When $b_1(E^4) = 3$ and $n \neq 0$ an explicit complex structure on $E^4$ can be constructed as follows. Let $X, Y, Z, T$ be the parallelization dual to $\alpha, \beta, \gamma, \eta$; the only nonzero bracket is $[X, Y] = -nZ$. Now define an almost complex structure $J$ on $E^4$ by $JX = Y, JZ = T$. A direct calculation shows that the Nijenhuis tensor of $J$ vanishes; consequently $J$ is complex. A similar construction yields a complex structure on an $E^4$ with $n = 0$.

None of these $E^4$ can possess a positive definite Kähler metric since $b_1(E^4)$ is odd. (There are also nonzero Massey products.) Nonetheless an indefinite Kähler metric $\phi$ for the complex structure $J$ can be constructed as follows. Let $\Omega$ be a symplectic form which has type $(1,1)$ with respect to $J$; for example we can take $\Omega = \alpha \wedge \gamma + \beta \wedge \eta$. Then put $\phi(U, V) = \Omega(U J V)$ for vector fields $U, V$ on $E^4$.

In general suppose that $\Omega$ is Hermitian with respect to an almost complex structure $J$ so that the metric $\phi$ is given by $\phi(x, y) = \Omega(x J y)$. For vector fields $X, Y, Z$ we have that

$$2\nabla_X(\Omega)(Y, Z) = d\Omega(X, Y, Z) - d\Omega(X, JY, JZ) - \phi(X, S(Y, JZ)),$$

where $S$ denotes the Nijenhuis tensor of $J$ [Gra, formula (4.8)]. It follows that if $J$ is integrable and $\Omega$ is symplectic, then $\phi$ is Kählerian, but possibly indefinite.

REMARK. The Kodaira-Thurston example belongs to case (ii); explicitly it is $\Gamma_{-1} \setminus H_{-1} \times S^1$. As a complex manifold it has the following description. For each Gaussian integer $n$ let

$$G_n = \begin{cases} \left( \begin{array}{ccc} 1 & \bar{z} & w/n \\ 0 & 1 & z \\ 0 & 0 & 1 \end{array} \right) & \text{if } z \text{ and } w \text{ are complex} \end{cases}.$$
G_n is a complex manifold and as a Lie group it is left holomorphic but not right holomorphic. Let \( \Psi_n \) be the subgroup of \( G_n \) consisting of all those matrices whose elements are Gaussian integers. Then \( E^4 = \Psi_n \backslash G_n \) is a nilmanifold and a complex manifold (but not a complex nilmanifold). The Kodaira-Thurston example is \( \Psi_1 \backslash G_1 \).

The proof of (iii) is obvious.

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