Addendum

Mark J. Gotay, Janusz Grabowski and Hendrik B. Grundling,
An Obstruction to Quantizing Compact Symplectic Manifolds,
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We have redone the proof of Theorem 4. While the original proof remains valid, the proof given here is simpler. The discussion beginning with Theorem 4 and ending two paragraphs after the proof of Theorem 4 should be replaced by the following:

Theorem 4 *Let M be a nonzero coadjoint orbit in $\mathfrak{b}^*$, where $\mathfrak{b}$ is a compact, simple Lie algebra. Then M admits $\mathfrak{b}$ as a basic algebra.*

Proof. We first observe that since $\mathfrak{b}$ is compact and semisimple, its adjoint group B is compact [**Va**, Thm. 4.11.7]. Consequently M is compact.

Now the elements of $\mathfrak{b}$, regarded as (linear) functions on $\mathfrak{b}^*$, form a finite-dimensional space of observables on M closed with respect to the Poisson bracket $\{\ ,\ \}$. Their Hamiltonian vector fields are complete due to the compactness of M; moreover, transitivity is automatic since M is an orbit. Thus $\mathfrak{b}$ satisfies (B1) and (B2). We will show below that $\mathfrak{b}$ is minimal with respect to these properties. Given this, let $\mathcal{B}$ be a subspace of $\mathfrak{b}$ which is minimal amongst all such satisfying (B2). Then $\mathcal{B}$ is a basic set and generates $\mathfrak{b}$, for otherwise the transitive Lie algebra generated by $\mathcal{B}$ would be smaller than $\mathfrak{b}$, which contradicts the minimality of $\mathfrak{b}$. It follows that $\mathfrak{b}$ is a basic algebra.

Identifying $\mathfrak{b}$ with $\mathfrak{b}^*$ by means of the Ad-invariant inner product (1), we can identify adjoint and coadjoint orbits. Thus $M = O_h$ for some $h \in \mathfrak{b}$. With respect to this identification $\{f, g\}(k) = \langle [f, g], k\rangle$ for $f, g, k \in \mathfrak{b}$. It is easy to see that the isotropy subalgebra $\mathfrak{b}_h = \{f \in \mathfrak{b} \,|\, [f, h] = 0\} = [\mathfrak{b}, h]^\perp$, so we have the decomposition

$$\mathfrak{b} = \mathfrak{b}_h \oplus [\mathfrak{b}, h] \tag{4}$$

with

$$[\mathfrak{b}_h, [\mathfrak{b}, h]] \subset [\mathfrak{b}, h]. \tag{5}$$

Suppose now that $\mathfrak{b}$ is not minimal, and let $\mathfrak{a} \subset \mathfrak{b}$ be a minimal subalgebra for the compact orbit O_h. Proposition 1 implies that $\mathfrak{a}$ is compact and semisimple, and hence has a commutative Cartan subalgebra $\mathfrak{c}$. We extend $\mathfrak{c}$ to a Cartan subalgebra $\mathfrak{k}$ of $\mathfrak{b}$. Now fix a Cartan subalgebra of $\mathfrak{b}$ containing h; since this subalgebra is commutative it must be contained in $\mathfrak{b}_h$. As all Cartan subalgebras of a compact semisimple Lie algebra are conjugate under B [**Va**, Thm. 4.12.2], we may map $\mathfrak{k}$ onto this subalgebra by some conjugation. Since O_h is an adjoint orbit, the image of $\mathfrak{a}$ under this conjugation is also basic for O_h. Thus without loss of generality we may suppose that $\mathfrak{c} \subset \mathfrak{k} \subset \mathfrak{b}_h$.

Passing to complexifications we have the root space decomposition

$$\mathfrak{a}_{\mathbb{C}} = \mathfrak{c}_{\mathbb{C}} \oplus \left(\sum_{\delta \in \Delta} \mathfrak{a}_\delta \right)$$

where $\Delta \subset {\mathfrak{c}_{\mathbb{C}}}^*$ is the set of roots of $(\mathfrak{a}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$. The inclusion $\mathfrak{c} \subset \mathfrak{k}$ implies that every root space $\mathfrak{a}_\delta$ of $(\mathfrak{a}_{\mathbb{C}}, \mathfrak{c}_{\mathbb{C}})$ is also a root space $\mathfrak{b}_{\delta'}$ of $(\mathfrak{b}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ for some unique $\delta' \in {\mathfrak{k}_{\mathbb{C}}}^*$ with $\delta' \,|\, \mathfrak{a} = \delta$. We can therefore write

$$\mathfrak{a}_{\mathbb{C}} = \mathfrak{c}_{\mathbb{C}} \oplus \left(\sum_{\delta \in \Delta} \mathfrak{b}_{\delta'} \right). \tag{6}$$

By transitivity $[\mathfrak{a}, h] = T_h O_h = [\mathfrak{b}, h]$. Recalling that $h \in \mathfrak{k}$ so that $[\mathfrak{b}_{\delta'}, h] \subset \mathfrak{b}_{\delta'}$, we compute using (6),

$$[\mathfrak{b}_{\mathbb{C}}, h] = [\mathfrak{a}_{\mathbb{C}}, h] = [\mathfrak{c}_{\mathbb{C}}, h] \oplus \left(\sum_{\delta \in \Delta} [\mathfrak{b}_{\delta'}, h] \right) \subset \sum_{\delta \in \Delta} \mathfrak{b}_{\delta'} = \sum_{\delta \in \Delta} \mathfrak{a}_\delta \subset \mathfrak{a}_{\mathbb{C}}.$$

Thus

$$[\mathfrak{b}, h] \subset \mathfrak{a}. \tag{7}$$

Let $\mathfrak{h}$ be the Lie algebra generated by $[\mathfrak{b}, h]$; then from (4) and (5) we see that $\mathfrak{h}$ is an ideal in $\mathfrak{b}$. From (7) it follows that $\mathfrak{h} \subseteq \mathfrak{a}$. But as $\mathfrak{b}$ is simple, $\mathfrak{h} = \mathfrak{b}$ and this forces $\mathfrak{a} = \mathfrak{b}$, whence $\mathfrak{b}$ is minimal. ■

When $\mathfrak{b}$ is merely semisimple this result need not hold. For instance, S^2 is a coadjoint orbit in $\mathrm{su}(2)^* \oplus \{0\} \subset \mathrm{su}(2)^* \oplus \mathrm{su}(2)^* \cong \mathrm{so}(4)^*$ with basic algebra su(2), not so(4). (More generally, if there are several simple components, say $\mathfrak{b} = \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_K$, then there will be low-dimensional orbits for which $\mathfrak{b}$ is not basic; for instance, for h a regular element in a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{b}_1$ we have $\mathfrak{b}_h = \mathfrak{c} \oplus \mathfrak{b}_2 \oplus \cdots \oplus \mathfrak{b}_K$, and the corresponding orbit is $B/B_h \cong B_1/C$ with basic algebra $\mathfrak{b}_1$, not $\mathfrak{b}$.)

When M has *maximal* dimension in $\mathfrak{b}^*$, however, $\mathfrak{b}$ will be a basic algebra on M. Indeed, it is a well-known result of Duflo and Vergne that the isotropy algebras of maximal coadjoint orbits are abelian (see, e.g. [**MR**, pps. 278-280]). But, referring back to the proof of Theorem 4 it follows from this fact, (4), (5), and the semisimplicity of $\mathfrak{b}$ that $[\mathfrak{b}, h]$ generates $\mathfrak{b}$, i.e. $\mathfrak{h} = \mathfrak{b}$. So again (7) yields $\mathfrak{a} = \mathfrak{b}$.

Additional References

[**Va**] Varadarajan, V. S. [1984] *Lie Groups, Lie Algebras and Their Representations.* (Springer-Verlag, New York).