A Collection of Problems, with Solutions and Comments

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Introduction

We hope that instructors will find this a convenient and useful resource. We have tried to create a collection of interesting problems that can be attacked by high school students. Most of the problems stress ideas and techniques from school mathematics which, we believe, should be reinforced. The problems also at times dip into the puzzle tradition.

The solutions provided are meant to convey a set of attitudes: that mathematics deals with very concrete matters; that there can be many ways to attack a problem; that the calculator can be a foe or a friend; that structure and symmetry are often the key; and that in the beginning is the idea. The comments after the solutions are made in order to provide some historical background, point forward to related mathematics, or suggest related problems. Some comments are made to express an opinion about how certain material could be presented. There is, for example, frequent emphasis on diagrams. A few of the problems presented here look forward to more advanced material, and could serve as the beginning of longer explorations. Material is included that can be helpful in preparation for mathematics contests, in particular the *Fermat* and the *Euclid*, but it cannot replace systematic practice with previous contests.

The Intended Audience

The problems in this collection are meant for senior high school students, though in many cases the mathematical concepts required for their solution begin to be explored in the junior years. There are entire chapters (*Word Problems*, *Counting*, and *Number Theory*) in which every problem could be successfully tackled by a determined grade 9 student. The *Probability* chapter might require knowledge of grade 10 or grade 11 material, but has no problems that require grade 12 material. Only the *Trigonometric Functions* and *Maxima and Minima* chapters can be regarded as accessible primarily to grade 12 students. Knowledge of calculus is not needed. Almost all of the references to calculus are instructions not to use it.

The solutions are in the main directed to instructors who want to present some of the problems to regular classes or to special problem sessions. The solutions could be described as script outlines meant to identify certain themes that are worth emphasizing (such as attention to symmetries, the physical context, elegance, and the fact that we are dealing with something real, not manipulating symbols), to describe possible modes of approach, and to serve as suggestions that invite improvement. Although in most cases a good deal of detail is supplied—only some routine calculations are omitted—the solutions of some problems are written in a style too telegraphic to be read with comfort by all students.

Computation vs. Proof

Most of the problems are computational, in the sense that they ask for a number or numbers. There are only a couple of dozen instances where a proof is *explicitly* asked for. This is partly because the word "proof" can be intimidating. And often the distinction between what is obvious and what needs to be justified is subtle and context-dependent.

But whether a problem asks for a number or a proof, the mental processes required are similar. Computations and proofs are in fact indistinguishable in a logical sense: In the end something must be demonstrated. The explanations that accompany the calculation of an area, or of a probability, can be thought of as a proof that the number found is correct. And all calculations ought to be presented with attention to the logical flow of the argument, in complete sentences.

Most of the problems are meant to be solved using traditional tools—the trained imagination, assisted by paper and pencil. But a simple scientific calculator will be useful, and occasionally a graphing calculator can provide insight or even a satisfactory answer. Calculator-related issues come up fairly often in the solutions and comments, particularly in the chapter *Problems in Computation.*

The Level of Difficulty

Many of the problems in this collection are not easy. If a question were easy, how could it be a problem? The problems are intended to be more challenging than the usual questions asked on Provincial Exams, where students are asked to answer too many too easy questions in too little time. But these problems should not be treated like questions on multiple choice exams, where what is not completely right is wrong. A good diagram, and/or numerical experimentation, often represent considerable progress, for they are a sign that the question is thought of as dealing with something *real*, and that a genuine assault has begun: Mathematics is a contact sport.

Organization

The problems are divided by topic into ten chapters, with an additional chapter of problems with short solutions. Although a division into topics cannot be exact, it was done so that the user might search for a particular kind of question without having to scan five hundred of them. Within chapters, problems do not necessarily get harder, although there is *some* clustering of the harder ones toward the end of each chapter. Some of the early problems in a chapter are solved in a more leisurely style and are more heavily commented than later problems.

What Remains to be Done

The material is still in a fairly raw state. The comments could be substantially expanded, both in number and in length. But more importantly, there are fewer than four hundred problems, and for almost all of them only one or two solutions. Several solutions for more of the problems should be written if the reader is to see several ideas at work more often. We believe that time will bring these new solutions along. In the classroom, students and others working on these problems will come up with arguments that are nicer, or more natural to them, than the arguments presented here. In this process, the parts that are obscure, difficult, or even wrong will become apparent.

Pictures have been supplied only for about half of the solutions that need them. In the remaining cases, the required diagrams are verbally described in the solution, but that is not adequate because diagrams are an indispensable tool for visualization. The picture, the soul of the idea, cannot effectively be replaced by strings of words.

Interesting solutions and suggested improvements are very welcome. These will be acknowledged in the next edition. Comments can be directed to:
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August 22, 1999

Chapter 1

Word Problems

Introduction

These problems are of the kind sometimes called "word problems," that is, problems in which a situation is described in words, using a minimum of mathematical terms, and the solver needs to decide how to bring mathematics to bear. Word problems can produce more apprehension than even quite complicated symbolic manipulations.

The first step toward a solution is to produce a *mathematical model* of the situation described by the words. That's not always easy, and often simplifying assumptions need to be made: a mathematical model and the physical reality are different things. In some of the solutions attention is drawn to possible deficiencies of the model used.

The technical background needed to solve these problems is modest, just basic algebra, and often not even that. But before a word problem is solved, it has to be *read,* and these word problems have sometimes been made a little wordier than usual. The equations that arise are in the main linear, with an occasional quadratic. Once a method of attack has been found, filling in the details is not difficult—but that does not mean that the problems are all easy.

This chapter, unlike the others, has many historically significant problems and puzzles. In a number of cases, two sorts of solutions are given: modern "algebraic" ones, and more old-fashioned "rhetorical" solutions that use a minimum of machinery.

Problems and Solutions

I-1. An exam was given to 25 students, and each student got a mark between 0 and 100. Let μ be the mean of the marks, and let m be the median. It turned out that $|\mu - m|$ was as large as possible. What can be said about μ?

Solution. Let the marks be $x_1, x_2, \ldots, x_{25}$, where $x_1 \le x_2 \le \cdots \le x_{25}$. Then

$$\mu = \frac{x_1 + x_2 + \cdots + x_{25}}{25} \qquad \text{and} \qquad m = x_{13}.$$

Subtract, and temporarily drop the denominator 25. We study y, where

$$y = x_1 + x_2 + \cdots + x_{12} - 24x_{13} + x_{14} + \cdots + x_{25}.$$

For any fixed value of x_{13}, y is maximized by making $x_1 = x_2 = \cdots = x_{13}$, and $x_{14} = x_{15} = \cdots = x_{25} = 100$. Thus given x_{13}, the largest value of y is $1200 - 12x_{13}$. This is maximized by taking $x_{13} = 0$. That gives $|\mu - m| = 48$.

In the same way, we can show that the smallest value of y is obtained by taking $x_1 = x_2 = \cdots = x_{12} = 0$ and $x_{13} = x_{14} = \cdots = x_{25} = 100$, so the minimum value of $\mu - m$ is -48.

Since $|\mu - m|$ was as large as possible, we conclude that either there were thirteen 0's and twelve 100's, for a mean of 48, or twelve 0's and thirteen 100's, for a mean of 52.

Comment. If the marks $x_1, x_2, \ldots x_{25}$ make $\mu - m$ equal to y, then by symmetry the marks $100 - x_{25}$, $100 - x_{24}$, $\ldots$, $100 - x_1$ make $\mu - m$ equal to $-y$. So the second computation was unnecessary: given that $|\mu - m|$ is as large as possible, there are always two possible means, and they add up to 100.

I-2. After competing in the school's Sports Day, Ms. Z and her class—there were between twenty and thirty of them—joined as a group the lineup for muffins and drinks. After a while more classes got in line behind them. At a certain point one-fifth of the line was in front of Ms. Z and her class, and twelve-seventeenths of the line was behind them. How many students are there in Ms. Z's class?

Solution. Altogether, the people in front of Ms. Z and her class plus the ones behind them make up a fraction $1/5 + 12/17$ of the whole line. Bring the fractions to the common denominator 85 and add: the result is $77/85$, so Ms. Z and her class make up the fraction $8/85$ of the whole line. There are between 20 and 30 of them altogether, so there must be $3 \cdot 8$, and therefore there are 23 students in the class.

I-3. Alphonse and Beth did a lap on the school 400 meter track, starting at the same time and place but running in opposite directions. Beth finished

27.2 seconds after they passed each other, and Alphonse finished 42.5 after they passed. Each ran at constant speed. How long did Beth take to run the lap?

Solution. Let t be Beth's lap time. Then Alphonse's time is $t+(42.5-27.2)$. Their running speeds are therefore

$$\frac{400}{t} \qquad \text{and} \qquad \frac{400}{t+15.3}.$$

The distances they ran after they passed are

$$27.2\frac{400}{t} \qquad \text{and} \qquad 42.5\frac{400}{t+15.3}.$$

These distances add up to 400. After simplifying, we obtain the equation $t^2 - 54.4t - (15.3)(27.2) = 0$. By the quadratic formula, $t = 61.2$.

Another way: More generally, suppose that Alphonse finishes a seconds after they pass and Beth finishes b seconds after they pass. Let p be Alphonse's speed and let q be Beth's, and let t be the time they run *before* they meet. So when they meet Alphonse has run a distance pt, and Beth has run a distance qt. To finish, Alphonse needs to run a distance qt, which takes him time qt/p. Similarly, it takes Beth time pt/q to finish after they pass. We were told that

$$\frac{qt}{p} = a \qquad \text{and} \qquad \frac{pt}{q} = b.$$

Multiply. We get $t^2 = ab$, so $t = \sqrt{ab}$. Thus Beth's running time is $\sqrt{ab}+b$. Finally, put $a = 42.5$ and $b = 27.2$. Beth ran the lap in 61.2 seconds.

Comment. Nice run! The first solution is efficient, but we prefer the second for its greater symmetry. Working with the "general" a and b is often more informative (and simpler!) than working with particular numbers.

I-4. When train A goes at full speed, it takes 8 seconds to pass a certain point on the track. Train B takes 12 seconds to pass the same point. The trains are going at full speed in opposite directions on parallel tracks. What can be said about the time it takes them to pass each other?

Solution. Let a be the speed of train A, in kilometers per second, and let b be the speed of B. So the length of A is $8a$, the length of B is $12b$, and their relative speed is $a+b$.

If t is the time it takes them to pass each other, then

$$t = \frac{8a+12b}{a+b} = 8 + \frac{4b}{a+b}.$$

Let $a = xb$. Then $4b/(a+b) = 4/(x+1)$. . As x increases from 0, the function $4/(x+1)$ travels from 4 toward 0. We conclude that $8 < t < 12$.

Comment. We can't say more without some knowledge about physical properties of trains. The ratio of the maximum speeds of real trains travels not over the interval $(0, \infty)$, but over a much smaller interval, maybe something like $[1/8, 8]$. There are also constraints on the lengths of real trains.

I-5. A man carrying rice passes through three local customs posts. At the first, he has to give up one-third of his rice. At the second, he gives up one-fifth of what remains, and at the third, one-seventh of what remains. He ends up with 5 measures of rice. How much did he start with?

Solution. We can work backwards. The 5 measures are six-sevenths of what he had before going through the third post. So he arrived at the third post with $5(7/6)$ measures. The same argument shows that he arrived at the second post with $5(7/6)(5/4)$ measures, and at the first with $5(7/6)(5/4)(3/2)$, that is, $10\frac{15}{16}$ measures.

Another way: If he had started with 105 measures of rice, then he would have 70 after going through the first post, 56 after the second, and 48 after the third. But in fact he only has 5, so he must have started with $105(5/48)$.

Another way: Let x be the amount of rice he starts with. After going through the three posts, he is left with $x(2/3)(4/5)(6/7)$. Set this equal to 5 and solve for x.

Comment. This problem, adapted from the *Jiuzhang suanshu,* testifies eloquently to the rapacity of officials. There are dozens of variants from India, the Greek world, and Medieval and Renaissance Europe.

I-6. One morning, Alan set out to walk the 2 kilometers to school. Three minutes after he left, his mother Beth realized that he had forgotten his math homework and set out after him. She caught up to Alan and immediately headed back home. Beth got home at exactly the same time as Alan got to school. Beth walked at a brisk 8 kilometers per hour. How fast did Alan walk?

Solution. Let d be the distance from home to where Beth caught up to Alan, and let s be Alan's walking speed. It took Beth $d/8$ hours to walk back home. In that time Alan walked the remaining $(2 - d)$ kilometers to school, so

$$\frac{d}{8} = \frac{2-d}{s},$$

which simplifies to $ds = 16 - 8d$, or equivalently $d = 16/(s + 8)$.

Beth took 3 minutes less than Alan to cover distance d, so

$$\frac{d}{s} = \frac{d}{8} + \frac{1}{20}.$$

Substitute $16/(s+8)$ for d in the above equation. Things look messy at first, but by multiplying through by $20s(s+8)$ we arrive at $s^2+48s-320=0$. The quadratic formula yields $s=-24+\sqrt{896}$, so Alan walked at about 5.9 kilometers per hour.

Another way: The above two variable approach seems natural. Nothing very simple can work, for a quadratic equation is unavoidable. For variety we describe an equivalent approach that uses geometric language.

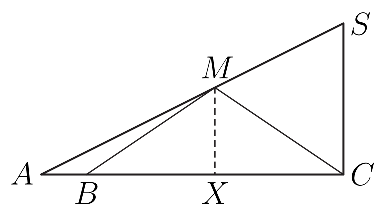

Figure 1.1: Going to School

In Figure 1.1, the horizontal axis represents time and the vertical axis represents space. The point A is Alan's starting point in space-time, B is Beth's, M is the event of their meeting, S is Alan's arrival at school, and C is Beth's arrival back home. So AS is Alan's path in space-time, and BMC is Beth's.

Note that $AB = 0.05$ (three minutes) and $SC = 2$. Let $t = BX$. So t is the time it took for Beth to catch Alan. Since Beth walked at 8 km per hour, $MX = 8t$. Triangles AMX and ASC are similar, and therefore

$$\frac{8t}{0.05+t} = \frac{2}{0.05+2t}.$$

Simplify. We obtain the quadratic equation $160t^2 - 16t - 1 = 0$, which has the solution $t = (16+\sqrt{896})/320$. Thus AC, the time Alan took to get to school, is about 0.337, so his speed is about 5.9 km per hour.

I-7. There are some rabbits and pheasants in a cage. In total there are 35 heads and 94 feet. How many of each animal are there?

Solution. Let r be the number of rabbits, and p the number of pheasants. From the two given totals we obtain the equations

$$r+p=35 \qquad \text{and} \qquad 4r+2p=94.$$

There are several ways to solve this system. For example, draw the lines $r+p=35$ and $4r+2p=94$, find out where they appear to meet, and check that what seems to be the solution really is.

Or else note that the first equation is equivalent to $2r+2p=70$, and therefore

$$(4r+2p)-(2r+2p)=94-70,$$

from which we conclude that $r = 12$ and therefore $p = 23$.

Another way: There is no need of symbols. Guess that there are 35 pheasants and no rabbits. The number of heads is then right, but the number of feet is 70, wrong by a lot. Start trading in pheasants for rabbits. For each pheasant traded in, we gain 2 feet. In order to gain 24 feet, we need 12 rabbits.

Another way: Ask the rabbits to stand on their hind feet and dance. Each animal now has two feet on the ground, for a total of 70. There are $(94 - 70)$ feet in the air, 2 per rabbit, so there are 12 rabbits.

Comment. Versions of this problem have appeared in problem collections for more than two millenia. This one is taken from the *Jiuzhang suanshu* (Nine Chapters on the Mathematical Art), which was probably compiled around -200 from older sources.

I-8. A person remarked that upon his wedding day the proportion of his own age to that of his bride was as 3 to 1; but fifteen years afterwards the proportion of their ages was 2 to 1. What were their ages upon the day of their marriage?

Solution. Let the bride's age on marriage be x; the groom's was therefore $3x$. Fifteen years later their ages are $x + 15$ and $3x + 15$. We are told that $2(x + 15) = 3x + 15$. Thus $x = 15$, and their ages on marriage were respectively 15 and 45.

Comment. This problem comes from *The Ladies' Diary,* which published puzzles and problems for an English upper income readership from 1704 to 1841. A modern version of the problem would presumably use different numbers.

I-9. The water supply of a desert outpost is stored in a cistern from which water evaporates at a constant rate. There is enough water in the cistern to supply 60 people for 30 days, or 30 people for 50 days. How long will the water supply hold out if there are 10 people at the outpost?

Solution. Let a be the amount of water lost to evaporation each day, and let b be the amount a person uses per day. It is convenient to choose our unit of volume in such a way that $b = 1$. So the sun drinks as much water each day as a people.

There is enough water to take care of 60 people (and the sun) for 30 days. So there are $30(a + 60)$ units of water in the cistern. But the water would take care of 30 people for 50 days, so

$$30(a + 60) = 50(a + 30).$$

Thus $a = 15$ and there are 2250 units of water in the cistern. That will take care of 10 people for $2250/(15 + 10)$ days.

Comment. Newton's *Arithmetica Universalis* (1707) has a mathematically equivalent problem. If 12 oxen eat up $3\frac{1}{3}$ acres of grass in 4 weeks, and 21 oxen eat up 10 acres in 9 weeks, how many oxen will eat up 24 acres in 18 weeks, the grass being at first equal on every acre, and growing uniformly.

I-10. Fresh mushrooms are 90% water. Dried mushrooms are only 15% water. (a) How many kilograms of fresh mushrooms do we need to make 10 kg of dried mushrooms? (b) How many kg of dried mushrooms do we get from 12 of fresh?

Solution. (a) The 10 kg of dry are 85% "mushroom powder" and 15% water, so there are 8.5 kg of powder. That represents 10% of the fresh weight, which is therefore 85 kg. (b) Note from part (a) that 10 kg of dry come from 85 of fresh, so 12 of fresh yield 12(10/85) of dry.

I-11. (Leonardo of Pisa, 1202) One man had three loaves of bread, the other had two. When they sat down to eat, a soldier joined them and shared their meal. Each person ate the same amount. When all the bread was eaten, the soldier left 5 bezants to pay for his meal. How should the money be shared between the two men?

Solution. They should go to an inn and use the 5 bezants to share a jug of wine. We get a different answer if we take a crude monetary point of view. The soldier ate 5/3 loaves, which he valued at 5 bezants. Thus 1/3 of a loaf is worth 1 bezant.

The first man started out with 3 loaves and ate 5/3 loaves, so he contributed $(3 - 5/3)$ loaves, that is, 4/3 loaves, worth 4 bezants, to the soldier's meal. The second man's contribution is only worth 1 bezant.

I-12. A certain lion would eat a sheep in 4 hours; the leopard would eat a sheep in 5 hours; and it takes the bear 6 hours. It is asked, if one sheep had been thrown among them, how many hours would they have taken to devour it.

Solution. The eating rates of lion, leopard, and bear are respectively 1/4, 1/5, and 1/6 sheep per hour. Thus their combined eating rate is $1/4 + 1/5 + 1/6$. Let x be the number of hours it takes them companionably to eat one sheep. Then their combined rate is $1/x$, and therefore

$$\frac{1}{x} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6}.$$

We can use a calculator to add the fractions, and then invert. Or else bring the right-hand side to the common denominator 60. We conclude that $1/x = 37/60$ and therefore $x = 60/37$.

Another way: Suppose that many sheep are available, and feasting goes on at constant rate for 60 hours. The lion will have disposed of 15 sheep, the leopard of

12, and the bear of 10, for a total of 37 sheep in 60 hours. Thus when they have dinner together it takes them 60/37 hours to eat one sheep.

Comment. This problem is taken from Leonardo of Pisa's *Liber abbaci* (1202). Mathematically equivalent problems have a long history, and can still be found in school books. The first surviving variant comes from Heron of Alexandria (100?). A cistern can be filled by a large pipe in two hours, by a medium sized pipe in three hours, and by a small pipe in six. How long does it take to fill the cistern using all three pipes?

The "rates" machinery used in the first solution is quite abstract. It isn't obvious that one should add rates and not hours.

I-13. Yorick went on a crash diet. His weight dropped from 140 pounds to 120 pounds in 60 days. Every day he lost an amount proportional to his weight at the start of that day. How much weight had Yorick lost after 30 days of dieting?

Solution. If Yorick's weighs w pounds at the start of a certain day, then he loses kw that day, where k is a constant, so he weighs $(1-k)w$ pounds at the start of the next day. It follows that he weighs $(1-k)^2w$ pounds at the start of the day after that, and so on. After 60 days he weighs $(1-k)^{60}w$ pounds. We are told that

$$120 = 140(1-k)^{60}.$$

Thus $(1-k)^{60} = 120/140$. Yorick's weight after 30 days is $140(1-k)^{30}$, that is, $140(120/140)^{30/60}$, or more simply $\sqrt{120 \cdot 140}$. It's time to use the calculator. In the first 30 days he lost about 10.4 pounds.

Comment. Someone might guess that halfway into his diet Yorick would have lost 10 pounds. A little thought shows that can't be right, since he loses weight at a faster rate when he is heavier.

What's interesting is how nearly right it is. Yorick's weight after half the time is not the *arithmetic mean* $(140+120)/2$ of the two weights, but their *geometric mean* $\sqrt{140 \cdot 120}$. If two numbers have ratio not far from 1, then their arithmetic mean and geometric mean are nearly equal.

We can get a dramatic illustration of this fact by using a graphing calculator. Let a be positive; for example take $a = 10$. Let u and v be positive numbers that add up to a. Their arithmetic mean is $a/2$, and their geometric mean is $\sqrt{u(a-u)}$. Ask the graphing calculator to graph the half-circle $y = \sqrt{x(a-x)}$. The display shows that y is nearly constant as x ranges over numbers not too far from $a/2$—the circle is nearly flat on top, just like the surface of a calm sea is locally flat.

I-14. A hare starts with a lead of 100 paces. A dog then starts chasing it. When the dog has run 250 paces, it is still short of the hare by 30 paces. How many more paces must the dog run to catch the hare?

Solution. Assume that each animal runs at constant speed (this is undoubtedly contrary to fact). The dog gained 70 paces while running 250. To gain another 30 it must therefore run an additional 250(30/70) paces. The problem is solved, but maybe we should calculate: the answer is about 107 paces. One reason to calculate is that we can quickly decide whether the answer makes physical sense, and thus get a partial check on the correctness of the solution.

We get essentially the same solution by letting a be the additional amount that the dog must run, and noting that $a/30 = 250/70$.

Another way: Symbolic algebra can play a larger role—we can invent symbols for various unknowns, find equations that link the unknowns, and solve. For example, let h be the speed of the hare, and d the speed of the dog. While the dog runs 250 paces, the hare runs $250 - (100 - 30)$, and therefore $d/h = 250/180$.

Let x be the *total* distance the dog runs until it catches the hare. That is 100 more than the distance that the hare runs, and therefore $x/(x - 100) = d/h = 250/180$. Simplification yields the linear equation $180x = 250x - (100)(250)$. Finally, solve for x and subtract 250 to find the answer.

Comment. Problem I-14 is adapted from one of the beginning problems in the *Jiuzhang suanshu* (Nine Chapters on the Mathematical Art) which according to Liu Hui (250?) is based on ancient learning that was set down again around −200. *Nine Chapters,* like most of the Chinese mathematical documents that have survived, was designed as a resource for those in the business of training students.

Nothing *very* ancient has survived intact. This is sometimes attributed to the fact that in −213 the Emperor Shih Huang-ti ordered all books burned and all scholars buried. In any case, the humid climate of much of settled China was not kind to paper documents. By way of contrast, the desert conditions and burial practices of Egypt preserved many papyrus scrolls. And the Babylonians wrote on clay that they then baked. The resulting tablets can last virtually forever.

I-15. An internet service provider allows its customers a certain number of "free" hours per month and charges for each hour beyond that. A, B, and C have separate accounts with this company.

This month, the *sum* of the hours that A and B were logged on was 105. Each ended up paying a surcharge, and between them they had to pay a total surcharge of \$10. C all by herself used 105 hours, and had to pay a \$26 surcharge. What is the charge for each extra hour?

Solution. Let f be the number of free hours, and s the surcharge per extra hour. Let a and b respectively be the number of hours that A and B were logged on, and u and v the surcharges they paid.

The information we were given is summarized by the equations

$$a + b = 105; \quad u + v = 10; \quad s(a - f) = u; \quad s(b - f) = v; \quad s(105 - f) = 26.$$

"Add together" the third and fourth equations. (Strictly speaking, we can't add equations, for an equation is an assertion that two things are equal.) Then use the first two equations to simplify. We obtain $s(105 - 2f) = 10$. This, together with the fifth equation, gives $sf = 16$. If we substitute in the fifth equation, we find that $105s = 42$, and therefore the surcharge is 40 cents per hour.

Another way: The preceding solution used five variables. Here is a solution at the other extreme—we find the answer using no algebra at all. Between them, A and B paid \$16 less than C in surcharges. Since both A and B used up all of their free hours, the \$16 must be the value of one account's free hours. Thus if free hours were suddenly charged for at the same rate as extra hours, then C would have to pay a total of \$42 for her 105 hours. It follows that extra time costs \$0.40 per hour.

Comment. The first approach was quite mechanical—which of course doesn't mean bad, indeed there is much to be said in favour of mechanical. We invented variables for everything in sight, found equations that link the variables, and then solved. The solving had to be tackled in a specific order. Not enough information is provided to determine how much A had to pay in surcharges. So a and b can't be separated, and neither can u and v. Thus adding the third and fourth equations was essentially a forced move.

The idea for this problem was borrowed from a standard problem about excess luggage on an airplane, which of course was borrowed from an earlier source. Charles Smith's *A Treatise on Algebra,* published in 1888, has an excess luggage problem.

I-16. A floor refinisher can strip the old wax from 35 square meters of floor per shift. Once the wax is stripped, she can apply new wax to 65 square meters of floor per shift. How many square meters should the refinisher strip so as to work the entire shift and leave no stripped floor unwaxed?

Solution. It takes $x/35$ of a shift to strip x square meters of floor. To put new wax on these x square meters uses up an additional $x/65$ of a shift, and therefore the total time needed is $x/35 + x/65$ shifts. If precisely one shift is to be used, then $x/35 + x/65 = 1$. Solve for x in the usual way: $x = (35 \cdot 65)/(65 + 35) = 22.75$.

Another way: We can find the answer without any x. It takes $(1/35 + 1/65)$ of a shift to strip and wax 1 square meter, so the number of square meters that can be stripped and waxed in one shift is $1/(1/35 + 1/65)$.

Another way: Two hundred years ago, the problem would probably have been solved by using *regula falsi,* the "Rule of False Position." The idea is that you guess an answer, test it, then use the result of the test to get, in one step, the *right* answer. The Rule of False Position is completely different from the "guess and test" procedures sometimes encouraged in the schools. School "guess and test" is based on the fact that answers are engineered to be small integers.

Guess that the refinisher should strip $35 \cdot 65$ square meters. Let's see whether we got lucky. The $35 \cdot 65$ square meters will take 65 shifts to strip, and 35 to wax, for a total of 100 shifts. Not a very good guess! But all is not lost: since stripping and waxing $35 \cdot 65$ square meters requires 100 shifts, she can strip and wax $(35 \cdot 65)/100$ square meters in 1 shift.

The fact that numbers worked out nicely is built into the method: like users of *regula falsi* were taught to do, we chose $35 \cdot 65$ because dividing this by 35 or 65 is easy. Another good guess is $5 \cdot 7 \cdot 13$, the least common multiple of 35 and 65.

Comment. We tacitly assumed that the amount of work done in time t, for example the area stripped, is a constant times t. This assumption is almost certainly false.

Regula falsi methods—there are several—were used in ancient Egypt and India, and were taught in Europe for hundreds of years, finally disappearing in the twentieth century.

Why was *regula falsi* so popular? Because it helped with two common fears—fear of fractions and fear of abstraction. The use of symbols for unknowns only dates back, in Western Europe, to the seventeenth century, and took a long time to diffuse beyond the scientific community. So the now standard algebraic methods were unavailable. And maybe the user of *regula falsi* retains continuous concrete grasp of the problem more than the x-manipulator.

I-17. A punctual mathematician cycles to her office every day along the path beside the railway track, at 12 km/h. When the morning train is on schedule, it reaches a certain crossing at the same time as she does. One day the train set out 10 minutes behind schedule, but ran at its usual speed. The train caught up to the mathematician 2.4 km past the crossing. Find the speed of the train.

Solution. Let v be the speed of the train. Imagine that the mathematician lives along the track, at distance d from the crossing. So ordinarily when the mathematician starts, the train is at distance $dv/12$ from the crossing. Since the train is 10 minutes late, this time it is $dv/12 + v/6$ from the crossing.

The mathematician travels the distance $(d + 2.4)$ in time $(d + 2.4)/12$. This is the same as the time it takes the train to travel $dv/12 + v/6 + 2.4$. Thus

$$\frac{d + 2.4}{12} = \frac{dv/12 + v/6 + 2.4}{v}.$$

Simplify and solve for v. The terms in d cancel, as they must, since the value of d is irrelevant—we could have assumed from the beginning that her house is *at* the crossing. It turns out that $v = 72$.

Another way: The train reached the crossing 10 minutes after the mathematician did. When the train reached the crossing, the mathematician was $12(10/60)$ km beyond the crossing, so she was 2 km ahead of the train.

While the train travelled 2.4 km, the mathematician travelled $(2.4 - 2)$ km. So the speed of the train is $(2.4/0.4)(12)$, that is, 72 kilometers per hour.

Another way: Draw a space-time diagram. In Figure 1.2, the x-coordinate of a point represents time, and the y-cooordinate is distance from the crossing. Let ℓ be

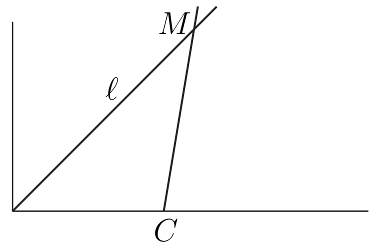

Figure 1.2: The Mathematician and the Train

the line with slope 12 that passes through the origin. (For convenience, the scales on the two axes are quite different.) Then ℓ represents the mathematician's path in space-time. Let M be the point on ℓ that has y-coordinate 2.4, and let C be the point on the x-axis with x-coordinate 1/6 (10 minutes). Then the line CM represents the train's path in space-time, and the slope of CM is the train's speed. This can be measured from the graph, or computed by noting that the x-coordinate of M is 0.2.

I-18. A rich merchant died, and left gold coins to his children as follows. The eldest got 100 coins, plus one-fifteenth of what remained of the coins after that. The next got 200 coins, plus one-fifteenth of what remained after that. And the next got 300 coins, and one-fifteenth of what remained, and so on. As it turned out, all the gold coins were distributed, and everyone got exactly the same amount. How many children were there, and how much did each get?

Solution. Let n be the number of children. For simplicity, we work with bags of 100 coins. *If* it is indeed possible for each child to get the same amount, with nothing left over—that will have to be checked—then each child got n bags. This follows from the fact that the last child got n bags plus one-fifteenth of what remained, but nothing remained. The total fortune is therefore n^2.

The first child got $1+(n^2-1)/15$ bags, and they all got n, so $(n^2-1)/15 = n-1$. The wording of the problem doesn't allow $n = 1$, so we conclude that $n = 14$ and everyone got 1400 coins. We cannot yet conclude that the answer is 1400, for this number was obtained by assuming that the division described in the problem is *possible.* Maybe it isn't; problem posers make mistakes.

We check that in general a fortune of n^2 units can be distributed equally among n children, with the k-th child receiving k units plus the $(n + 1)$-th part of what remains.

The first child receives $1 + (n^2 - 1)/(n + 1)$, and this is equal to n. Suppose now that children 1, 2, …, $k - 1$ have each received n. First give k to the k-th child. There remains $n^2 - (k - 1)n - k$, that is, $(n+1)(n-k)$. The $(n+1)$-th part of that is $n-k$, and $k+(n-k)=n$, so the k-th child also receives n units. We conclude that all children get n units.

Comment. Versions of this inheritance problem can be found in Fibonacci's *Liber abbaci* (1202), Chuquet's *Triparty en la Science des Nombres* (1484), and Euler's *Algebra* (1770). In Euler's version, it turns out that there are nine children. Euler himself had thirteen children.

Fibonacci probably borrowed this problem, like most of the others in *Liber abbaci,* from an Islamic source. Bequest problems take up almost half of al-Khwārizmī's *al-jabr w'al muqābala*, the book whose title gave us the word *algebra.* The complexity of the Koranic laws of inheritance inspired Islamic writers to make up elaborate word problems.

I-19. A bowl held 10 liters of non-alcoholic punch. A chemistry major took out a pitcherful, replaced it with a pitcherful of pure alcohol, and mixed thoroughly. She took a tiny sip, then removed another pitcherful and replaced it with a pitcherful of alcohol. The final mixture was 15% alcohol by volume. How big is the pitcher?

Solution. Let P be the capacity of the pitcher in liters. After the first pitcherful of alcohol is put in, the bowl contains P liters of alcohol, so the amount of alcohol per liter of spiked punch is $P/10$.

When the second pitcherful is taken out, the amount of alcohol removed is therefore $P(P/10)$, so at the end there are $2P-P^2/10$ liters of alcohol in the bowl. But this is 15% of 10 liters, and therefore $2P-P^2/10=1.5$.

We multiply through by 10 and obtain $P^2-20P+15=0$. Why multiply? Probably in the hope that the resulting quadratic will factor over the integers. It doesn't—most quadratics don't—but the quadratic formula works just fine. One root is greater than 10, so irrelevant. Thus $P=10-\sqrt{85}$, about 0.78 liters.

Comment. We took it for granted that volumes add like numbers do. But when different liquids are mixed, the volume of the mixture is ordinarily *not* equal to the sum of the volumes—that happens in particular with alcohol and water. Taking this effect into account doesn't change the answer very much.

I-20. Alma walked in a charity walkathon. If she had made 20 cents less per kilometer, she would have had to walk 5 more kilometers to raise as much money as she did. If she had made 10 cents more per kilometer, she would have had to walk 2 fewer kilometers to raise as much as she did. How much money did she raise?

Solution. Let a be the amount, in cents, that Alma made per kilometer, and d the distance she walked. Thus she raised ad cents. If she had walked 5 more kilometers at rate of pay 20 cents less, she would have raised $(a-20)(d+5)$. We conclude that $ad = (a-20)(d+5)$. Similarly, $ad = (a+10)(d-2)$. These equations simplify to

$$5a - 20d = 100 \qquad \text{and} \qquad -2a + 10d = 20.$$

Solve for a and d by graphing the lines with the above equations and seeing where they meet, or by algebraic methods. It turns out that $a = 140$ and $d = 30$, and therefore she raised \$42.00.

Another way: In the above solution, the ad terms cancelled and we got linear equations. We should be able to argue our way to these directly.

Imagine that Alma has finished walking distance d, and is told that a sponsor withdrew, and her rate of pay has gone down by 20 cents. So she has lost $20d$. She is determined to make that up and walks another 5 km, earning $5(p - 20)$. That yields the equation $20d = 5(a-20)$. Similarly (new last minute sponsor), we obtain the second linear equation, and solve as before.

Or make a rectangle, label the base "kilometers," the height "cents." The amount of money raised is just the area of the rectangle. In another colour, increase the base by 5, shrink the height by 20. The area doesn't change, so what is lost "on top" is exactly balanced by what is gained "on the right," and therefore $20d = 5(a - 20)$.

I-21. Every weekday, A, B, and C cycle to the university along the same road, each at unvarying speed. Yesterday, B passed C, and A went by C one minute later. A caught up to B six minutes after passing C. This morning, A passed B, and five minutes later passed C. When did B catch up with C?

Solution. Only the *relative speeds* of the riders matter. Since C is the slowest, we use a frame of reference that travels with C, or equivalently imagine that C is standing still, that C is a street Corner.

Yesterday, when A caught B they were both a certain distance past the Corner, and A had covered this distance in 6 minutes while B took 7. So today it took B 5(7/6) minutes to do what A did in 5. We conclude that B passed C five minutes and 50 seconds after B was passed by A.

Another way: We can be more explicitly algebraic. Let a, b, c be the speeds, in units per minute, of A, B, and C. Let P be the point where B passed C yesterday.

When A went by C, the latter had travelled a distance c from P, while B had travelled a distance b from P. At that time A was $b - c$ behind B. It took 6 more minutes for A to catch B, so $b - c = 6(a - b)$. But $a - b = (a - c) - (b - c)$, and therefore $b - c = 6(a - c) - 6(b - c)$. It follows that $7(b - c) = 6(a - c)$. The first solution obtained this relationship more easily by using a frame of reference that travels with C.

We next analyze what happens today in terms of a, b, and c. Let Q be the place where A passes B. When this happens, C is $5(a-c)$ from Q. Let t be the time it takes for B to pass C. Then $t(b-c) = 5(a-c)$, and therefore $t = 5(a-c)/(b-c) = 5(7/6)$.

Comment. Yesterday and today can be made to look more similar by playing today's videotape backwards.

For a character analysis of the people in this problem, see Stephen Leacock's essay "A, B, and C," in *Literary Lapses.* Leacock points out that in algebra books, A is always the fastest at walking, ditch digging, cutting wood, and poor C is always the slowest.

I-22. A 600 pound pumpkin has been entered in a contest. When it arrived, it was 99% water. After sitting in the warm sun for several days, it lost some weight (water only), and is now 98.5% water. How much does it weigh now?

Solution. The pumpkin had 6 pounds of "solid" (non-water) matter. These 6 pounds make up 1.5% of the new weight, so the new weight is 6/0.015, that is, 400 pounds. We obtain essentially the same solution by letting the new weight be w and arguing that $w - 0.985w = 600 - (0.99)(600)$.

Comments. 1. Surprisingly many people get the wrong answer. It is tempting to pick up the calculator and start to multiply, or divide, or whatever. In this problem, as in many others, the right approach involves looking at what doesn't change—in mathematical jargon, what remains *invariant.* Here the amount of "dry" pumpkin matter remains invariant.

2. The human body is also largely water, and one can achieve remarkable, albeit temporary, weight loss by losing water. Boxers and wrestlers have long used dehydration to go down in weight class, with sometimes fatal consequences.

One way to get temporarily dehydrated is to eat little for a few days, and/or to eat mainly protein. But the body soon adjusts, and dieters, after losing seven pounds in the first week of the miracle diet, find that in the second week their weight doesn't go down further, and may even go up.

I-23. American driving guidelines recommend that the distance between your front bumper and the car in front of you should be at least one car length for every 10 miles per hour of speed. Assume that drivers follow this recommendation. If the speed limit on a freeway is reduced from 65 miles per hour to 55 miles per hour, by what percentage is the carrying capacity reduced?

Solution. Assume that all cars go at exactly the speed limit. That's false, but even simple models can yield useful knowledge, serving as points of departure toward more sophisticated models. Assume also that all cars have the same length—this is in fact not needed.

Let the (average) length of a car be L miles. At 65 miles per hour, a car occupies its own length L, together with a distance of at least $65L/10$ to the rear bumper of the car in front. So if the freeway is being used to capacity, the elapsed time between a car passing a certain point and the next car doing so is $7.5L/65$ hours.

Thus the capacity of any lane is $65/(7.5L)$ cars per hour. Similarly, capacity is $55/(6.5L)$ at 55 miles per hour. Capacity is thus reduced by the factor

$$\frac{65/(7.5L) - 55/(6.5L)}{65/(7.5L)},$$

a bit under 2.4%, not as much as one might guess—after all, speeds have gone down by more than 15%.

Comment. We used a simple *deterministic* model. A good deal of sophisticated mathematics has gone into producing and analyzing *probabilistic* models. Related questions come up when we study the carrying capacity of telephone networks, internet service providers, and even supermarket checkout lines. Studies indicate that, because the accident rate tends to go up as speeds increase, and an accident can cause massive delays, the *average* carrying capacity of a freeway may well increase if the speed limit is lowered!

I-24. Canadian driver handbooks recommend the "two-second rule," which says that the distance between your front bumper and the next car should be at least equal to the distance travelled in two seconds at current speed. (a) Change the two-second rule to read that the distance between *front bumpers* should be at least two seconds. Assume that drivers always obey this rule, and always drive at the speed limit. How is the carrying capacity of a lane affected if the speed limit is lowered from 100 km per hour to 90 km per hour? (b) How is the carrying capacity affected under the real two-second rule? Assume that cars are 4.5 meters long.

Solution. (a) Under the modified two-second rule, when the lane is used to capacity 2 seconds go by from the time that the front of a car passes a certain point to when the front of the next car does. So the capacity—how many cars pass the point in an hour—is independent of speed until cars form a continuous ribbon of metal.
(b) Let the speed limit be v. At capacity, the space between front bumpers is the length 0.0045 of a car plus the "two-second" gap $2v/3600$. In each lane,

$$\frac{v}{0.0045 + 2v/3600}$$

cars pass a given point per hour. When $v = 100$, this is about 1665. When $v = 90$, it is about 1651. Carrying capacity has decreased by about 0.8%.

I-25. Alfred and Betty did an endurance run, circling a 400 meter oval track 125 times. They started at the same time and place, and each ran at constant speed. Alfred took 2 minutes per lap, and Betty took 1 minute and 50 seconds. How many times after the start did Betty pass Alfred?

Solution. Alfred runs at 1/120 laps per second, while Betty runs at 1/110 laps per second, so their relative speed is $(1/110 - 1/120)$, that is, 1/1320. Thus Betty gains a lap every 1320 seconds. It took Betty $125 \cdot 110$ seconds to finish the run. Since $(125 \cdot 110)/1320$ is about 10.4, Betty passed Alfred 10 times.

Another way: Betty covers 12 laps in the time it takes Alfred to do 11. Thus she passes him every 12 laps. Since 125/12 lies between 10 and 11, she passed him 10 times.

I-26. If Alphonse and Beth work together, they can paint a room in 9 hours. If Beth and Gamal work together, it takes them 10 hours, while Alphonse and Gamal working together take 15 hours. How many hours does it take for Alphonse, Beth, and Gamal working together to paint 5 rooms?

Solution. Let α, β, and γ be the *rates* (in rooms per hour) at which Alphonse, Beth, and Gamal work. So the rate at which Alphonse and Beth work in tandem is $\alpha + \beta$. We obtain similar expressions for the other combined rates. Thus

$$\alpha + \beta = \frac{1}{9}; \qquad \beta + \gamma = \frac{1}{10}; \qquad \alpha + \gamma = \frac{1}{15}.$$

Add up, divide by two. We find that $\alpha + \beta + \gamma = 5/36$, and therefore it takes 36 hours to paint 5 rooms.

I-27. I am twice the age that you were when I was your age. When you get to be my age our ages will total $76\frac{1}{2}$ years. How old am I?

Solution. Let "my" age be x, and "your" age be y. Clearly $x > y$. I was y years old $x - y$ years ago. At that time your age was $y - (x - y)$, and I am now twice that old, so $x = 2(2y - x)$, that is, $3x = 4y$.

In $x - y$ years, you will reach my (current) age x, I will be $x + (x - y)$, and these two numbers add up to 76.5, so $3x - y = 76.5$. Since $3x = 4y$, we conclude that $y = 25.5$ and $x = 34$.

Comment. There are endlessly many age problems in the puzzle literature. Students are sometimes encouraged to solve them by "guess and test." That's a largely pointless activity unless there is a real-world need for the ability to find small integer answers to artificial problems.

The wording in age problems is often intended to confuse. For that reason, the discipline involved in setting up variables and equations is useful even if more informal solutions are possible.

I-28. Andrew bought a new watch at five o'clock one afternoon, and the salesperson set it to exactly the right time. But the watch was defective, losing 3 seconds every hour. He only noticed about a week later, when he arrived—he thought—just in time for a meeting and was told he was 8 minutes late. At what time of the day was the meeting scheduled to start?

Solution. Interpret losing 3 seconds per hour to mean that when 1 real hour elapses, the watch claims that the elapsed time is 1 hour minus 3 seconds (there are other interpretations).

For clarity, we use algebraic notation. Let T be the number of true hours between the time the watch was bought and the time the meeting was called for. At the time the meeting started, Andrew thought that $T(1-3/3600)$ hours had elapsed. So he waited, by his watch, another $3T/3600$ hours. That's $(3T/3600)/(1-3/3600)$ true hours, which turned out to be 8 minutes. Solve for T. We find that T is 6 days, 15 hours, and 52 minutes, so the meeting was called for 8:52 in the morning.

I-29. Sara's investment portfolio is made up of tech shares and gold mining shares. Last year, the value of her tech shares went up by 10%, and the value of her gold mining shares went down from \$10,000 to \$9,000. Overall, the total value of her investments went up by 6%. Find the value of her tech shares at the end of last year.

Solution. Calculation is marginally simpler if we let x be the value of the tech shares at the *beginning* of last year. The total value of the investments at that time was $x + 10000$.

After one year, the tech investment had grown to $x(1.10)$, so the total stock value was $x(1.10) + 9000$. That represents 6% growth, and therefore

$$x(1.10) + 9000 = (x + 10000)(1.06).$$

Simplify and solve: $x = 40000$. At the end of last year, the tech stock was worth $40000(1.10)$, that is, \$44,000.

I-30. Three merchants find a treasure. The first says "If I take this treasure, I will be twice as rich as the two of you together." The second says, "If I take it, I will be three times as rich as the two of you together," and the third says "If I take it, I will become five times as rich as the two of you together." How large is the treasure, and how large the fortune of each merchant?

Solution. Let x, y, and z be the fortunes of the three merchants, and a the size of the treasure. The words of the merchants translate to the equations

$$a + x = 2(y + z); \qquad a + y = 3(x + z); \qquad a + z = 5(x + y).$$

We can solve by systematic elimination or by a trick. Let $t = x + y + z$. Add $2x$ to both sides of the first equation: we get $a + 3x = 2t$. Similarly, $a + 4y = 3t$ and $a + 6z = 5t$. Multiply both sides of the first of these three equations by 4, both sides of the second by 3, and of the third by 2, and add. We get $9a = 15t$. Finally, solve for x, y, and z in terms of a. The result is $x = a/15$, $y = 3a/15$, $z = 5a/15$.

We can do no better: ratios are determined, but actual money amounts are not. The smallest solution in integers is $x = 1$, $y = 3$, $z = 5$, and $a = 15$.

Comment. The problem comes from Mahāvīra, a ninth-century mathematician and astronomer. A version that has two persons but very similar wording can be found in Chuquet's *Triparty* (1484). There are many variants in the puzzle literature. The earliest known example, attributed, probably wrongly, to Euclid, has a donkey and a mule comparing their loads.

I-31. Alicia started up the Grouse Grind at 4:30. Fred and Janet started 30 minutes later. Janet passed Alicia at the halfway point, and Fred passed Alicia 16 minutes afterwards. Janet got to the top 12 minutes before Fred. Everyone climbs at unvarying speed. At what time did Alicia reach the top?

Solution. Alicia takes 30 minutes more than Janet to get to the halfway point, and therefore 60 minutes more for the full hike. But Janet takes 12 minutes less than Fred, so Alicia takes 48 minutes more than Fred.

Fred got to the halfway point 6 minutes after Janet did. When Fred caught up to Alicia, she had climbed for 16 minutes beyond the halfway point while Fred had climbed for 10. Thus Alicia's Grind time is Fred's time multiplied by $16/10$. But the difference between their times is 48 minutes, so six-tenths of Fred's time is 48 minutes. It follows that Fred's time is 80 minutes and Alicia's is 128 minutes. She arrived at 6:38.

Another way: What variables should we introduce? Maybe the speeds of the participants. Or maybe (better) their Grind hike times. Let these be a, f, and j. We are told that $f - j = 12$. Also, Janet takes 30 minutes less than Alicia to do half the Grind, so $a - j = 60$.

Alicia's speed is $1/a$ Grinds per minute, so when Fred gets to the halfway point Alicia has a lead of $6/a$. Their relative speeds are $1/f - 1/a$, and therefore $(1/f - 1/a)(16) = 6/a$. If we simplify, we get $16a = 10f$. We now have three linear equations in three unknowns, and the rest is easy.

Another way: The same ideas are illustrated in Figure 1.3. We compute exactly, but in principle the picture can be drawn and an approximate answer obtained by measuring. The horizontal axis represents time and the vertical axis distance. The

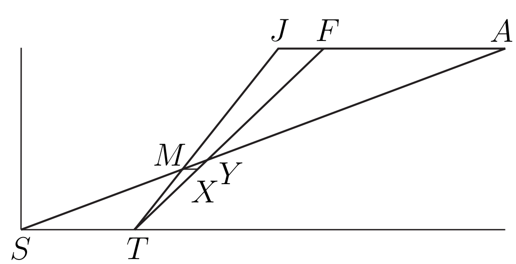

Figure 1.3: The Grouse Grind

point S represents the space-time position of Alicia at the start of her climb, and

T is the space-time position of Fred and Janet when they started. The points J, F, and A represent the space-time positions of the three climbers at the end of their climb, and M is the event "Janet passes Alicia." Because M is midway in vertical distance between S and A, the triangles MST and MAJ are congruent. In particular, since the time between S and T is 30 minutes, so is the time between J and A. We are given that $JF = 12$, so $FA = 18$.

The line segment MX is 1/2 of JF, so $MX = 6$. Since $\triangle MXY$ is similar to $\triangle AFY$, we conclude that the vertical distance between M and Y is 1/3 of the vertical distance between Y and A, so the vertical distance between M and Y is 1/8 of the total Grind distance. It took Alicia 16 minutes to cover this, so the whole Grind took her 128 minutes.

Comment. The first solution shows that it is possible to argue "rhetorically," without algebra. But that may require too much concentration. A nice thing about algebra is that an equation, once written down, serves as a permanent written encapsulation of the idea that led to it, freeing up brain space for further thinking.

I-32. Alfonso borrowed \$3000 from Beti at an interest rate of 8% compounded annually. He paid part of what he owed a year later, the rest a year after that. The second payment was twice as large as the first. What was the first payment?

Solution. Let the first payment be P dollars. Just *before* the first payment, Alfonso's debt had grown to \$3000 plus (3000)(0.08) in interest, that is, to (3000)(1.08). Just *after* the first payment, the debt was $(3000)(1.08) - P$. After one more year, that debt had grown to $[(3000)(1.08) - P]\,(1.08)$, but was paid in full with the final payment of $2P$. We obtain the linear equation

$$[(3000)(1.08) - P]\,(1.08) = 2P,$$

which yields $P = (3000)(1.08)^2/(1.08 + 2)$. To the nearest cent, the first payment is \$1136.10.

Comment. Many solvers use the calculator not only at the end, but repeatedly during the solution process. In this question, they obtain the equation $3499.2 - 1.08P = 2P$. That kind of on the fly calculation is sometimes a good idea. But it can hide structural information: when we write 3499.2, its origins as $(3000)(1.08)^2$ are no longer visible. And structural insight, a sensitivity to patterns, is often the key to solving a problem.

I-33. Two hikers came out of the bush onto an East-West road. They needed to go West, but were tired and decided to catch the bus. They argued about which was closer, the first bus stop to the East or the first one to the West. So Esther headed East at 6 km per hour, while Wesley headed West at 5 km per hour. Esther got to the bus stop just in time to catch the bus. And

Wesley got to her stop at the same time as the bus that Esther was on. The bus travels at an average speed of 60 km per hour between stops. Who walked further, Esther or Wesley?

Solution. Let t_e be the amount of time that Esther walked, and t_w how long Wesley walked. The sum of the distances they walked is $6t_e + 5t_w$. This is the distance between the bus stops.

From the time that Esther caught the bus to the time it arrived at Wesley's stop, $t_w - t_e$ hours elapsed, and therefore the bus travelled a distance $60(t_w - t_e)$. Thus

$$6t_e + 5t_w = 60(t_w - t_e), \quad \text{so} \quad 6t_e = 5t_w.$$

Esther and Wesley walked equal distances.

Comment. The equality is an "accident." If the bus speed is 59, then Esther walked the lesser distance. If it is 61 then Esther walked the greater distance. The *actual* distances they walked can't be found from the facts provided.

I-34. Cyclists A, B, and C are riding in a race in which they go 36 times around a 1500 meter circuit. Cyclist A pedals at a steady 750 meters a minute, B at 875 meters a minute, and C at 925 meters a minute. They start together. (a) When will A and B first be together again? (b) When will A, B, and C all be together again?

Solution. (a) Cyclist B gains 125 meters every minute, so she gains 1500 meters, that is, one whole circuit, every 12 minutes.

(b) We saw that A and B are together every 12 minutes. A similar calculation shows that B and C are together every 30 minutes. Is there a time $t > 0$ when they are *all* together? Such a t must be simultaneously an integer multiple of 12 and an integer multiple of 30.

The least common multiple of 12 and 30 is 60, so it looks as if the answer is 60 minutes. Not quite. Cyclist C takes about 58.38 minutes to complete the 36 circuits—at the 60 minute mark she is no longer on the track.

Comment. The cyclist question is less frivolous than it looks. The Sun, the Moon, and the planets display various forms of cyclic behaviour as they travel through the sky. In trying to understand eclipses and other conjunctions, astronomers and astrologers in the Near East, India, and China contributed significantly to the development of mathematics.

I-35. Four jewellers possess respectively 8 rubies, 10 sapphires, 100 pearls, and 5 diamonds. Each gives one of his own stones to each of the others as a token of regard. After that, they all possess stock of exactly the same value. Find the relative prices of rubies, sapphires, pearls, and diamonds (Bhāskara, 1150).

Solution. Let r be the value of a ruby, s the value of a sapphire, and so on. We are tacitly asked to assume that all the stones of the same type have the same value—unlikely, but so are the actions of the jewellers.

After the exchange of gifts, the person who owned rubies has stock of value $5r + s + p + d$, the one who owned sapphires has stock of value $r + 7s + p + d$, and so on. These values are all equal, so

$$5r + s + p + d = r + 7s + p + d = r + s + 97p + d = r + s + p + 2d.$$

A quick way to solve this system is to subtract $r + s + p + d$ from each expression, obtaining $4r = 6s = 96p = d$. A ruby is therefore worth 24 pearls, a sapphire 16 pearls, and a diamond 96 pearls.

Another way: Imagine that each jeweller also gave a gift to himself. Each now has four gift stones, together with what is left of his initial stock. These leftovers must then all have equal value, and therefore 4 rubies are equivalent to 6 sapphires, to 96 pearls, to 1 diamond. Note that this solution is fundamentally the same as the algebraic one—when we concentrate on the "leftovers," we are in effect subtracting $r + s + p + d$.

Comment. In the Bakhshālī manuscript (200?) three merchants possess respectively 7 *asavas*, 9 *hayas*, and 10 camels. Each gives to each of the others two of his animals. All three are then equally well-off. We are asked to compare the relative values of the three kinds of animal.

I-36. A water tank has a flat bottom and vertical sides. The floor area of the tank is $5000\,\text{cm}^2$, and there is water in the tank to a depth of 20 cm. A cubical concrete block with sides 25 cm is placed on the floor of the tank. (a) By how much does the water level rise? (b) A second identical block is then placed on the floor of the tank. How deep is the water now?

Solution. Figure 1.4 indicates what will happen, but to show these sketches are roughly correct requires calculation. (a) There are $100000\,\text{cm}^3$ of water in the tank.

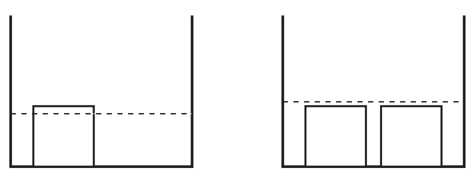

Figure 1.4: The Level of the Water

Imagine first that the concrete block has base 25 cm by 25 cm, but is quite tall. The available area of bottom is $4375\,\text{cm}^2$, and therefore the depth of water would be 100000/4375, roughly 22.857 cm. That's not high enough to cover the actual block, so the water level rises by about 2.857 cm.

(b) This time the blocks are submerged, since the water level clearly rises by more than 2.857 cm. Together, the blocks have volume 2×25^3 cubic centimeters. Imagine that the blocks are made of water. Then the total amount of "water" in the tank is 131250. So the depth of the water is 131250/5000, that is, 26.25.

Comment. This problem, with different numbers, comes from Henry Dudeney's *536 Puzzles & Curious Problems*.

Here is a harder variant. Put into the tank a concrete block that has a 25×25 square base and has the property that vertical cross-sections perpendicular to one of the edges of the base are all isosceles triangles of height 25—the block looks like an A-frame tent. How much does the water level rise?

Or else use a concrete Egyptian-style pyramid with base 25×25 and height 25. We end up having to solve a cubic equation. That can be done, to excellent accuracy, with the *Solve* key that can be found on the more sophisticated calculators, or more crudely by graphing.

I-37. Only 20% of the light gets through a one-quarter centimeter layer of tinted glass. How much does a three-sixteenth centimeter layer of the same kind of glass let through? Assume that when light strikes glass of given thickness, an amount proportional to the intensity gets through, and none is reflected.

Solution. Imagine light of intensity I falling on a pane of glass 1/16 cm thick. Then pI gets through, for a certain number p. Put two of the panes together, making a 1/8 cm pane. The amount of light that gets through is $p(pI)$, that is, p^2I. For example, if a 1/16 cm pane lets 70% through, then the second pane attenuates the light by another 30%, so 49% of the light gets through the two panes.

An amount p^3I gets through three panes, and p^4I gets through four—that is, through a layer 1/4 cm thick. We know that $p^4 = 0.20$, and want to calculate p^3. Since $p = (0.20)^{1/4}$, it follows that $p^3 = (0.20)^{3/4}$. Now that we know the answer, it is time to use the calculator: approximately 30% of the light gets through.

I-38. The Fahrenheit scale of temperature calls the freezing point of water 32°, and the boiling point 212°. Celsius suggested dividing the interval between boiling point and freezing point into 100 equal parts. Not many people know that Celsius chose 0 as the boiling point of water, and 100 the freezing point! After a few years he changed his mind.

Call the initial suggestion of Celsius the *old* Celsius scale. What temperature is the same under the old Celsius scale and the new Celsius scale? What temperature is the same under the Fahrenheit scale and the old Celsius scale?

Solution. The answer to the first question is clearly 50. We answer the second question by producing a formula for converting Fahrenheit temperatures to old Celsius. Because the (new) Celsius scale is familiar, we convert first to new Celsius.

There are 100° Celsius between freezing point and boiling point, and 180^{circ} Fahrenheit. The Fahrenheit temperature x is $x - 32$ degrees Fahrenheit "above" freezing ($x-32$ can be negative) and therefore the temperature is $(x-32)(100/180)$ degrees Celsius. It follows that the old Celsius temperature that corresponds to x degrees Fahrenheit is $C(x)$, where

$$C(x) = 100 - \frac{(100)(x-32)}{180}.$$

Set $C(x) = x$ to find the temperature that is the same in both scales, simplify and solve. We get $x = 530/7$, about 75.7.

Another way: The problem can be solved graphically. In Figure 1.5, Fahrenheit temperatures are measured along the x-axis, old Celsius temperatures along the y-axis, and the scales on the two axes are the same. Draw the line that joins the point $(32, 100)$ to the point $(212, 0)$. This line has equation $y = C(x)$, where $C(x)$ is the old Celsius temperature that corresponds to the Fahrenheit temperature x. We want to solve the equation $C(x) = x$. To do this graphically, just draw the

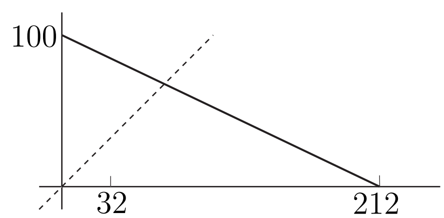

Figure 1.5: The Old Celsius Scale

dashed line $y = x$, and read off where it meets the line $y = C(x)$.

Comment. In the days before calculators, engineers had elaborate graphical ways of solving problems by joining points on carefully printed scales. The technique was called *nomography.*

I-39. You are doing your usual long morning commute on the freeway, which has three lanes in each direction. Traffic is moving steadily but slowly, at 20 km per hour. The radio says that traffic is bumper to bumper all the way to downtown. Then you see a sign saying that the left lane is closed for the next 6 km. There is no exit or entrance for a long way. How long can you expect to spend in the two-lane stretch?

Solution. Let L be the average distance between the front of a car and the front of the car behind it. On the three lane stretch, traffic is flowing at $(3 \cdot 20)/L$ cars per hour. Let v be the speed on the two lane stretch. Assume, perhaps wrongly, that the average distance between cars is still L. Then traffic on that stretch is flowing at $2v/L$ cars per hour.

Traffic is bumper to bumper for a long way, so we are in a *steady state* situation. Thus the flow rate through the three-lane stretch is the same as the flow rate through the two-lane stretch. It follows that $60/L = 2v/L$, that is, $v = 30$. The 6 km two-lane stretch should take about 12 minutes.

Comment. It is tempting to guess that traffic moves more slowly on the two-lane stretch. To see that this isn't true, recall that where a stream narrows, the water flows more quickly.

We assumed that the average space between cars on the two-lane stretch is the same as on the three-lane stretch despite the fact that speeds are higher. That may be reasonable at the very low speeds involved here, and the radio did say "bumper to bumper."

It is instructive to solve the problem under the assumption that drivers obey the American guidelines of one car length for every 10 miles per hour. The answer is dramatically different! This new model is good for situations in which beyond the narrowing there is uncongested highway.

I-40. Assume that the force of gravity on the Moon is one-sixth of the force on Earth. High jumper J can, on Earth, clear a bar set at 2.06 meters. What height can J clear on the Moon? J's standing center of mass is 1.1 meters off the ground. J can clear the bar in such a way that J's center of mass passes 0.05 meters *under* the bar.

Solution. Assume that J generates the same take-off energy in both places, and that this energy is used to raise the center of mass a suitable amount. On Earth, the center of mass was raised by $(2.06 - 0.05 - 1.1)$ meters. On the Moon, the center of mass is raised by 6 times as much, namely 5.46. Add to this 1.1 for the normal height of the center of mass, and 0.05 for the fact that the center of mass passes under the bar. J should clear 6.61 meters.

Comment. The center of mass of a skilled high jumper or pole vaulter *does* pass below the bar: by the time the mid-body clears the bar, head, arms, and shoulders are well below the bar.

More and more mathematics is entering sports. Mathematical models have been used to refine technique, to determine optimal batting order, and to design sports equipment ranging from pole vault poles to racing boats.

I-41. A stone is dropped down a deep shaft. Five seconds later the sound of the stone hitting bottom is heard. Assume that the stone falls a distance $gt^2/2$ meters in t seconds, where $g = 9.8$, and that the speed k of sound at the ambient temperature is 346 meters per second. How deep is the shaft?

Solution. Let d be the depth of the shaft, and t the travel time of the stone to the bottom. That leaves leaves $(5-t)$ seconds for the sound to travel back up. It follows that $k(5-t) = d = gt^2/2$. The quadratic formula yields

$$t = \frac{-k + \sqrt{k^2 + 10gk}}{g}$$

(the negative root was discarded). It turns out that t is about 4.688, and therefore d is about 108 meters.

Comment. We sketch a method of *successive approximation* which for this problem is harder, but gives a truer picture of informal mathematical thinking in the sciences.

The stone travels less than 5 seconds, so its terminal speed is less than $5g$, and its average speed is less than $(5g)/2$, which is about 24.5. Sound travels much faster, so most of the 5 seconds are taken by the stone going down. More precisely, the ratio r of the time for the stone to go down to the time for the sound to travel up is greater than $346/24.5$. It follows that the time for the sound to go up is less than $5/(1+r)$. Multiply this estimated time by 346. We conclude that the depth of the well is less than 114.4 meters.

Using the standard formula $s = gt^2/2$, we find that the time for a stone to drop 114.4 meters is about 4.832 seconds. So the stone's average speed was actually less than $(4.832)(9.8)/2$, which is about 23.676. That means that the ratio r of downward travel time to upward travel time is greater than 14.6. As in the preceding paragraph, we conclude that the depth of the well is less than 110.8.

We can go through one or two more computational cycles, or decide that the calculations are inherently inexact, in that for example they neglect air resistance, so an answer of about 110 is reasonable.

I-42. Alan can split a cord of wood in eight and a half hours, and Allison can do it in six. Alan started splitting, but got tired after a while. So Allison took over, and finished splitting the cord 7 hours after Alan had started. Who worked longer, Alan or Allison?

Solution. Suppose that Alan worked for t hours. When Alan stopped, he had split $t/8.5$ cords. Then Allison worked for $7-t$ hours, and therefore split $(7-t)/6$ cords.

The total amount of wood split is 1, and therefore

$$\frac{t}{8.5} + \frac{7-t}{6} = 1.$$

Simplify and solve: $t = 3.4$, so Allison worked a bit longer than Alan.

Another way: We computed too much: after all, we weren't asked how long Allison worked. Suppose Alan had worked for exactly half the time. Then the amount of wood split would be $3.5/8.5 + 3.5/6$. The calculator shows that this is about 0.995, not quite a full cord. And things get worse if Alan works more than 3.5 hours. So Allison must have worked longer.

I-43. Sarah was hiking up Grouse Mountain. She got 3/4 of the way up in 43 minutes and 12 seconds. She took just as long to do the first half as the second half, and took a minute and 36 seconds more to do the third quarter than the fourth quarter. How long did the entire hike take?

Solution. Suppose that the fourth quarter took x minutes. Then the second half (and hence the first half) took $x + (x + 1.6)$ minutes. The first three-quarters therefore took $3x+3.2$ minutes. But we know that took 43.2 minutes, and therefore $3x = 40$. So the fourth quarter time was 13 minutes and 20 seconds. It follows that the entire hike took 56 minutes and 32 seconds.

I-44. A return ticket on the Grouse Mountain Skyride costs \$15. Some people prefer to hike up and ride down. A ticket just to go down costs only \$5. Last Saturday, Skyride receipts were \$16250. On Sunday, twice as many people hiked up and rode down than on Saturday, and 160 more people took the Skyride both ways than on Saturday. Total receipts were \$19500. How many people hiked up and rode down on Saturday?

Solution. Suppose that a people rode both ways on Saturday, and that b people rode down only. Then $15a + 5b = 16250$. The Sunday results tell us that $15(a + 160) + 10b = 19500$. Solve: $b = 170$.

I-45. With the cold water tap and the hot water tap both turned full on, it takes 4 minutes to fill a sink. If the cold water is full on and the hot is off, it takes 2 minutes less to fill the sink than if the cold water is off and the hot is full on. How long does it take to fill the sink from the hot water tap alone? Assume that the cold and hot water flow rates are independent of each other (that is roughly true in an apartment building).

Solution. Let t be the time it takes for the hot water tap alone to fill the sink. Then the time for the cold water alone is $t - 2$. The rates of flow of the two taps are, respectively, $1/t$ and $1/(t - 2)$ sinkfuls per minute.

If the taps are both on, their combined rate is the sum of the individual rates. We were told that the combined rate is 1/4, and therefore

$$\frac{1}{t} + \frac{1}{t-2} = \frac{1}{4}.$$

Since t can't be 0 or 2, we get an equivalent equation by multiplying both sides by $4t(t - 2)$. After simplifying, we obtain $t^2 - 10t + 8 = 0$. The solutions are $t = 5 \pm \sqrt{17}$. The smaller root is not sensible, for it is less than 1, meaning that the cold water tap fills the sink in negative time!

I-46. On an airless planet far far away, the local Galileo tried to drop an apple and a feather simultaneously from a 100 meter tower. But Galileo's timing was off, and the feather had already travelled 1 centimeter when she dropped the apple. How far above the feather was the apple when the feather hit the ground? Under constant acceleration a, an object initially at rest is moving at speed at when time t has elapsed, and has travelled distance $at^2/2$.

Solution. Let T be how long the feather takes to reach the ground. Then $aT^2/2 = 100$, where a is the local gravitational acceleration. Suppose that when the apple is released, the feather has been travelling for time t. During this time, the feather travelled from rest through a distance of 0.01 meters, and therefore $at^2/2 = 0.01$.

The gap between feather and apple when the feather hits the ground is $aT^2/2 - a(T-t)^2/2$. Now calculate. Everything could be done on a simple calculator if we knew a, but we don't. We could *hope* that since the acceleration due to gravity was not supplied, the answer is independent of a. But we can't assume that, so it looks as if we will have to carry a along.

Actually, we don't have to carry a. For the only information we have been given is a couple of lengths, and the question asks for another length, so we are free to choose the unit of time however we like. Choose it so that $a = 2$.

Then $T^2 = 100$, and $t^2 = 0.01$. The required distance is $T^2 - (T-t)^2$, that is, $t(2T - t)$. Thus the distance between feather and apple when the feather hits the ground is 1.99 meters.

Comment. Most students know the formula $s = at^2/2$, but do not know *why* it is true. We give a justification that reproduces in a continuous setting the usual trick for adding up numbers in arithmetic progression. This explanation was inspired by a movie chase scene on top of a train.

A long train has a flat top. The train is initially at rest, and travels with constant acceleration a, from time 0 to time t. So at the end the velocity is at. On top of the train is a car, which travels at initial velocity at, and constant acceleration $-a$, both relative to the train.

The car has 0 acceleration relative to the ground, so it travels at constant speed at relative to the ground. In time t, the car travels a distance at^2 relative to the ground. By symmetry, the train moves half that distance relative to the ground.

I-47. A stroboscopic light produces very brief flashes 21 times a second. A dark turntable with a reflective spot on its edge is spinning at 2400 revolutions per minute. The sole illumination in the room is provided by the strobe. At what rate does the spot appear to rotate?

Solution. The turntable spins at 40 revolutions per second, say clockwise. In the interval between flashes, the turntable goes through 40/21 revolutions, that is, 2/21 revolutions short of 2 full revolutions. The brain interprets this to mean (more colloquially, it looks as if) the reflective spot has moved through 2/21 revolutions *counterclockwise*. That apparent motion took 1/21 seconds, so the apparent number of revolutions per second is (2/21)/(1/21), and the spot appears to rotate at 120 revolutions per minute in a direction opposite to the real spin.

Comment. The same phenomenon can be seen in movies, which are simply rapid sequences of still images. If a wagon wheel is rotating at suitable speeds relative to the frame rate, the wheel appears to rotate backward while the wagon moves forward. Sometimes the wheel even appears nearly to stand still, with a small wobble.

I-48. A and B were walking together from A's house to B's house, and were 30% of the way to B's house when they turned around and saw in the far distance a vicious dog pursuing them.

A ran back toward her house at 12 km per hour, while B ran toward *her* house at 12 km per hour. A reached safety just in time. So the dog kept running, and reached B's house precisely when B did. How fast is the dog?

Solution. Let s be the dog's running speed, and h the distance between A's house and B's house. Let d be the distance that the dog is from A's house when A and B first see it.

The dog reaches A's house in time d/s, while A reaches it in time $(0.30)h/12$. We obtain the equation $d/s = (0.30)h/12$. A similar calculation for B gives $(d + h)/s = (0.70)h/12$. Subtract and simplify. It turns out that $s = 30$.

Another way: We can solve the problem without algebra. The dog and A reached A's house at the same time. Since A and B run at the same speed, B was already 60% of the way home when the dog got to A's house. And while the dog ran from A's house to B's, B covered the remaining 40% of the distance between the two houses. Thus B's speed is 40% of the dog's speed, and therefore the dog runs at $12/(0.4)$ kilometers per hour.

Another way: Use the space-time Figure 1.6. The horizontal axis represents time and the vertical axis represents space. Let S be the space-time event "A and B saw

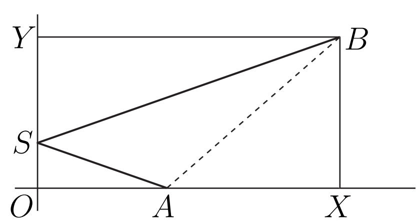

Figure 1.6: The Vicious Dog

the dog," A be the event "A reached home" and B the event "B reached home." The dashed line represents part of the space-time path of the dog.

Because A and B run at the same speed, $\angle SAO = \angle SBY$. It follows that $\triangle SAO$ is similar to $\triangle SBY$. Since OS is 30% of OY, we conclude that $YB/OA = 70/30$, and therefore $AX = (40/30)(OA)$. But $XB/OS = 100/30$, and therefore $XB = (100/30)(OS)$. Thus XB/AX, the slope of AB, is $(100/40)(OS/OA)$.

Since OS/OA is the speed of A, we conclude that XB/AX, the speed of the dog, is (100/40)(12).

Chapter 2

Counting

Introduction

The basic theme here is counting through systematic listing—so in theory at least we make a complete list, and then count. Sometimes the numbers involved are so small that careful counting is enough, though being organized always helps. But the numbers are often either large or or unspecified ("n"). An explicit list is then not possible, but still systematic *structured* lists must be imagined and then counted.

Most of these problems are not technically difficult—once we know how to do them! But it can be hard to reach the insight that makes the problem fall apart. And sometimes minor-seeming changes of wording can turn a simple problem into a difficult one. But it is almost always possible to make progress through experimentation. If the numbers involved are too large for that, we can work with smaller numbers: after a while, patterns magically appear, and everything becomes clear.

We will often use the *Multiplication Principle*: If task A can be done in a different ways, and for *every one* of these ways task B can be done in b ways, then there are $a \times b$ ways of doing task A followed by B. We also need to know how many ways there are of selecting r objects, where order of selection doesn't matter, from a collection of n distinct objects. Calculators and school textbooks denote this number by ${}_n\mathrm{C}_r$, or occasionally $\mathrm{C}(n, r)$. Most mathematicians use the symbol $\binom{n}{r}$, whose English pronunciation is "n choose r." It will be necessary to know that

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-r+1)}{r!}.$$

"Closed form" expressions are stressed in the schools. Here and in the

Sequences chapter we also pay attention to *recurrence formulas,* though admittedly mainly as a way to reach closed form formulas. With the advent of high speed computing, recurrence formulas are of increasing importance.

Problems and Solutions

II-1. Let $\mathcal{P}$ be a regular n-sided polygon. Call a triangle *good* if all its vertices are vertices of $\mathcal{P}$ but none of its sides is a side of $\mathcal{P}$. How many good triangles are there?

Solution. Begin by playing with small n. If $n = 3$, 4, or 5, there are no good triangles. If $n = 6$ there are 2 good triangles. If $n = 7$ there are 7 good triangles—already here care is needed. So let's tackle the general case! Let $n \geq 6$.

We can choose 3 vertices of $\mathcal{P}$ in $\binom{n}{3}$ ways. Some of these choices lead to *bad* triangles, that is, triangles that have one or more edge in common with $\mathcal{P}$.

There are two kinds of bad triangle: *very bad* (two edges in common with $\mathcal{P}$) and *somewhat bad* (one edge in common with $\mathcal{P}$). The two edges a very bad triangle shares with $\mathcal{P}$ meet in a point. Once this point is chosen, the triangle is determined, so there are n very bad triangles.

Now count the somewhat bad triangles. Such a triangle has two neighbouring vertices and an isolated one A. There are n ways to choose A. The first vertex that we meet as we travel counterclockwise from A can be in any one of $n-4$ places. The third vertex is then determined. Thus there are $n(n-4)$ somewhat bad triangles, so the number of good triangles is

$$\binom{n}{3} - n - n(n-4)$$

which simplifies to $n(n-4)(n-5)/6$.

Another way: To make a good triangle, choose a vertex A (n ways) and colour it blue. Call any vertex other than A and its two neighbours *safe*.

Choose the remaining two vertices from the $n-3$ safe points. Of the $\binom{n-3}{2}$ ways to do this, $n-4$ choices give neighbouring pairs and must be removed. Thus there are $n\binom{n-3}{2} - n(n-4)$ good triangles with one vertex coloured blue. Calculate: the number turns out to be $\binom{n-4}{2}$.

Any triangle gives rise to exactly three triangles with one vertex coloured blue. So to find the number of triangles divide $n\binom{n-4}{2}$ by 3.

Comment. We can see directly that there are $n\binom{n-4}{2}$ triangles with one blue vertex. After picking the blue vertex A, *choose* two edges from the $n-4$ that join safe vertices and colour them red. Let B be the first red point as we travel counterclockwise from A, and C the first as we travel clockwise.

There are many ways to count, and it is all too easy to make a mistake. The first approach was probably the safest. Counting in two different ways serves as a partial check. And we should almost always check any general result against actual careful counts for small n.

II-2. A rectangle is divided into 24 small rectangles by drawing 3 lines parallel to its base and 5 lines perpendicular to its base. Find the *total* number of rectangles of all sizes.

Solution. Since the numbers aren't large, we can and should count directly. Draw a picture like Figure 2.1. Note that in the picture the 24 little rectangles are squares. The problem didn't even specify that lines are equally-spaced, but the spacing doesn't affect the count, and the picture might as well look nice. For now pay no special attention to the two thicker lines.

It is natural to organize the count by rectangle size. Start with "large" ones—there are fewer of them. There is 1 six by four rectangle, 2 six by three, 3 six by two, and 4 six by one. Continue by counting the five by four, the five by three, and so on.

We show how to figure out for example the number of four by three rectangles (width four, height three). Start with a four by three wedged into the Northeast corner of the rectangle, and see what freedom of movement it has: 0, 1, or 2 units to the West and/or 0, 1 or two units South, for a total of $3 \cdot 2$ possibilities.

Another way of putting it is that a four by three is completely determined once we specify its Southwest corner. And the Southwest corner of a four by three has to lie in the Southwest rectangle determined by the two thick lines in Figure 2.1. The picture shows that there are $3 \cdot 2$ choices for that Southwest corner. After a

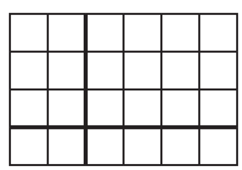

Figure 2.1: Counting Rectangles

while we get that the total number of rectangles is

$$(1+2+3+4)+2(1+2+3+4)+3(1+2+3+4)+\cdots+6(1+2+3+4).$$

This number can be rewritten as $(1+2+3+4)(1+2+3+4+5+6)$. It turns out to be 210.

The idea generalizes. If (including the sides of the rectangle) there are m East–West lines and n North–South lines, the count proceeds in exactly the same way, with result

$$[1+\cdots+(m-1)]+2\,[1+\cdots+(m-1)]+\cdots+(n-1)\,[1+\cdots+(m-1)]\,.$$

Another way: Instead of organizing the count by size, give it structure by counting for each possible P the number of rectangles that have P as Southwest corner.

Start with P the Southwest corner of the whole rectangle. Why start there? We have a coordinate system in mind, and think of the Southwest corner as $(0, 0)$.

To make a rectangle with Southwest corner P, we need to pick its Northeast corner Q. A look at Figure 2.1 shows that there are $6 \cdot 4$ ways of picking Q. Now move P one unit North. Why North? We are increasing the y-coordinate by 1. There are then $6 \cdot 3$ rectangles with P as Southeast corner. Go on like that, systematically. We get a total of $6(4+3+2+1)$ rectangles whose Southwest corner is on the y-axis.

Now start with $P = (1, 0)$. There are $5 \cdot 4$ rectangles that have P as bottom left corner. Move to $(1, 1)$, then $(1, 2)$, and so on. We get a total of $5(4+3+2+1)$ rectangles with Southwest corner on the line $x = 1$. Continue.

Another way: We think the above approach is best, but there is a slicker way. Suppose that when we include the sides of the rectangle there are m East–West and n North–South lines. We produce the rectangles by *choosing* two East–West lines and two North–South lines to form its boundary. The East–West lines can be chosen in $\binom{m}{2} ways. For each such way, the North - - South lines can be chosen in \binom{n}{2}$ ways. Thus there are $\binom{m}{2}\binom{n}{2}$ rectangles. When $m = 5$ and $n = 7$ there are 210 rectangles.

II-3. In a chess tournament, every player plays every other player once. The organizers had planned to invite every ranked player in the city, but realized that would involve too many games. So they didn't invite the 3 weakest ranked players, thereby saving a total of 45 games. How many did they invite?

Solution. If k people play in a round robin tournament, then the number of games is the number of ways of choosing 2 people from k, namely $\binom{k}{2}$, that is, $k(k-1)/2$.

Let n be the number who were actually invited. Then there are $n+3$ ranked players in the city. If they had all been invited, there would have been $(n+3)(n+2)/2$ games. Only n were invited, so there were $n(n-1)/2$ games. Since 45 games were saved,

$$45 = \frac{(n+3)(n+2)}{2} - \frac{n(n-1)}{2} = 3n + 3,$$

and therefore $n = 14$.

Another way: We can do it with less machinery. Let n be the number actually invited. If 3 more had been invited, the following additional games would have been needed: (i) 3 games between the extra players and (ii) for each extra player, n games with the players actually invited. Thus $3n+3$ extra games would be needed, and therefore $3n + 3 = 45$.

II-4. There are four judges at a wine tasting. Each judge assigns the wine a mark from 1 to 10 and the marks are added together. In how many different ways can a wine get a total score of 20? (The Albanian judge giving 9 and the Bolivian 6 is not the same as 6 from the Albanian and 9 from the Bolivian.)

Solution. Count first the number of ways of getting various total scores from *two* judges, A and B. This is basically the same problem as counting the number of ways of getting various sums when we toss two dice.

Write (a, b) to mean a from A and b from B. We list all the ways of getting, for example, a sum of 6: $(1, 5)$, $(2, 4)$, $(3, 3)$, $(4, 2)$, and $(5, 1)$. So there are 5 ways that the sum can be 6. Similarly, we can see that if n is any number from 2 to 10, there are $n - 1$ ways of getting a sum of n.

We could continue counting for $n = 11$ to $n = 20$. But note instead that

$$a + b = n \quad \text{if and only if} \quad (10 - a) + (10 - b) = 20 - n.$$

So there are just as many ways of getting a sum of $20 - n$ as there are of getting a sum of n, there is mirror symmetry across $n = 10$.

We now tackle the four judges problem, by thinking of the ways that the total score can be 20. The scores of A and B can add up to 2 and the scores of C and D to 18, or A and B can add up to 3 and C and D to 17, and so on. We summarize the possibilities as $(2; 18)$, $(3; 17)$, $\dots$, $(9; 11)$, $(10; 10)$, $(11; 9)$, $\dots$, $(18; 2)$.

The pattern $(2; 18)$ can happen in $1 \cdot 1$ ways, $(3; 17)$ can happen in $2 \cdot 2$ ways, and so on. So the total number of ways is

$$1^2 + 2^2 + \cdots + 8^2 + 9^2 + 8^2 + \cdots + 2^2 + 1^2.$$

Add up. There aren't many numbers, so it is probably not worthwhile to develop general theory, though it is worth noticing that $1^2 + 2^2 + \cdots + 8^2$ occurs twice. The sum is 489.

Comment. The same analysis works if each judge assigns a mark from 1 to m and we want to count the number of ways of getting a total score of $2m$. The answer is

$$\left[1^2 + \cdots + (m - 2)^2\right] + (m - 1)^2 + \left[(m - 2)^2 + \cdots + 1^2\right].$$

Using the standard formula $1^2 + 2^2 + \cdots + k^2 = k(k + 1)(2k + 1)/6$ (see VI-37) we find that the number of ways is

$$\frac{(m - 1)(2m^2 - 4m + 3)}{3}.$$

More generally, we can compute the number of ways to get total score s. The results are particularly nice for $s = 2m \pm 1$. We could also ask similar questions about three judges.

II-5. A rectangle is split into 26×27 identical rectangles by drawing 25 lines parallel to a pair of sides, and 26 lines parallel to the other pair. Draw a diagonal of the rectangle. How many of the small rectangles does it travel through?

Solution. We can draw a picture and then count. But even if, like Figure 2.2, it is drawn with precise drawing software, the counting requires care. However, by *thinking,* we can easily solve a more general problem. Choose as origin O the

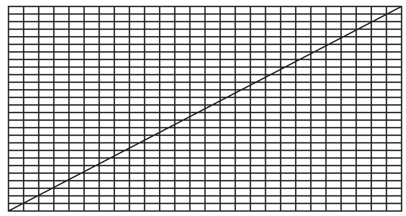

Figure 2.2: The Number of Rectangles Crossed by a Diagonal

bottom left corner of the big rectangle, let the axes be along the two sides that meet at O, and let P be the corner opposite to O. Let the rectangle have base a and height b. We first show that the diagonal OP can only touch the corner of a small rectangle at O and at P.

The coordinates of any corner C of a small rectangle are of the shape $(sa/26, tb/27)$ where $s \le 26$, $t \le 27$ and s and t are integers. If $C \ne O$ and C is on the diagonal, then OC has slope b/a. It follows that

$$\frac{b}{a} = \frac{tb/27}{sa/26}$$

and therefore $27s = 26t$. But since 27 and 26 have no factor greater than 1 in common, that forces $s = 26$ and $t = 27$, that is, $C = P$.

Now follow the diagonal as it travels upward. The diagonal enters a new small rectangle every time it pierces a horizontal line or a vertical line, and it can't simultaneously pierce a horizontal and a vertical. So it must cross $26 + 25$ lines. Add in the small rectangle at the bottom left: 52 small rectangles are crossed.

Comment. Imagine colouring in red the rectangles that the diagonal traverses. The red rectangles form a zig-zag path from the bottom left rectangle to the one at top right. Think of the path as the travels of a chess King who can only move up or to the right. The King must take a total of 25 steps to the right and 26 up (even if he were allowed to stray from the diagonal), so he must have visited his starting rectangle and $25 + 26$ others.

The same argument shows that if we have m instead of 26, and n instead of 27, and m and n have no common factor greater than 1, then $m + n - 1$ rectangles are crossed by the diagonal. More generally, if d is the greatest common divisor of m and n, then the number of rectangles crossed is $d(m/d + n/d - 1)$, that is, $m + n - d$.

II-6. (a) Three runners compete in a 100 meter race. How many possible orders of finish are there, if ties are allowed? (b) What about if there are four runners?

Solution. (a) Count first the orders of finish with no ties. The winner can be chosen in 3 ways. For each of these ways, the second-place runner can be chosen in 2 ways. Now the order of finish is fully determined, so there are $3 \cdot 2$ possibilities of this type.

There is 1 three-way tie. There are $3 \cdot 2$ two-way ties, for the runner who *won't* be tied can be chosen in 3 ways, and for each choice there are 2 possible orders of finish. There are therefore 13 possible orders of finish.

(b) Count first the orders of finish with no ties. Use the "multiplication principle" twice. There are $4 \cdot 3$ ways to assign the first two places. For each of these, third place can be assigned in 2 ways, giving $(4 \cdot 3) \cdot 2)$ possible orders of finish.

There is 1 four-way dead heat. To count three-way ties, note that the three runners can be chosen in $\binom{4}{3}$ ways and for each choice there are 2 orders of finish, tie for first or tie for last, for a total of 8.

Now count the ways that exactly two runners can finish tied. The tied runners can be chosen in $\binom{4}{2}$ ways, and for each choice there are 6 possible orders of finish (tie the tied runners together, now three "people" are running), for a total of 36.

Finally, count the double ties. The pair who finish tied for first can be chosen in $\binom{4}{2}$ ways. Once this choice is made, everything is determined, so there are 6 possible double ties. (The double ties are the the easiest to get wrong.) Add up: the total is 75.

II-7. How many ways are there to colour the faces of a cube using red, white, blue, green, brown, and yellow paint, if the faces are to be of different colours? (Two colourings are different if one can't be obtained from the other by a rotation of the cube.)

Solution. Imagine that the cube has been coloured and place it on a desk red face down. There are 5 possibilities for the colour of the top face. We will count the colourings that have red face down and, say, white face up and multiply the result by 5.

Leaving red face down and white face up, rotate the cube so that the blue face points toward you. Then the face to the right could be any one of 3 colours (green , brown, or yellow), and for each such choice the face to the left could be any one of 2 colours. Once the right and left colours are chosen, the colour of the back face is determined. So there are $3 \cdot 2$ colourings with red at the bottom and white on top, and therefore 30 colourings altogether.

II-8. Let X be one of the corners of a cube, and Y the corner diagonally opposite to X. How many ways are there to walk from X to Y along the edges of the cube without ever going through the same point twice?

Solution. Use the stylized cube of Figure 2.3 as a guide. The first step takes us to A, B, or C. When we reach one of these 3 vertices, there are 2 choices for the next vertex to visit. So the first two steps can be taken in 6 different ways. Up to now,

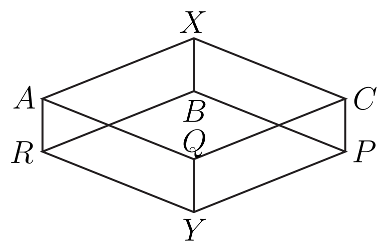

Figure 2.3: Walking the Cube

the situation is fully symmetric, so we count the paths that begin with XAR and multiply the result by 6.

Given start XAR, we can finish with Y, BPY, or $BPCQY$, 3 ways only. Thus there are 18 paths from X to Y.

II-9. A convex nonagon (9 sides) has the property that no point in the interior of the 9-gon lies on three or more diagonals. How many intersection points of diagonals lie in the interior of the 9-gon?

Solution. We could and should make a sketch. Maybe we could even use the sketch to count. But if we draw all 27 diagonals the picture may be too messy to be useful. There is also a theoretical difficulty: the question is about all convex 9-gons, and it is not yet clear that the answer is independent of shape. The arguments that follow are motivated by pictures but do not depend on them.

Call a diagonal a 1-diagonal if it joins vertices A and B that are close to each other, in the sense that there is only one vertex between A and B. Similarly, we can define 2-diagonal and 3-diagonal. The 9-gon has 9 each of these three kinds of diagonal.

Look first at a 1-diagonal. It has 1 vertex on one side and 6 on the other. That gives $1 \cdot 6$ diagonals that meet the given 1-diagonal inside the 9-gon, and therefore 6 intersection points.

Look now at a 2-diagonal. There are 2 vertices on one side, 5 on the other, giving $2 \cdot 5$ intersection points. Similarly, each 3-diagonal produces $3 \cdot 4$ intersection points.

So we obtain a total of $9(1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4)$ intersection points. Not quite! Note that for any two diagonals ℓ_1 and ℓ_2 which intersect in the interior of the 9-gon, the intersection point has been counted twice, once when we focus on ℓ_1 and again when we focus on ℓ_2. So we need to divide by 2. The result is 126.

Another way: We solve the problem for all convex n-gons such that no three diagonals meet at a point. Colour four of the vertices of the n-gon blue. The blue vertices determine a unique convex quadrilateral. If we draw all possible lines that join blue vertices, we can see that they determine exactly one intersection point in the interior of the n-gon. Thus there are just as many interior intersection points as there are ways of choosing four vertices, namely $\binom{n}{4}$.

Comment. The first calculation also generalizes to n-gons, and yields an expression for the number of internal intersection points. This expression must be equal to $\binom{n}{4}$, so we obtain an interesting formula as a bonus for counting things in two ways. And counting can be a tricky business. If we count in two different ways and get the same answer, we get some reassurance that the answer is right.

II-10. There are 7 tickets in a box, with the numbers 1 to 7 written on them. The tickets are taken out of the box one at a time. In how many different ways can this be done if at any stage of the process the numbers already taken out have to be a collection of consecutive integers? For example, one of these ways is withdrawal in the order 4356271.

Solution. Imagine that as we take out the tickets in a permitted way, we place them in their natural order. If, for example, we take them out in the order 4356271, we obtain successively 4, 34, 345, 3456, 23456, 234567, 1234567.

Imagine running the videotape backwards: start with 1234567 and put the tickets back into the box. So at any stage we must remove a ticket either from the left end or the right end of the sequence. Any way of doing this is completely specified by a word of length 6 made up of the letters L (for left) and/or R. There are 2^6 such words.

II-11. How many numbers are there of the shape

$$\pm 1 \pm 2 \pm 4 \pm 8 \pm 16 \pm 32 \pm 64 \pm 128 \pm 256 \pm 512?$$

Solution. Let's experiment with shorter sums. There are 2 numbers representable as ± 1. There are 4 numbers representable as $\pm 1 \pm 2$. And there are 8 numbers representable as $\pm 1 \pm 2 \pm 4$. It is natural to conjecture that 2^{10} numbers can be represented by using all the powers of 2 from 2^0 to 2^9 suitably decorated with plus signs and/or minus signs.

There are 2^{10} different ways of making a string of length 10 made up of the "letters" -1 and 1. So in order to show that 2^{10} numbers are representable, it is enough to show that different strings produce different numbers. So suppose that

$$s_0 + 2s_1 + \cdots + 512s_9 = t_0 + 2t_1 + \cdots + 512t_9,$$

where the s_i and t_i are each -1 or 1. Bring everything to one side. We get

$$2(u_0 + 2u_1 + 4u_2 + \cdots + 512u_9) = 0$$

where each u_i is -1, 0, or 1. Then each of the u_i must be 0, meaning that the strings are the same. For if they are not all 0, let u_k be the first non-zero one. Then the left-hand side is divisible by 2^{k+1} but by no higher power of 2, while the right-hand side is divisible by every power of 2.

Another way: There is a more concrete and therefore probably better approach. The numbers $\pm 1 \pm 2 \pm \cdots \pm 512$ are odd and range from -1023 to 1023. We give a *recipe* for expressing any of the 1024 odd numbers in this interval.

Any non-negative integer is a sum of distinct powers of 2 (possibly none). Just keep taking away the largest possible power of 2 until there is nothing left. For example, $355 = 256 + 99$, but $99 = 64 + 35$, $35 = 32 + 3$ and $3 = 2 + 1$ so $355 = 256 + 64 + 32 + 2 + 1$.

If x has been expressed as a sum of distinct powers of 2 chosen from 1, 2, 4, …, 512, then $1023 - x$ is the sum of the powers of 2 that we *didn't* use, so

$$x - (1023 - x) = \pm 1 \pm 2 \pm 4 \pm \cdots \pm 512$$

for some choice of plus and/or minus signs. So we only need to show that any odd integer n between -1023 and 1023 can be expressed as $2x - 1023$ for some x between 0 and 1023. That's easy: just let $x = (n + 1023)/2$. Since n is odd, x is an integer, and as n ranges from -1023 to 1023, x ranges from 0 to 1023.

II-12. How many different numbers do we get by putting parentheses in the following expression in all possible ways?

$$2 \div 3 \div 5 \div 7 \div 11 \div 13 \div 17 \div 19$$

Solution. Instead of working with specific numbers, we work with letters, starting with three letters. Note that $a \div (b \div c)$ simplifies to ac/b while $(a \div b) \div c = a/bc$. So there are 2 possibilities.

Now look at $a \div b \div c \div d$. First form $(a \div b) \div c \div d$. By the three-term case, we can insert further parentheses and make d end up "on top" or "below". Note that a will be on top, while b and c will be below. Start again and form $a \div (b \div c) \div d$. By the three-term case, we can insert further parentheses and make d end up on top or below. Note that a and c will be on top, and b below.

In summary, a ends up on top and b below—that's unavoidable—but c and d can end up on top or below, together or apart, 4 possibilities in all.

For five terms, first form $(a \div b) \div c \div d \div e$, and insert further parentheses. Note that c ends up below, but by the four-term case d and e can end up on top or below, together or apart. Then form $a \div (b \div c) \div d \div e$. Again by the four-term case, d and e can end up anywhere we like, and c ends up on top. In summary, a ends up above and b below, that can't be helped, but c, d, and e can end up on top or below, in every conceivable combination, and there are 8 possibilities.

In *exactly* the same way, we can use the conclusion for the five-term case to show that when there are six terms, the last four can end up on top or below, in every conceivable combination, for a total of 2^4 possibilities. And the conclusion for the six-term case produces in the same way an analysis of the seven-term case, and so on.

If we start with n terms, there are 2^{n-2} possible results. The original question had 8 terms. Because they are distinct primes, different simplified expressions represent different numbers, so the answer is 64.

Comment. It can be visually comforting to take logarithms. If we do that, we end up putting parentheses in expressions of the form

$$a_1 - a_2 - a_3 - \cdots - a_n.$$

The argument doesn't really change, but subtraction is more familiar than division, so things *feel* simpler.

II-13. Twenty people have entered a table tennis tournament. They are to be divided into ten pairs to play in the first round. In how many ways can this be done?

Solution. List the players, say alphabetically. Take the first player in the list, and pair her off with one of the other players. That other player can be chosen in 19 ways. Take the first player who is still unpaired, pair her off with one of the remaining players. That can be done in 17 ways. So the first two pairings can be done in $19 \cdot 17$ ways. Take the first player who remains unpaired, pair her off with .... We conclude the number of ways to do the pairing is

$$19 \cdot 17 \cdot 13 \cdot \cdots \cdot 5 \cdot 3 \cdot 1.$$

(the final 1 is there to make things look nicer).

Another way: We pair up two people, then pair up two more, and so on. The first pair can be chosen in $\binom{20}{2}$ ways. For each such pair, the second pair can be chosen in $\binom{18}{2}$ ways, and so on. So it looks as if there are

$$\binom{20}{2}\binom{18}{2}\binom{16}{2}\cdots\binom{4}{2}\binom{2}{2}$$

ways to do the pairing. That's *not* true. What we have just counted is the number L of ways of pairing people up and *lining up* the pairs in a row. Let P be the number of ways of pairing people up. For every way of pairing people up, there are 10! ways of lining up the pairs in a row. We conclude that $L = 10!P$. So if we take the (correct) expression for L obtained above and divide by 10!, we get P. Simplify. There is a lot of cancellation, and we get $19 \cdot 17 \cdot \cdots \cdot 3 \cdot 1$.

II-14. Explain why in any group of 100 people there are at least two people who have the same number of friends in the group. Assume that if B is a friend of A, then A is a friend of B.

Solution. Suppose first that somebody, say Alphonse, has no friends. Then no one has 99 friends, for no one likes Alphonse. Make boxes labelled 0, 1, 2, ..., 98. Into the box labelled k, put all people who have exactly k friends. There are 99 boxes and 100 people, so some box contains 2 or more people, that is, there are at least two people who have the same number of friends.

Suppose on the other hand that everyone in the group has at least 1 friend. Make boxes labelled 1, 2, 3, ..., 99, and put people into boxes as before. Again there are 99 boxes but 100 people, so some box contains at least 2 people.

Comment. The idea that we used is called the *Dirichlet Box Principle,* or the *Pigeonhole Principle.* It has surprisingly many applications. For a more serious use of the Dirichlet Box Principle, see IX-3.

II-15. A league has eight basketball teams, T_1, T_2, $\ldots$, T_8. This year, each team met every other team twice. For $i = 1, 2, \ldots, 8$ let a_i be the number of games that team T_i won, and b_i the number of games it lost—there are no draws. Explain why

$$a_1^2 + a_2^2 + \cdots + a_8^2 = b_1^2 + b_2^2 + \cdots + b_8^2.$$

Solution. To prove that two numbers are equal, it is enough to show that their difference is 0. We will show that

$$(a_1^2 - b_1^2) + (a_2^2 - b_2^2) + \cdots + (a_8^2 - b_8^2) = 0.$$

Every team played 14 games, so

$$a_i^2 - b_i^2 = (a_i + b_i)(a_i - b_i) = 14(a_i - b_i),$$

and therefore

$$(a_1^2 - b_1^2) + \cdots + (a_8^2 - b_8^2) = 14\left[(a_1 - b_1) + \cdots + (a_8 - b_8)\right].$$

But $a_1 + a_2 + \cdots + a_8 = b_1 + b_2 + \cdots + b_8$, since the total number of wins is the same as the total number of losses. Thus

$$(a_1 - b_1) + (a_2 - b_2) + \cdots + (a_8 - b_8) = 0,$$

and the result follows.

II-16. Let $\lfloor x \rfloor$ denote the greatest integer which is less than or equal to x. How many positive integers n below 11000 satisfy the equation

$$\left\lfloor \frac{n}{2000} \right\rfloor - \left\lfloor \frac{n}{2001} \right\rfloor = 1?$$

Solution. The smallest solution is 2000, the next two are 4000 and 4001, followed by 6000, 6001, 6002, and so on, for a total of 15 below 11000.

II-17. Math classes A and B have the same number of students. There are boys and girls in each class. More students passed in class A than in class B.

Is it possible that a greater percentage of the boys in class B passed than in class A, and also a greater percentage of the girls in class B passed than in class A?

Solution. It *is* possible! Here is an example. Class A has 2 boys and 28 girls; 1 boy and 27 girls passed. Class B has 28 boys and 2 girls; 25 boys and both girls passed.

In A, 28 out of 30 passed, while in B only 27 out of 30 did, so A is "better" than B. But in B, 25 out of 28 boys passed, while in A only 50% of the boys did. So B is better for boys. A similar calculation shows that B is also better for girls.

II-18. In how many different ways can we represent 7 as a sum of positive integers, where the order of summation matters? For example, $1+3+2+1$ is different from $2+3+1+1$.

Solution. There aren't many representations, and they can be counted by careful listing. Work with numbers smaller than 7 to get started. The number 1 has 1 representation, while 2 has 2 of them, and 3 has 4 representations (3, $2+1$, $1+2$, and $1+1+1$). We can continue to experiment, finding that 4 has 8 representations. On the basis of this limited evidence, we might conjecture that n has 2^{n-1} representations.

Given a representation of n, we can get a representation of $n+1$ in two different ways: (i) add 1 on the right and (ii) add 1 to the rightmost number in the representation. For example, starting from the representation $1+3$ of 4, by using (i) we get the representation $1+3+1$ of 5, while using (ii) we get $1+4$.

All representations of $n+1$ arise in this way, for (i) gives the representations of $n+1$ that have a 1 at the right end, and (ii) gives the rest. So $n+1$ has twice as many representations as n. It follows that 4 has 8 representations, 5 has 16, 6 has 32, and finally 7 has 64. In general, n has 2^{n-1} representations.

Another way: The next approach is less efficient, but maybe more natural. Let $f(k)$ be the number of representations of k. How many representations of n start with 1? We get such a representation by following the 1 with any representation of $n-1$, so there are $f(n-1)$ representations of this type.

Similarly, $f(n-2)$ representations start with 2, $f(n-3)$ start with 3, and so on up to $f(1)$ that start with $n-1$. And 1 representation of n has n all by itself. We have proved the recurrence formula

$$f(n) = f(n-1) + f(n-2) + f(n-3) + \cdots + f(2) + f(1) + 1.$$

By direct listing, we saw that $f(3) = 4$, $f(2) = 2$, and $f(1) = 1$. Thus $f(4) = 4+2+1+1 = 8$, $f(5) = 8+4+2+1+1 = 16$, $f(6) = 32$, and $f(7) = 64$. The recurrence formula can be used to prove that in general $f(n) = 2^{n-1}$.

Another way: The following geometric idea is useful in a variety of problems. Imagine seven x's lined up in a row, like this:

$$x \quad x \quad x \quad x \quad x \quad x \quad x.$$

There are six "gaps" between x's. Look at each gap in turn, and decide whether or not to insert a separator. Once the separators are inserted, count the number of x's between gaps. So for example the pattern $x|xxx|xx|x$ corresponds to the representation $7 = 1 + 3 + 2 + 1$. There are just as many representations of 7 as a sum as there are ways of placing separators.

At the first gap there are 2 choices—place a separator or don't. For *each* such choice, there are 2 choices at the second gap, so there are $2 \cdot 2$ ways to decide what happens at the first two gaps. Go on in this way. We conclude that there are 2^6 different ways of placing the separators. The same argument shows that if order matters then n can be expressed in 2^{n-1} ways as a sum of positive integers.

II-19. Find the sum of the three-digit numbers all of whose digits are odd.

Solution. Since there are 5 odd digits, there are 5^3 three-digit numbers all of whose digits are odd. Listing them all and adding up is unappealing. Instead, we *imagine* listing and adding.

There are 25 numbers in the list with last digit 1, 25 with last digit 3, and so on. Thus the sum of all the last digits is $25(1 + 3 + \cdots + 9)$, that is, 625. Similarly, the sum of all the "tens" digits is 625, as is the sum of the "hundreds" digits. The sum of all the numbers is therefore $625 + 625 \cdot 10 + 625 \cdot 100$, that is, 69375.

Another way: The first number is 111 and the last is 999. Note that they add up to 1110. So do 113 and 997. In general, let n be a three-digit number with odd digits a, b, and c. Then the number with digits $10 - a$, $10 - b$, and $10 - c$ also has odd digits, and the sum of the two numbers is 1110.

Thus our numbers can be paired up so that numbers in each pair add up to 1110, with 555 as the only unpaired number. There are 62 pairs, and therefore the sum is $62 \cdot 1110 + 555$. Alternately, the numbers have average value 555, so their sum is $125 \cdot 555$.

II-20. Montreal and Toronto are playing in the baseball World Series. A game can't end in a tie, and the first team to win four games wins the Series. How many different outcomes are possible? Here are a couple: MMMM (Montreal won in four straight) and MMMTTM (Montreal won the first three, Toronto came back to win two, but Montreal won the deciding sixth game).

Solution. List and count the patterns in which *Montreal* wins the Series, then multiply the result by 2.

There is 1 pattern in which Montreal wins four straight. For patterns in which Montreal takes five games to win the Series, we need to produce all five-letter words, made up of the letters M and T, that end in M and have exactly one T among the first four letters. Start listing: TMMMM, MTMMM, ... Where the T goes can be chosen in 4 ways, so there are 4 such words.

For the patterns in which Montreal wins in six, we must make a six-letter word that ends in M and has two T's among the first five letters. So we must *choose* the location of the two T's from the five available places. But 2 places can be chosen from 5 in $\binom{5}{2}$, namely 10 ways. Similarly, there are $\binom{6}{3}$, namely 20 patterns in which Montreal wins in seven. The number of possible outcomes is therefore $2(1+4+10+20)$.

Another way: Again we count the patterns in which Montreal wins the Series and double the result. Think of the seven scheduled game dates. We need to *choose* four of these dates as Montreal wins. Once we have done that, the Series outcome is determined. (The dates not chosen that come before the last chosen date are Toronto wins. The other dates not chosen are games that were not played because the Series was over.) We conclude that the number of ways Montreal can win the Series is $\binom{7}{4}$, namely 35, so there are 70 possible outcomes.

Comment. Which way is better? Perhaps the second way is *too* slick. Like power steering, it deprives one of the feeling of driving. But looking at the problem in two ways reassures us that the count is right, and also brings a nice bonus.

First, we generalize. The greedy owners and television networks have decided that the first team to win $n+1$ games will win the Series. How many different ways can Montreal win? If we count as in the first way, we get

$$\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{2n}{n}.$$

(we wrote $\binom{n}{0}$ instead of 1 to make things look nicer). If we count as in the second way, we get $\binom{2n+1}{n+1}$. Since the two counts are equal, we obtain a pretty identity. Counting things in two different ways can be a powerful technique.

II-21. An integer is "lucky" if its decimal representation has two or more identical neighbouring digits. So 2337 is lucky, as is 11145, but 2464 is not lucky. How many lucky numbers are there from 1000 to 9999 inclusive?

Solution. We could count the lucky numbers directly, but it is easier to count the unlucky numbers. Let's manufacture unlucky numbers by choosing their digits, from left to right. The leftmost digit can be any digit from 1 to 9.

For *each* choice of leftmost digit, there are 9 ways of choosing the next digit, since it can be any digit other than the leftmost one. Thus there are 9^2 ways of choosing the first two digits. For *each one* of these 9^2 choices, there are 9 ways of choosing the next digit, and so on. So there are 9^4 unlucky numbers. Our interval contains 9000 numbers, and therefore there are $9000-9^4$ lucky ones, that is, 2439.

II-22. A test consists of 10 multiple choice questions. On any question, you get 5 points if you answer the question right, 0 points if you answer it wrong, and 1 if you leave it blank. Find the *smallest* positive integer that is impossible to get as a grade.

Solution. One approach is to start at 0, check whether it is a possible grade, then check 1, 2, and so on until we bump into an impossible number. But we can save some time.

Suppose that we get q questions right, and leave b blank. Our grade on the test is then $5q + b$. Note that $0 \leq b \leq 10 - q$. If $q \leq 6$, then b can be as large as 4, so every grade less than or equal to $6 \cdot 5 + 4$ can be achieved. It is also easy to produce 35, 36, 37, and 38. To get 39, we need $q = 7$. But then $b \leq 3$, so 39 is impossible.

Comment. We can generalize slightly without much effort. Suppose that each correct answer is worth a points, where a is an integer greater than 1. The rest of the grading scheme doesn't change, but there are k questions on the test. If n is a positive integer, divide n by a, obtaining quotient q and remainder b. As long as $q \leq k - (a - 1)$, we can achieve a mark of n by having q right, b blank, and the rest wrong. If $q = k - (a - 1) + 1$, we run into trouble: there aren't enough questions to produce a remainder of $a - 1$. The smallest impossible grade turns out to be $ak - a^2 + 3a - 1$.

II-23. Every grade 11 student wrote three exams. Exactly 100 students got an A on one or more exams. Exactly 60 got an A on two or more exams, and 20 got an A on all three. How many A exams were written?

Solution. The students who got three A's account for $3 \cdot 20$ A exams. The $(60 - 20)$ students who got two A's account for $2 \cdot 40$ A exams. Finally, the $(100 - 60)$ students who got one A account for $1 \cdot 40$ A exams. Add up: we get 180.

There were 100 students with one or more A's, 60 with two or more, and 20 with three. Note that $100 + 60 + 20 = 180$. Is this a numerical coincidence? Repeat the argument with letters instead of numbers. Suppose that p students got one or more A's, q got two or more, and r got three A's. Then the r three A's people account for $3r$ A exams. The $q - r$ people who got two A's account for $2(q - r)$ A exams. And finally the $p - q$ people with one A account for $p - q$ A exams. Add up. We get $p + q + r$. It wasn't a coincidence!

Another way: We justify the "coincidence" not by algebra, but by combinatorial thinking, by finding a better way to count the A exams. Imagine that the teacher gives out a red star for every A exam, as follows. First she gives a star to everyone who got at least one A. That's 100 stars. Next she gives an additional star to everyone who got two or more A's. That accounts for 60 more stars. Finally she gives an additional star to everyone who got three A's. That accounts for 20 more stars, so the total number of A's is $100 + 60 + 20$.

Comment. Both solutions generalize readily. If no more than n exams were written by anyone, and p_1 people got one or more A's, p_2 got two or more, p_3 got three or more, and so on, with p_n getting exactly n A's, then there are $p_1 + p_2 + \cdots + p_n$ A papers.

II-24. You have four attractive brass digits, two 4's, one 5, and one 7. How many different house numbers can you form by using some or all of these digits?

Solution. We should count in a systematic manner, so as not to miss anything. There are 3 possible one-digit house numbers. What about two-digit numbers? If we don't repeat a digit, the first digit can be chosen in 3 ways, and for *each* such way there are 2 ways of choosing the second digit, for a total of 6 numbers. Add to that the house number 44 and we have 7.

Now count the three-digit numbers. There are 6 numbers with all digits different. If 4 is to repeat, there are 3 places for the other digit, and that digit can be chosen in 2 ways. So there are 12 three-digit numbers.

We can count the four-digit numbers in a similar way. But it is easier to notice that every three-digit number can be completed in only one way to a four-digit number by putting the unused digit on the right. So there are also 12 four-digit numbers. Add up: the total is 34.

Another way: There are other ways of counting systematically. For example, count the house numbers that begin with 5. This is equivalent to counting the numbers we can form using only 4, 4, and 7, where we are allowed to use the "empty" number with no digits. A quick scan shows that there are $1+2+3+3$ numbers. Similarly, there are 9 numbers that begin with 7. The situation with 4 is a little different, but the same sort of analysis shows that there are $1+3+6+6$ numbers. So the total is 34.

II-25. One hundred people were asked to list all the kinds of ice cream they liked. Three times as many people had vanilla on their list as had strawberry. Twelve people listed both vanilla and strawberry, and twenty people listed neither. How many people had strawberry on their list?

Solution. Let's "guess" that 12 people like strawberry. Then 36 people like vanilla. That includes all the people who like strawberry. When we add in the 20 who like neither, we get 56 people, 44 short of the 100 who were questioned.

So we need to add people who like strawberry but not vanilla. For each such person, we need to add three people who like vanilla but not strawberry, for a total of 4 people. Since we are short of 100 by 44, we need to have 11 people who like strawberry but not vanilla. Thus $12 + 11$ people like strawberry.

Another way: Let s be the number of people who like strawberry. Then $3s$ like vanilla. There are $100 - 20$ people who like strawberry or vanilla or both. Of these, $3s-12$ like vanilla alone, 12 like both vanilla and strawberry, and $s-12$ like strawberry alone. It follows that $(3s-12)+12+(s-12) = 80$, so $s = 23$.

Comment. A nicely labelled Venn diagram should be drawn. As we go through the calculations, we can insert the numbers or expressions in the appropriate regions. If there is no picture, even the short argument given above can seem like a jumble of symbols and digits.

II-26. Pizzas have just been delivered for a Math Club event. Each pizza has exactly three different toppings chosen from bacon, ham, mushroom, onion, and red pepper. Exactly 12 pizzas have bacon as one of the toppings and 27 pizzas have ham. No pizza has both bacon and ham—that would be too piggy. There is mushroom on 28 pizzas, onion on 29, and red pepper on 30. How many vegetarian pizzas are there?

Solution. Since each pizza has three different toppings, if we add together the number of pizzas that have bacon, the number that have ham, the number that have mushroom, and so on, the result is three times the total number of pizzas. So there are $(12 + 27 + 28 + 29 + 30)/3$ pizzas. Of these 42 pizzas, exactly 12 have bacon, 27 have ham, and there is no overlap. Thus 39 of the 42 pizzas contain some meat; there are 3 vegetarian pizzas.

Comment. It is not immediately obvious that the situation described is indeed possible. How many different ways are there to order pizzas that satisfy the above constraints?

II-27. A digital clock shows only the hour and the minute. For what fraction of the time between 12:00 noon and 12:00 midnight is there at least one 1 showing on the display? Assume that transitions are instantaneous.

Solution. There are 720 minutes in a complete cycle of the clock. At least one 1 is showing for the four hours from 12:00 to 2:00 and 10:00 to 12:00, that is, for 240 minutes. During the remaining 480 minutes, the minutes display lies between 10 and 19 one-sixth of the time, giving an additional 80 minutes. In the remaining 400 minutes, the second digit of the minutes display is 1 one-tenth of the time, giving an additional 40 minutes. The total is 360, exactly 1/2 of the time.

II-28. The *integer lattice* consists of the points in the plane whose coordinates are both integers. Let $A = (-20, 6)$ and $B = (100, 150)$. How many points of the integer lattice lie on the line segment AB, including both ends?

Solution. It's always nicer when one of the points is the origin, so push everything to the right by 20 and down by 6. Then A is pushed to the origin O, while B is pushed to P, where $P = (120, 144)$. Lattice points go to lattice points, so we now ask how many lattice points there are on OP.

Let $X = (x, y)$, with $X \neq O$. Then X is on the line OP if and only if the slope of OX is equal to the slope of OP, that is, if and only if $x/y = 120/144 = 5/6$. Because the fraction $5/6$ is in lowest terms, if x and y are to be integers, we must have $x = 5t$, $y = 6t$ for some integer t. Let t go from 1 to 24. That takes care of everything but O, while $t = 0$ takes care of O, so there are 25 lattice points.

Comment. The first step, "pushing" A to the origin—equivalently, choosing a new origin A and new coordinate axes—is unnecessary, but things go more smoothly if we do it. Instead of working with points A and B, we end up working with the vector $B - A$.

Let $P = (a, b)$ be a lattice point other than O. Using the same argument as in the concrete case $a = 120$, $b = 144$, we can show the line segment OP has $d + 1$ lattice points, where d is the greatest common divisor of a and b.

II-29. Imagine listing, in order, the positive integers that can be represented as a sum of one or more distinct powers of 3. The first number in the list is 1, since $1 = 3^0$; the second is 3 and the third is 4. Find the 80-th number in the list.

Solution. The list consists of numbers of the shape

$$3^0a_0 + 3^1a_1 + 3^2a_2 + \cdots + 3^na_n,$$

where each a_i is 0 or 1 and not all the a_i are 0. These are just the positive integers that can be written in base 3 using only the "digits" 0 and 1, and avoiding 2. Since "digit" is etymologically connected with "decem," Latin for ten, we should use another word, like "trit," which goes nicely with the standard term "bit."

Let N be the 80-th number of this shape. There are $2^6 - 1$ numbers of the right shape that have 6 or fewer trits, and $2^7 - 1$ with 7 or fewer, so N has 7 trits. Since 3^6 is the first 7 trit number in the list, it follows that 3^6 is the 64-th number. Thus $N = 3^6 + x$, where x is the 16-th number in the list. Similar reasoning shows that $x = 3^4$, so $N = 810$.

Comment. The calculation can be generalized. Let N be the k-th number in the list. Write k in binary notation, so let $k = a_0 + 2a_1 + 2^2a_2 + \cdots$, where each a_i is 0 or 1. Then $N = a_0 + 3a_1 + 3^2a_2 + \ldots$. The same idea works for sums of distinct powers of b where b is any integer greater than 1.

II-30. How many combinations of flavours are possible if we have ingredients of 6 different tastes, namely sweet, pungent, astringent, sour, salty, and bitter?

Solution. Count first the dishes that have a single ingredient. Obviously there are 6 possibilities. Then there are the dishes with two ingredients. We must select 2 ingredients from the 6 available. We can list all of the possibilities, or perhaps imagine the ingredients lined up on a shelf in jars beside the fire, and call them 1, 2, 3, 4, 5, and 6. If we use flavour 1, there are 5 ways of selecting the other flavour. If we use flavour 2 but not 1, there are 4 ways of selecting the other flavour. If we use 3 but not 1 or 2, there are 3 ways, and so on, so in total we get $5+4+3+2+1$, namely 15.

What about 3 ingredients? The counting goes in the same way. If we use ingredient 1, the remaining 2 ingredients can be selected in $(4 + 3 + 2 + 1)$ ways, and so on. We get a total of 20 ways.

What about 4 ingredients? There are just as many ways of picking 4 ingredients as there are of picking 2 (whenever 4 are chosen, 2 are chosen to be left out). So there are 15 ways. And there are 6 ways of picking 5, and only 1 way of picking 6. The total is 63.

We have used only basic counting techniques. Equivalently, we can write down the answer as

$$\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}.$$

Another way: Imagine the ingredients lined up on a shelf. The cook makes a dish by saying Yes or No to the first ingredient, then Yes or No to the second, and so on. So the different combinations available can be represented by "words" of length 6 made up of letters chosen from Y and N. The first letter can be chosen in 2 ways, and for each of these ways the second letter can be chosen in 2 ways, so the first two can be chosen in $2 \cdot 2$ ways, and so on. We get a total of 2^6. Not quite! Among these 2^6 words is the word "NNNNNN," which most cooks would disallow. A mathematician might call it the empty dish.

Comment. This problem is taken from Bhāskara's *Līlāvatī* (ca. 1150). The same question had been discussed by the physician Sushruta around −600.

In both India and Medieval Europe, scholars computed how many combinations of attributes various gods could have. Mathematics has applications beyond commerce or science.

II-31. The set $\{1, 2, 3, 4, 5, 6\}$ has 2^6 subsets. For any such subset A, let $\mathcal{S}(A)$ be the sum of all the numbers in A. Find the sum of all these sums.

Solution. The number 2^6 is relatively small, so we could list all 64 subsets, add up the numbers in each, then add again. But this is unpleasant, and would become painful if we were dealing with the 2^{12} subsets of $\{1, 2, 3, \dots, 11, 12\}$. So we look for a shortcut.

Every number from 1 to 6 occurs in exactly 2^5 subsets, so it gets added in 2^5 times. Thus the required sum is

$$2^5(1 + 2 + 3 + 4 + 5 + 6).$$

Another way: For any subset A of $\{1, 2, 3, 4, 5, 6\}$, let A' be the collection of numbers from 1 to 6 that are *not* in A. The set A' is usually called the *complement* of A. Note that the complement of the complement of A is A. So the 2^6 subsets can be divided into 2^5 complementary pairs. Note that $\mathcal{S}(A) + \mathcal{S}(A')$ is simply the sum of the numbers from 1 to 6, that is, 21. So the sum of all the $\mathcal{S}(A)$ is $21 \cdot 2^5$.

II-32. The positive integer n has the property $\mathcal{D}$ if $\lfloor \sqrt{n} \rfloor$ is a factor of n ($\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x). How many of the integers from 1 to 10000 have the property $\mathcal{D}$?

Solution. Any perfect square d^2 has the property $\mathcal{D}$, for $\lfloor \sqrt{d^2} \rfloor = d$, so the 100 perfect squares from 1^2 to 100^2 have property $\mathcal{D}$. Are there any others?

Suppose that n has the property $\mathcal{D}$. Let $d = \lfloor \sqrt{n} \rfloor$; then $n = d(d + k)$ for some non-negative integer k. We determine the possible values of k.

Since $d = \lfloor \sqrt{n} \rfloor$, we must have $n < (d + 1)^2$, that is, $n \le d^2 + 2d$. So n has property $\mathcal{D}$ if and only if $k = 0$, 1, or 2. The case $k = 0$ has been dealt with. For each of $k = 1$ and $k = 2$, there are 99 possibilities, from $d = 1$ to $d = 99$.

It intuitively reasonable—and not very hard to verify—that no number corresponds simultaneously to two different k. We conclude that 298 of the numbers from 1 to 10000 have the property $\mathcal{D}$.

II-33. In Figure 2.4, the chessboard on the right is made up of sixty-four 1×1 squares. The *tromino* on the left is made up of three 1×1 squares. In how many different ways can this tromino be placed on the chessboard

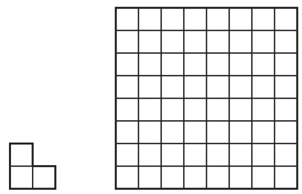

Figure 2.4: Placing a Tromino on a Chessboard

so that it covers 3 squares of the chessboard exactly? Generalize.

Solution. We list all possibilities and then count. Concentrate on the position of the central square of the tromino. This square could be placed on any one of the 4 corners squares. Then the position of the other two squares is completely determined, so we get 4 possibilities.

The central square could be on an edge square which is not a corner square. There are 24 of these, and for each of them the tromino can be placed in 2 ways, giving 48 possibilities. Finally, the central square could be on one of the 36 "inside" squares. For each of these, the tromino can be placed in 4 different ways, giving 144 possibilities. The total is 196.

Repeat the argument for an $n \times n$ square. We get the answer

$$4 + 2(4n - 8) + 4(n - 2)^2, \qquad \text{that is,} \qquad 4(n - 1)^2.$$

What a nice-looking answer! We might as well tackle the $m \times n$ rectangle. The same argument gives, after we simplify, $4(m-1)(n-1)$. Such nice-looking answers deserve a less clumsy argument, they shout that we should look for

Another way: Concentrate on the inner corner point of the tromino. This point can be placed on any one of the $(m-1)(n-1)$ inner points of the $m \times n$ generalized chessboard. Once the point is placed, the tromino itself can be rotated into 4 positions. So there are $4(m-1)(n-1)$ ways of placing the tromino.

Another way: We sketch an approach, based on the game *Tetris,* that is visually effective but awkward to describe in words.

Place the tromino so that it is wedged into the top left corner of the chessboard. There are 4 such starting positions. Now let it fall like a *Tetris* piece. Every starting position gives rise to 7 tromino placements, for a total of $4 \cdot 7$.

But we could have moved the tromino one square to the right before letting it fall, or two squares to the right, up to 6 squares to the right. Thus there are $(4 \cdot 7) \cdot 7$ ways of placing the tromino. The same argument works generally.

Comment. There are stranger solutions—here is one that a topologist might find natural. In the first solution, there were three types of squares. Each type required separate treatment; it would be nice to avoid that. Imagine that the chessboard is made of thin plastic, and make it into a cylinder, *overlapping* the top and bottom rows of squares. Then make the cylinder into a doughnut, overlapping the left and right "columns" of squares. There is now symmetry, no position is special, and there are $(m-1)(n-1)$ squares. If the center square of the tromino is placed on one of these, the rest of the tromino can be placed in 4 ways, for a total of $4(m-1)(n-1)$. Unroll the doughnut. We get all the ways of placing a tromino on the chessboard!

II-34. How many ordered triples (x, y, z) of integers are there such that $xyz = 10000$? Note that $(1, -5, -2000)$ is not the same as $(1, -2000, -5)$.

Solution. Let's first count the triples of *positive* integers. If there are N, then there are $4N$ triples of *integers,* because for any triple (x, y, z) of positive integers, we also want the 3 triples that come from decorating two of x, y, and z with $-$ signs.

Note that $10000 = 2^4 \cdot 5^4$. So the triples (x, y, z) of positive integers are given by $x = 2^p 5^s$, $y = 2^q 5^t$, $z = 2^r 5^u$ where p, q, r, s, t, u are non-negative integers and $p + q + r = 4$, $s + t + u = 4$.

Maybe $p = 0$; then q ranges from 0 to 4, and once q is chosen r is determined. Or $p = 1$, in which case q ranges from 0 to 3, and so on. Thus there are altogether $5+4+3+2+1$, that is, 15, choices for (p, q, r). For each of these, there are 15 ways of choosing (s, t, u). So there are altogether 225 triples of positive integers (x, y, z) such that $xyz = 10000$, and therefore 900 triples in all.

II-35. How many ordered pairs (x, y) of integers satisfy the equation

$$\sqrt{x} + \sqrt{y} = \sqrt{2000}?$$

Solution. Rewrite the equation as $\sqrt{y} = \sqrt{2000} - \sqrt{x}$ and square both sides, obtaining

$$y = 2000 - 2\sqrt{x}\sqrt{2000} + x.$$

The original equation is satisfied if and only if $x \le 2000$, $\sqrt{8000x}$ is an integer, and y is chosen as above. But $8000x = (40^2)(5x)$, and therefore $\sqrt{8000x}$ is an integer if and only if x is of the form $x = 5n^2$ for some integer n. Since $0 \le x \le 2000$, n must range from 0 to 20, and there are 21 solutions.

II-36. How many ways are there to give change for a fifty-dollar bill using only two-dollar coins, one-dollar coins, and quarters?

Solution. We can use 25 two-dollar coins, or 24, or 23, ..., or 1, or 0. If we use 25 two-dollar coins, the job is done. If we use 24 two-dollar coins, then we need to produce \$2 from one-dollar coins and quarters, and there are 3 options (2 one-dollar coins, 1, or 0). If we use 23 two-dollar coins, there remain 5 options, any number of one-dollar coins from 4 down to 0. Continue all the way down to 0 two-dollar coins. In that case, there are 51 options left.

It follows that there are $1 + 3 + \cdots + 49 + 51$ ways of making change. The arithmetic progression has sum 26^2, so there are 676 ways. The answer has such an attractive shape that it is probably worthwhile to explore

Another way: We illustrate the general idea by counting the number of ways of giving change for a ten-dollar bill. Draw a 6×6 square array of dots, as in Figure 2.5. There is only 1 way of making change if we use 5 two-dollar coins, represented by

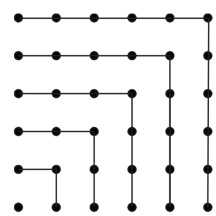

Figure 2.5: Making Change

the dot at the bottom left. If we use 4 two-dollar coins, there are 3 ways, represented by the three linked dots. Go on in this way. The total number of ways of making change is the number of dots in the picture, clearly 6^2.

Basically the same picture shows that there are $(n+1)^2$ ways of making change for a $2n$-dollar bill. A closely related picture shows that there are $(n + 1)(n + 2)$ ways to make change for a $2n + 1$-dollar bill. The coin-changing problem can be solved without pictures, but it provides an opportunity to make connections.

Comment. In the same way, we can count the ways of giving change for a fifty-dollar bill using one-dollar coins, quarters, and dimes, except that since a dime does not "divide" a quarter, the counting goes a little differently—the number of quarters must be even. Problems that involve four types of coin are also accessible.

Variants of the problem can be given even at the grade 5 level. We can ask how many ways there are of giving change for a ten-dollar bill using only one-dollar coins and/or two-dollar coins. Most students can find the answer, although their list is often unsystematic. Once they learn to make a systematic list, they can be asked about making change for a twenty-dollar bill, a fifty-dollar bill, and so on. The advantages of organization become clear: fairly quickly they can count without writing down a full list.

II-37. How many seven-digit numbers are divisible by 321 and have 321 as their last three digits?

Solution. The seven-digit numbers that end in 321 have shape $1000n + 321$, where $1000 \leq n \leq 9999$. And $1000n + 321$ is divisible by 321 if and only if n is divisible by 321.

If a and N are positive integers, then the number of positive multiples of a that are less than or equal to N is just $\lfloor N/a \rfloor$, the largest integer less than or equal to N/a. On the calculator, evaluation is simple: just divide and throw away the part after the decimal point. In our case, the number of possibilities is $\lfloor 9999/321 \rfloor - \lfloor 1000/321 \rfloor$, that is, 228.

II-38. Three movie fans were talking about the movies they saw last year at the local theatre. The first had seen 50, the second 60, and the third 70. Between them they saw every one of the 99 movies shown at the theatre last year. Call a movie *very popular* if all three saw it, and *unpopular* if only one of them saw it. How many more movies were unpopular than very popular?

Solution. Let x be the number of movies seen by only one person, y the number seen by two, and z the number seen by all three. We want $x - z$. We know that $x + y + z = 99$. Suppose that a ticket costs \$1. Then the theatre made $x + 2y + 3z$ from the three movie fans, and this is $50 + 60 + 70$, so $x + 2y + 3z = 180$. Eliminate the variable y using our two equations. We get

$$x - z = 2(x + y + z) - (x + 2y + 3z) = 198 - 180 = 18.$$

Comment. The information provided determines $x - z$ uniquely, but does not determine any of the quantities x, y, or z. We could ask "What are the possible numbers of very popular movies?" It turns out that z can be any integer from 0 to 40.

II-39. Divide 500 one-dollar coins between 9 bags in such a way that if you need to pay a bill for any integer number of dollars up to 500 you can do so exactly by just handing over some of the bags.

Solution. Any positive integer up to $2^{n+1} - 1$ can be expressed as a sum of distinct powers of 2 taken from 1, 2, 4, ..., 2^n. How can we see that? We can check directly that every positive integer up to 7 can be expressed as a sum of distinct numbers taken from 1, 2, 4. But then (add 8 or not) it follows that every positive integer from 1 to 15 can be expressed as a sum of distinct numbers taken from 1, 2, 4, 8, and therefore (add 16 or not) every positive integer up to 31 can be expressed as a sum of distinct integers taken from 1, 2, 4, 8, 16, and so on.

Thus any positive integer up to 511 is a sum of distinct numbers taken from 1, 2, 4, ..., 256, so bags of 1, 2, ..., 256 should do the job. Unfortunately that uses 511 coins.

A small modification will work. Use bags of 1, 2, ..., 128. That takes care of everything up to 255. Put the remaining 245 coins into the ninth bag. Note that 245, 246, ..., 255 end up having two different representations. There are several other solutions. For example we could use 1, 2, ..., 64, 127. That takes care of everything up to 254. Put the remaining 246 coins into the ninth bag.

II-40. How many twenty-letter "words" can be made with ten A's and ten B's if no two A's can be next to each other? Generalize.

Solution. We can list the words—there aren't many. In general, let $F(n)$ be the number of $2n$-letter words made up n A's and n B's, where no two A's can be next to each other. It is clear that $F(1) = 2$. Quickly we find that $F(2) = 3$ and $F(3) = 4$. On the basis of this limited evidence, let's conjecture that $F(n) = n+1$.

Let U be a word of the type we are considering, and look at the letters that make up U, starting from the left. Maybe U has shape $(AB)^n$, where by $(AB)^n$ we mean n repetitions of the two-letter word AB. If U doesn't have this shape, it must have shape $(AB)^s V$, where $0 \le s < n$ and the word V begins with B.

But if V begins with B, then V must have shape $(BA)^t$ for some integer t. For if in V a B were ever followed by a B, the B's would be "two ahead," and the A's could only catch up if there were two successive A's, which isn't allowed.

We conclude that our words all have shape $(AB)^s(BA)^t$, where $0 \le s \le n$ and $s+t = n$. Conversely, it is easy to see that any word of this shape satisfies our conditions. Since s can be chosen in $n+1$ ways, it follows that $F(n) = n+1$.

Another way: Take 10 (or n) A's and place them in a row like this:

$$A \quad A \quad A \quad A \quad A \quad A \quad A \quad A \quad A \quad A.$$

Now place the B's. We need to put a B in each of the $n-1$ gaps between the A's. We might decide to put an extra B into one of these gaps. There are $n-1$ ways to decide where that extra B goes. Or we might use just one B in each gap, in which case the last B goes in front or at the end. Thus there are $(n-1)+2$ ways to construct the word.

Comment. If we are feeling in a playful mood, we might observe that in fact $F(0) = 1$, for there is exactly one word with no letters, the *empty word*, and certainly no two A's are next to each other in the empty word. The notion of an empty word is useful when we study the algebra of words over an alphabet.

II-41. Find the sum of all the numbers from 1 to 2000 that are divisible by neither 4 nor 5.

Solution. Find the sum of all numbers from 1 to 2000. Subtract the sum of the numbers that are divisible by 4, that is, subtract $4(1+2+\cdots+500)$; subtract also $5(1+2+\cdots+400)$. We have taken away the multiples of 20 twice, so add back $20(1+2+\cdots+100)$. To complete the calculation, use the formula for the sum of an arithmetic progression. Unpleasant!

Another way: Note that n is divisible by neither 4 nor 5 if and only if $2000-n$ is divisible by neither 4 nor 5. Pair off 1 and 1999, then 2 and 1998, 3 and 1997, 6 and 1994, and so on. Each pair adds up to 2000.

Now find the number of pairs. The smaller number is less than 1000, so count the numbers from 1 to 1000 which are divisible by neither 4 nor 5. There are 250 divisible by 4, 200 divisible by 5, and 50 divisible by both. So there are $1000 - (250+200-50)$ divisible by neither. It follows that our sum is 1200000.

II-42. A point in the plane with integer coordinates is called a *lattice point.* How many lattice points are there inside or on the circle $x^2 + y^2 = 16$?

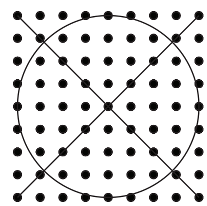

Figure 2.6: Counting Lattice Points

Solution. It is a good idea to draw a picture like Figure 2.6. The two axes and the lines $y = x$ and $y = -x$ have been drawn in order to point out symmetries. By taking advantage of the symmetries we can count quickly. But the answer can also be reached without a picture.

The pair (x, y) is included if $x^2 + y^2 \leq 16$. The numbers less than or equal to 16 that are expressible as the sum of two squares are 0, 1, 2, 4, 5, 8, 9, 10, 13, and 16. The number 0 can only be expressed in one way as the sum of two squares. The numbers 1, 4, 9, and 16 each can be expressed in four ways. For example, $9 = 0^2 + (\pm 3)^2 = (\pm 3)^2 + 0^2$. The numbers 2 and 8 also each can be represented in four ways as the sum of two squares. (A rotation through 90° takes us from one representation to another.)

The numbers 5, 10, and 13 each can be represented in eight ways, one in each of the eight pie slices of Figure 2.6. So there are $1 + 4 \cdot 4 + 2 \cdot 4 + 3 \cdot 8$, that is, 49 lattice points inside the circle or on its boundary.

Comment. Let $f(r)$ be the number of lattice points inside or on $x^2 + y^2 = r^2$. We have just shown that $f(4) = 49$. There is no nice formula for $f(r)$, but a considerable amount is known about this function.

For any lattice point P inside or on the circle, colour in red the 1×1 square that has P as its Southwest corner. The red region has area $f(r)$. This region roughly coincides with the circle, and the red stuff outside the circle has more or less the same area as the part inside the circle which isn't red. So it is reasonable to expect that $f(r)$ is roughly πr^2. When $r = 4$, πr^2 is about 50.2, remarkably close to 49.

With some work, we can *prove* that $f(r)/(\pi r^2)$ is nearly equal to 1 when r is large. A considerable amount of research effort has gone into estimating $|f(r) - \pi r^2|$. Much is known, but very difficult questions remain.

II-43. In how many ways can we select two squares on a chessboard if the squares can't be in the same row or the same column?

Solution. Let C_1 be the leftmost column of the chessboard, let C_2 be the next column, and so on up to C_8. The leftmost square chosen is in C_1 or in C_2, and so on up to C_7.

First count how many choices there are in which the leftmost square is in C_1. Since any column has 8 squares, there are 8 ways to choose this square. Once the square is chosen, there are 7 columns available for the other square, and for each column there are 7 rows in which to put that square, a total of $8 \cdot 7 \cdot 7$. Similarly, there are $8 \cdot 6 \cdot 7$ patterns in which the leftmost square is in C_2. Continue all the way to C_7 and add up. There are

$$(8)(7)(7 + 6 + \cdots + 1),$$

that is, 1568 ways to select the two squares.

Another way: We need to choose two columns. That can be done in $\binom{8}{2}$ ways, that is, in 28 ways. And for every choice of two columns we need to choose two rows. That can also be done in 28 ways. So the number of possible choices is $28 \cdot 28$, that is, 784.

Another way: The first square can be chosen in 64 ways. For each such choice, there are 15 squares forbidden to the second square, and therefore 49 allowed squares. That gives $64 \cdot 49$, namely 3136 choices.

We have a problem now: three different methods, three different answers! In fact, the last two methods are wrong, but fixable.

The second solution is incorrect. One of the column pair–row pair choices that we counted among the 784 was columns 1 and 3, rows 4 and 7. But there are *two* choices of squares that are consistent with this choice of pairs: (i) one square in column 1, row 4 and the other in column 3, row 7 and (ii) one square in column 1, row 7 and the other in column 3, row 4. *Each* of the 784 column pair–row pair choices gives rise to two choices of squares, so there are $2 \cdot 784$ such choices.

The third solution is also incorrect. The problem is that each possibility was counted twice, for we had no right to refer to a "first" and "second" chosen square. So the answer 3136 should be divided by 2.

Comment. The third solution is the correct approach to a different problem, namely "In how many ways can we place a white rook and a black rook on a bare chessboard so that they don't attack each other?" There are indeed 64 places for the white rook, and for each of these 49 places for the black rook.

It is all too easy to make errors in counting. Each of the two incorrect solutions was plausible-sounding.

II-44. How many ordered triples (a, b, c) of positive integers are there with $a < b < c$ and $a + b + c = 101$?

Solution. Maybe $a = 1$. That leaves 100 to be split between b and c. We need $2 \le b < c$, so b can travel from 2 to 49 and there are 48 solutions. Maybe $a = 2$. Then 99 must be split between b and c, and $3 \le b < c$, so b can travel from 3 to 49, and there are 47 solutions. Maybe $a = 3$. That leaves 98 to be split between b and b, with $4 \le b < c$. There are 45 solutions.

Let a be odd, with $a \le 31$. Then $(101 - a)$ must be split between b and c, with $a + 1 \le b < c$. So b can travel from $a + 1$ to $(101 - a)/2 - 1$, and there are $(101 - a)/2 - a - 1$ solutions. In the same way, we can show that if a is even, with $a \le 32$, there are $(101 - a - 1)/2 - a$ solutions.

We could deal with odd and even numbers separately. But it is convenient to combine terms. Let a be even. The total number of values of b for $a - 1$ and a together is $101 - 3a$. Add up, with a going from 2 to 32 through even values. We could add the 16 terms by hand, but note that we have an arithmetic progression with first term 95 and last term 5, so it has sum $16(95 + 5)/2$, that is, 800.

Another way: Drop temporarily the condition $a < b < c$. If $a = 1$, there are 99 choices for b, if $a = 2$ there are 98, and so on. So there are $1 + 2 + \cdots + 99 = 4950$ possibilities. Since 3 is not a factor of 101, the entries can't all be equal. Count the triples where two entries are equal. The equal entries can be anything from 1 to 50. Once that number is chosen, *where* the third entry goes can be chosen in 3 ways, giving 150 triples.

Thus there are 4800 triples with all entries distinct. But each triple (a, b, c) with $a < b < c$ gives rise to 6 ordered triples of distinct integers, so there are 4800/6 triples with $a < b < c$.

The count of 4950 can be found another way. Imagine 101 books on a long shelf. Then there are 100 "gaps" between books. Insert a file card into two of these gaps, let a be the number of books to left of the first card, b the number between the two cards, and c the number to the right of the second card. We have expressed 101 as a sum of three positive integers. There are just as many ways to do this as there are to *choose* the places where the file cards go, namely $\binom{100}{2}$.

II-45. (a) Let $N = 100$. How many positive integers less than N have digit sum equal to 9? (b) Repeat with $N = 1000$ (c) Repeat with $N = 10000$.

Solution. Depending on the context, it may be convenient to write a number like 72 as 072, or 0072. That doesn't affect the sum of the digits but makes for greater symmetry.

(a) The list is short: 09, 18, 27, ..., 90, one in each decade. There are 10.

(b) The first digit can be any of 0, 1, 2, ..., 9. If the first digit is 0, we need to append a number less than 100 with digit sum 9. By part (a), there are 10 of these. If the initial digit is 1, we need to append a number less than 100 with digit sum 8. By making a simple list, we find there are 9 of these. The pattern continues. There are $10 + 9 + \cdots + 1$, that is, 55 numbers.

(c) The reasoning of (b) shows that for $k = 0, 1, \ldots, 9$ a digit sum of k occurs for $(k + 1)(k + 2)/2$ numbers less than 1000. But any number less than 10000 can be made by taking a digit from 0 to 9 and appending a number less than 1000. So the answer can be found by summing $(k + 1)(k + 2)/2$ as k goes from 0 to 9. We could develop some theory (see VI-51) or simply add the 10 numbers, which is undoubtedly faster! The sum is 220.

Another way: There is a more elegant approach which we illustrate with 10000, but the idea is general. Place $9 + 3$ white pebbles in a row, then choose three of them to paint black. The black pebbles divide the white into four groups, some possibly empty. Write down one after the other the number of white pebbles in each group. We have made a four-digit number with digit sum 9. So the answer is $\binom{12}{3}$.

II-46. How many possible yearly calendars are there?

Solution. There are two kinds of year: ordinary and leap. For each kind, the calendar is completely determined if we know the day of the week January 1 falls on. So there are no more than $7 + 7$ calendars. That doesn't *quite* answer the question: we must verify that each of these 14 calendars can occur.

If an ordinary year begins on a certain day of the week, then the next year begins on the next day of the week. This follows from the fact that 364 is divisible by 7. If a leap year begins on a certain day, the next year begins two days of the week later.

Call the day that 2001 started Day 1 (it happens to be Monday). Then 2002 starts on Day 2, 2003 on Day 3, 2004 on Day 4, 2005 on Day 6 (note the jump), 2006 on Day 7, and then back to Day 1. Note that every four years, the day of the week appears to advance by 5. We get the following (broken up into groups of four for convenience):

$$1234 \quad 6712 \quad 4567 \quad 2345 \quad 7123 \quad 5671 \quad 3456.$$

Then things begin all over again with 1234. Now we can see that ordinary years can begin with any of 1, 2, ..., 7. So can leap years: the fourth number in the group takes on the values 4, 2, 7, 5, 3, 1, 6. There are indeed 14 different calendars.

Comment. It looks as if calendars travel through a 28 year cycle. That's not quite true, for 1900, 2100, and in general years divisible by 100 but not by 400 are not leap years. It turns out that the number of days in 400 years is a multiple of 7, so the true cycle length is 400.

How long do we need to wait until we can recycle a leap-year calendar? From our list, it looks as if we have to wait precisely 28 years. That's usually true, but because 2100 is not a leap year, it turns out that the 2096 leap year calendar will be reusable in 2108.

II-47. Let $S(n)$ be the number of perfect squares with exactly n digits. How does $S(n+1)/S(n)$ behave for large n?

Solution. Let $F(n)$ be the number of perfect squares with n *or fewer* digits. There are $\lfloor\sqrt{M}\rfloor$ perfect squares from 1 to M ($\lfloor x\rfloor$ is the largest integer which is less than or equal to x). Thus $F(n) = \lfloor\sqrt{10^n - 1}\rfloor$.

In particular, $F(n)$ is quite close to $\sqrt{10^n}$. More precisely, $F(n) = 10^{n/2} - e_n$, where e_n is an "error" term which is always positive and never greater than 1. Now

$$\frac{S_{n+1}}{S_n} = \frac{F_{n+1} - F_n}{F_n - F_{n-1}} = \frac{10^{(n+1)/2} - 10^{n/2} - e_{n+1} + e_n}{10^{n/2} - 10^{(n-1)/2} - e_n + e_{n-1}}.$$

Divide top and bottom of the above expression by $10^{(n-1)/2}$. The numerator now has shape $10 - 10^{1/2}$ plus an error term which is negligibly small when n is large—n doesn't even have to be particularly large!—while the denominator is, apart from a negligible error term, equal to $10^{1/2} - 1$. Simplify. For large n, $S(n+1)/S(n)$ is very close to $10^{1/2}$.

II-48. When $(x+1)(x+2)(x+3)\cdots(x+10)$ is expanded out, we get a polynomial $P(x)$ of the form $a_0x^{10}+a_1x^9+\cdots+a_9x+a_{10}$. Note that $a_0 = 1$ and $a_{10} = 10!$. Find $a_0 + a_2 + a_4 + \cdots + a_{10}$.

Solution. We could expand and add, but there is an easier way. Note that $P(1) = a_0 + a_1 + \cdots a_{10} = 11!$. Also, $P(-1) = a_0 - a_1 + \cdots + a_{10} = 0$. It follows that

$$a_0 + a_2 + \cdots + a_{10} = \frac{P(1) + P(-1)}{2} = \frac{11!}{2}.$$

Comment. The idea has many uses. For example, the Binomial Theorem says that if n is a positive integer, then

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

By substituting $x = 1$ and $x = -1$ into the above formula, we obtain

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n \quad \text{and} \quad \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n\binom{n}{n} = 0.$$

II-49. How many ways are there to arrange the ten digits from 0 to 9 in a row so that 1 is (somewhere) to the right of 0, 3 is to the right of 2, 5 to the right of 4, and so on. One such arrangement is 6274853019.

Solution. Make 10 slots to hold the digits. We pick two slots to hold 0 and 1. That can be done in $\binom{10}{2}$ ways. Once we have picked the two slots, we have no choice about who goes where. For every one of the $\binom{10}{2}$ ways, there are $\binom{8}{2}$ ways picking the slots that will hold 2 and 3, and then where each goes is forced. So there are $\binom{10}{2}\binom{8}{2}$ of placing the first four digits. Continue in this way. We conclude that there are

$$\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}$$

arrangements. Calculation shows that there are 113400.

Another way: There are 10! ways of arranging the digits 0, 1, . . . , 9 in a row. One-half of these have 1 somewhere to the right of 0, and half have 1 somewhere to the left of 0. So 10!/2 arrangements satisfy the first condition. Of these arrangements, half have 3 somewhere to the right of 2, and half have 3 somewhere to the left, so $10!/2^2$ arrangements satisfy the first two conditions. We go on in this way, finally concluding that $10!/2^5$ arrangements satisfy all five conditions.

Comment. We often use counting to evaluate probabilities. The second solution in a sense used probability in order to count. For let N be the number of arrangements that satisfy all five conditions. Then the probability that a randomly chosen arrangement satisfies all five conditions is $N/10!$. But this probability is the same as the probability of getting 5 heads in a row when tossing a fair coin, that is, $1/2^5$, so $N = 10!/2^5$.

II-50. Let N be the number whose decimal expansion has the shape $999\cdots 998$—one hundred 9's followed by an 8. Find the sum of the decimal digits of N^2.

Solution. We should experiment with a calculator, using a lot fewer 9's. A pattern emerges, and if we *believe* that the pattern will continue, we can come up with an answer. But we should aim to achieve certainty.

Since $N = 10^{101} - 2$,

$$N^2 = 10^{202} - 4 \cdot 10^{101} + 4.$$

Perform the subtraction, in the usual paper and pencil way. We will have to "borrow." More formally,

$$N^2 = (10^{202} - 10^{102}) + (10^{102} - 4 \cdot 10^{101}) + 4.$$

The first bracketed term has 100 9's followed by 102 0's. The second has a 6 followed by 101 0's. So the digit sum is $100 \cdot 9 + 6 + 4$, that is, 910.

II-51. Find the percentage of seven-digit numbers that contain the digit 1.

Solution. We take the common-sense view that for example 0001234 is not a seven-digit number. The seven-digit numbers range from 1000000 to 9999999, so there are 9×10^6.

Next we count the seven-digit numbers that *don't* have a 1. The leftmost digit of such a number can be chosen in 8 ways. For each such choice, the second digit can be chosen in 9 ways, so there are $8 \cdot 9$ ways to choose the first two digits. For every choice of the first two digits, there are 9 choices for the third. After a while we conclude that there are $(8)(9^6)$ seven-digit numbers with no 1's.

Divide. About 47.24% of the numbers don't have a 1, and therefore about 52.76% of them have a 1.

II-52. (a) A mathematics binder contains 1001 pages of notes. How many ways are there to insert three blue dividers in the binder? Dividers can't go at the ends, and there can't be two consecutive dividers. (b) Find the number of positive integer solutions of

$$s + t + u + v = 1001.$$

The solution $s = 300$, $t = 300$, $u = 1$, $v = 400$ is different from $s = 1$, $t = 300$, $u = 300$, $v = 400$. (c) Find the number of solutions in non-negative integers of the equation of part (b).

Solution. (a) There are 1000 "spaces" between pages. We wish to *choose* three of these spaces to put blue dividers into. There are $\binom{1000}{3}$ ways of doing this. That number turns out to be 166167000.

(b) Let s be the number of pages of notes from the beginning to the first divider, t the number of pages from the first divider to the second, u the number of pages from the second to the third, and v the number after the third. Then $s+t+u+v = 1001$. Conversely, is s, t, u, and v are positive solutions of our equation, we can insert dividers after the s-th page, after the $(s+t)$-th page, and after the $(s+t+u)$-th page. Thus there are just as many solutions as there are ways of placing three dividers, so the answer is the same as in part (a).

(c) We recycle the idea of part (b). There are just as many ways of expressing 1001 as a sum of four non-negative integers as there are of expressing 1005 as a sum of four positive integers. So by the method of part (b) there are $\binom{1004}{3}$ solutions.

II-53. How many solutions does $x + 2y + 3z = 60$ have in non-negative integers?

Solution. We count the non-negative solutions of $x + 2y + 3z = 6n$, where $n \geq 0$. With more effort we can count the solutions of $x+2y+3z = 6n+r$ for any specified r between 0 and 5.

For any given z between 0 and $2n$, we count the solutions of $x + 2y = 6n - 3z$. Let z be even, say $z = 2u$. We are solving $x + 2y = 6n - 6u$. Since y can travel from 0 to $3n - 3u$, there are $3n - 3u + 1$ possibilities for y. For each of these, x is determined.

Add up, with u going from 0 to n. We have an arithmetic progression with first term $3n + 1$ and last term 1, so the sum is $(n + 1)(3n + 2)/2$.

Let z be odd, say $z = 2u - 1$. We are solving $x + 2y = 6n - 6u + 3$. There are $3n - 3u + 2$ possibilities for y. Add up, with u going from 1 to n. The result is $n(3n + 1)/2$ Now add together the counts for odd and even values and simplify. The result is $3n^2 + 3n + 1$. When $n = 10$ this is 331.

Another way: Let $F(n)$ be the number of solutions of $x + 2y + 3z = 6n$ in non-negative integers. These solutions are of two types: (i) those where x, y, and z are positive and (ii) those where one or more of the variables is 0.

The solutions of type (i) can be obtained by finding non-negative solutions of $u + 2v + 3w = 6n - 6$ and letting $x = u + 1$, $y = v + 1$, $z = w + 1$. Thus there are $F(n - 1)$ solutions of type (i).

We count the solutions of type (ii) directly. We deal first with $z = 0$. Then x can be any even number from 0 to $6n$, so there are $3n + 1$ solutions. Similarly, there are $2n + 1$ solutions with $y = 0$. For $x = 0$, we are looking at $2y + 3z = 6n$. Note that y must be a multiple of 3, say $3s$, and z a multiple of 2, say $2t$. We get $s + t = n$, and so there are $n + 1$ solutions.

Add up: the sum is $6n+3$. This is not the right answer, since the 3 solutions in which *two* of the variables are 0 have been counted twice. So there are $6n$ solutions of type (ii). We have obtained the *recurrence relation*

$$F(n) = F(n - 1) + 6n.$$

Now we can program a computer to calculate anything we need. Note that $F(0) = 1$. Thus by the recurrence formula, $F(1) = 7$, $F(2) = 19$, $F(3) = 37$, and so on. We quickly get to $F(10)$.

We can also use the recurrence to find an explicit formula for $F(n)$. By using the recurrence repeatedly, we get

$$F(1) + F(2) + \cdots + F(n) = F(0) + F(1) + \cdots + F(n-1) + 6(1 + 2 + \cdots + n).$$

Almost everything cancels. Since $F(0) = 1$, we get

$$F(n) = 1 + 6(1 + 2 + \cdots + n).$$

It follows that $F(n) = 3n^2 + 3n + 1$. For other ways of finding $F(n)$, see VI-31.

Chapter 3

Geometry

Introduction

The geometric problems considered here fall mostly in a narrow range: Information is given about some features (sides, angles, areas) of a geometric object, ordinarily a triangle or some object that can be reduced to triangles, and questions are asked about the size of some other feature of the object. The questions usually involve particular numbers even when the method is obviously general.

Occasionally, the main hurdle in solving the problem is algebraic, in that there is an easy reduction to an equation or a system of equations, but solving the equations presents difficulties. Symmetries are systematically taken advantage of.

There are no proofs of the "two-column" type. All too often students end up being asked to include a statement such as "$A = A$," and to justify it by the "reflexive property of equality." It isn't surprising if they sometimes conclude that mathematics is not about understanding real *things*, but rather a formal exercise with *words*.

Not many tools are needed. The Pythagorean Theorem is used, explicitly or implicitly, in a majority of the solutions. The more general Cosine Law ($c^2 = a^2 + b^2 - 2ab \cos C$) is occasionally useful. A few times it is convenient to know how to find the area of a triangle given two sides and the angle between them (Area $= (ab/2) \sin C$).

Problems III-54 and III-92 give methods of trisecting the angle and duplicating the cube with compass and *marked* straightedge. They connect with construction problems that go back almost to the beginnings of Greek geometry.

Problems and Solutions

III-1. A *diameter* of an ellipse is a line that bisects all chords parallel to a given line. For the ellipse with equation

$$4x^2 - 6xy + 9y^2 - 2x - 5y - 100 = 0,$$

find (a) the diameter that bisects chords parallel to the y-axis; (b) the diameter that bisects chords parallel to the x-axis; (c) the center of the ellipse.

Solution. (a) View the equation as a quadratic in y. Let x_0 be a particular value of x, and imagine solving for y in terms of x_0. The sum of the roots is $(6x_0+5)/9$, so half the sum is $(6x_0 + 5)/18$. Thus if (x_0, y_0) is the midpoint of the chord that has equation $x = x_0$, then $y_0 = (6x_0 + 5)/18$, so the diameter has equation $y = (6x + 5)/18$.

(b) The same method shows that the diameter that bisects chords parallel to the x-axis has equation $x = (6y + 2)/8$.

(c) The diameters of parts (a) and (b) meet at $(11/18, 13/27)$.

III-2. A triangle has vertices a, b, and g. The ratio of side ab to side ag is as 3 to 5, the altitude ad is 5 feet, and the base bg is 20 feet. Find the remaining sides of the triangle.

Solution. The triangle of the problem is sketched in Figure 3.1. Let de be equal to bd. Take line segment eg to have length $2x$. Then be has length $20 - 2x$, and

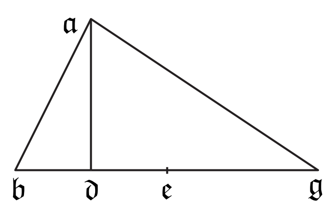

Figure 3.1: A Problem from Regiomontanus

therefore bd is $10 - x$ while dg is $10 + x$.

Square bd; the result is $x^2 + 100 - 20x$. Add to this the square of the altitude, namely 25, and you get $x^2 + 125 - 20x$, which by the Pythagorean Theorem is the square of ab. Similarly, the square of ag is $x^2 + 20x + 125$. But the square of ab is to the square of ag as 9 to 25. Multiply the first by 25 and the second by 9. We obtain after simplifying $16x^2 + 2000 = 680x$. Solve the quadratic equation. After some simplification we obtain

$$x = \frac{85 \pm 5\sqrt{209}}{4}.$$

The rest of the computation is straightforward, but more pleasant to do with a calculator than "exactly." It turns out that

$$ab = \frac{15\sqrt{34 \mp 2\sqrt{209}}}{4}$$

and of course $bg = (5/3)(ab)$.

Comment. The problem and method of solution are taken from Book II of Regiomontanus' *De Triangulis* (Concerning Triangles, 1464). This work of 137 pages is divided into five "books," the first two about triangles in the plane and the last three about spherical triangles.

Figure 3.1 is reproduced from the book. At the time, the first three letters of the alphabet were a, b, and g. Regiomontanus reached the verbal equivalent of the above quadratic equation—he used no symbols, our version is anachronistic.)

Regiomontanus didn't bother to solve the equation. If he had, he would have noticed that there are two admissible solutions, one of which gives a triangle very different in shape from the triangle of his diagram.

III-3. A teacher likes to assign group projects. This year, every student in her class is involved in 6 group projects. Any two projects have exactly one student in common. And for any two students, there is exactly one project that both students are on. How many students are there in the class? Hint: First show that any project involves exactly 6 students.

Solution. What is this problem doing in a geometry chapter? Instead of using the words "student" and "project," use the abstract words "point" and "line." This change of language will help, but we must be careful not to assume that line and point have the usual geometric properties.

Any point is on 6 lines, any two lines have exactly one point in common, and for any two points, there is exactly one line that they are both on. Because of the wording of the problem, we assume that any group project (line) has at least two points on it.

We first show that any line ℓ has exactly 6 points. There are lines other than ℓ. Any such line has exactly one point in common with ℓ, so there is a point P which is not on ℓ. Any point X on ℓ determines a unique line PX through P. Conversely, any line through P meets ℓ in one point X. So there are just as many points on ℓ as there are lines through P, namely 6.

Now let A be any point. There are 6 lines through A. Any point X lies on such a line, namely the line AX. No point other than A lies on more than one of these lines. Since each line has 6 points, each of the 6 lines through A contains 5 points other than A, for a total of 30. So including A there are 31 points. (There are also 31 lines.)

Comment. The "plane" in this problem differs from the usual Euclidean plane in two significant ways. Instead of having infinitely many points, it has only 31. Also, any two lines meet in a point. The Euclidean plane *almost* has this property, except that there are also parallel lines that don't meet at all.

A "plane" in which any two lines meet is called a *projective plane.* Projective planes arise naturally when we study the problem of drawing objects in correct perspective. Finite projective planes, like the one in this problem, have applications in coding theory and even in statistics.

It turns out that in a finite projective plane all lines have the same number $n+1$ of points. The number n is called the *order* of the plane. By the same argument as the one above, a plane of order n has $n^2 + n + 1$ points. There are many unsolved problems about finite projective planes. In the late twentieth century, a team led by Clement Lam proved that there is no projective plane of order 10, solving a problem that was more than 200 years old. It is not known whether there is a plane of order 12.

III-4. Let ℓ_t be the line with slope t that passes through the point $(-1, 0)$. Where does ℓ_t meet the circle $x^2 + y^2 = 1$?

Solution. The line ℓ_t has equation $y = t(x + 1)$. Substitute for y in the equation $x^2 + y^2 = 1$. We obtain

$$(1 + t^2)x^2 + 2tx + t^2 - 1 = 0.$$

The product of the roots is $(t^2 - 1)/(1 + t^2)$. Since one of the roots is -1, the other is given by $x = (1 - t^2)/(1 + t^2)$. Then $x + 1 = 2/(1 + t^2)$, and therefore $y = 2t/(1 + t^2)$.

Comment. If $x^2 + y^2 = 1$ and x and y are rational numbers (ratios of integers), then t is rational. So all rational points (x, y) on the unit circle, except for $(-1, 0)$, can be expressed as $x = (1 - t^2)/(1 + t^2)$, $y = 2t/(1 + t^2)$, where t is rational. The missing point $(-1, 0)$ can be thought of as corresponding to $t = \infty$.

The fact that $(\cos\theta, \sin\theta)$ travels around the unit circle as θ travels over the interval $[0, 2\pi)$ is extremely useful. The fact that as t travels over the reals, $(1 - t^2)/(1 + t^2), 2t/(1 + t^2)$ travels around the unit circle (missing $(-1, 0)$) also has a number of uses.

III-5. The consecutive vertices of a square are A, B, C, and D. Given that P is inside the square and $\angle PCD = \angle PDC = 15°$, show that $\triangle APB$ is equilateral.

Solution. We use the left-hand square in Figure 3.2. Let our square have side 2. To show that $\triangle APB$ is equilateral, it is enough to show that PAX is a $60°$ angle, or equivalently that $PX = \sqrt{3}$. A look at $\triangle PYD$ shows that $PY = \sin 15°$, so we need to show that $\tan 15° = 2 - \sqrt{3}$. The calculator says that there is equality to at

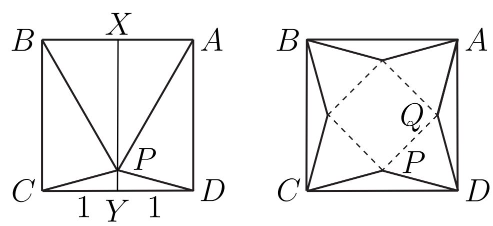

Figure 3.2: Angle Chasing in a Square

least 9 decimal places. Thus $\triangle APB$ is equilateral for all practical purposes. And it is very plausible that the triangle is fully equilateral.

To get certainty, note that

$$\sin 15^\circ = \sin(60^\circ - 45^\circ) = \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

and similarly $\cos 15^\circ = (1+\sqrt{3})/2\sqrt{2}$, so $\tan 15^\circ = (\sqrt{3}-1)/(\sqrt{3}+1)$. Finally, multiply "top" and "bottom" by $\sqrt{3}-1$ to obtain the desired result.

Alternately, note that $\cos 30^\circ = 1 - 2\sin^2 15^\circ$ and therefore $\sin^2 15^\circ = (2 - \sqrt{3})/4$. Similarly, $\cos^2 15^\circ = (2+\sqrt{3})/4$. Divide, and rationalize the denominator. We conclude that $\tan^2 15^\circ = (2-\sqrt{3})^2$.

Another way: We first show that *if* $\triangle APB$ is equilateral, *then* $\angle PCD = \angle PDC = 15^\circ$. A little angle chasing does it.

Suppose that $\angle PDC = \theta$. Then $\angle PDA = 90^\circ - \theta$. Since $\triangle APB$ is equilateral, $AP = AX = AD$, and therefore $\angle DPA = 90 - \theta$. But $\angle YPD = 90^\circ - \theta$, and $\angle APX = 30^\circ$, and therefore

$$(90^\circ - \theta) + (90^\circ - \theta) + 30^\circ = 180^\circ,$$

so $\theta = 15^\circ$.

This doesn't settle things: we need to show that *if* $\angle PCD = \angle PDC = 15^\circ$, *then* $\triangle APB$ is equilateral. In general, from the fact that if $\mathcal{S}$ then $\mathcal{T}$, we cannot conclude that if $\mathcal{T}$ then $\mathcal{S}$.

Imagine that the lines through P are made of stretched elastic. When $\triangle APB$ is equilateral, the angles at the bottom are exactly 15°. If P is pulled downwards, the angles at the bottom shrink below 15°, and if P is pushed up, the angles at the bottom grow beyond 15°. So if they are exactly 15° , then $\triangle APB$ must be equilateral.

Another way: Look at the right-hand square in Figure 3.2. For symmetry, $\triangle CDP$ has been replicated on the other three sides of the square $ABCD$. We obtain an inner square, with isosceles triangles erected outwards on its four sides. In each such triangle, the angle opposite the dashed side is 90° minus twice 15°, so the triangles are all equilateral. Thus $\angle AQP = 150^\circ$, and therefore $\triangle AQP$ and $\triangle AQD$ are congruent, so $AP = AD$ and the result follows.

III-6. The midpoints of the sides of triangle $\mathcal{T}_0$ are joined to make triangle $\mathcal{T}_1$. Then the midpoints of the sides of $\mathcal{T}_1$ are joined to make triangle $\mathcal{T}_2$, and so on forever. (a) Suppose that the line ℓ is a median of $\mathcal{T}_0$. Show that ℓ is a median of $\mathcal{T}_n$ for all n. (b) Show that the medians of $\mathcal{T}_0$ meet in a point.

Solution. (a) The situation is illustrated in Figure 3.3. Let P be the midpoint of the bottom side, and ℓ the line that passes through A and P. We first show

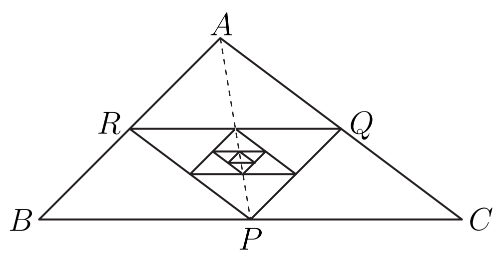

Figure 3.3: Nested Triangles

that ℓ is a median of $\mathcal{T}_1$, that ℓ bisects the line segment RQ. Since $ARPQ$ is a parallelogram, its diagonals bisect each other. But these diagonals are ℓ and RQ. Alternately, $\triangle ARQ$ is a scaled down version of $\triangle ABC$, so since ℓ bisects AB it also bisects RQ.

Now repeat the argument starting with $\mathcal{T}_1$. Since ℓ is a median of $\mathcal{T}_1$, it follows that ℓ is a median of $\mathcal{T}_2$, and so on forever.

(b) There is exactly one point X which is in all the triangles $\mathcal{T}_n$. But ℓ passes through $\mathcal{T}_n$ for every n, so ℓ must pass through X. This is also true for the other two medians of $\mathcal{T}$, so the three medians of $\mathcal{T}$ meet at X.

Comment. The fact that the medians of a triangle meet in a point is an important geometric theorem. There are many proofs.

III-7. (a) A 7×5 rectangle is divided into 35 equal squares as in Figure 3.4. Find the area of $\triangle ABC$. (b) Let $\mathcal{T}$ be a triangle whose vertices have integer coordinates. Show that the area of $\mathcal{T}$ is of the form $n/2$ where n is an integer.

Solution. (a) The rectangle has area 35, and is divided into four triangles, including $\triangle ABC$. The triangles other than $\triangle ABC$ are right-angled, with sides that can be read off from the picture. Their combined area is 20, so $\triangle ABC$ has area 15.

(b) If the coordinates of the vertices of $\mathcal{T}$ are integers, then, like $\triangle ABC$ of Figure 3.4, $\mathcal{T}$ can be inscribed in a rectangle $\mathcal{R}$ with sides parallel to the axes. The part of $\mathcal{R}$ outside $\mathcal{T}$ is made up of three right-angled triangles with integer legs. Each of these triangles has area of the form $m/2$ where m is an integer. Since $\mathcal{R}$

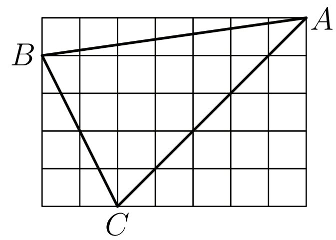

Figure 3.4: The Area of a Lattice Triangle

has integer area, it follows that the area of $\mathcal{T}$ is of the form $n/2$ where n is an integer.

Comments. 1. The *integer lattice* is the collection of points in the plane that have integer coordinates. A polygon whose vertices are lattice points is called a *lattice polygon.*

We can find the areas of complicated lattice polygons by using *Pick's Theorem,* which says that the area is $i+p/2-1$, where i is the number of lattice points *inside* the polygon, and p is the number of lattice points on the periphery of the polygon. (For $\triangle ABC$, we have $i = 12$ and $p = 8$.) Proving Pick's Theorem is a challenging but accessible task.

2. We cannot make an equilateral triangle all of whose vertices have integer coordinates. An equilateral triangle of side x has height $x\sqrt{3}/2$, so it has area $x^2\sqrt{3}/4$. By the Pythagorean Theorem, if triangle $\mathcal{T}$ has vertices with integer coordinates, then any side of $\mathcal{T}$ has length of the form $\sqrt{m}$ for some integer m. So if $\mathcal{T}$ were equilateral, it would have area $m\sqrt{3}/4$ for some integer m. By part (b), this area is half an integer, say $n/2$, so $m\sqrt{3} = 2n$. But $\sqrt{3}$ cannot be expressed as the ratio of two integers.

III-8. Let $\mathcal{S}$ be a collection of 101 points in the plane. Show that there is a point P in $\mathcal{S}$, and a line passing through P, such that 50 of the points in $\mathcal{S}$ are on one side of the line and 50 are on the other side.

Solution. Draw a line ℓ such that all the points in $\mathcal{S}$ are on one side of ℓ. Slowly move ℓ parallel to itself towards $\mathcal{S}$. After a while, the line moves past a point in $\mathcal{S}$, then another, then another. Keep moving until the line has moved past 50 of the points and just reaches the 51st, and we are done.

Not quite! The argument breaks down if as ℓ moves it can meet points of $\mathcal{S}$ in bunches. We will make the argument work by making sure that the moving line can't pass simultaneously through more than one point of $\mathcal{S}$.

There are only finitely many lines that contain two or more points of $\mathcal{S}$. In fact, there are at most $\binom{101}{2}$, that is, 5050, and possibly far fewer. These lines determine only finitely many slopes. Choose ℓ so that its slope is *different* from all the slopes determined by pairs of points in $\mathcal{S}$. Then the moving line can't ever contain two or more points of $\mathcal{S}$, so the original argument now works.

III-9. There is a square garden with a circular pool of water in the middle. The *land* area of the garden is 3300 square paces. The distance from any edge of the garden to the point on the pool furthest from that edge is 40 paces. Find the diameter of the pool.

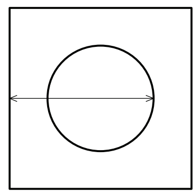

Figure 3.5: The Pool in the Garden

Solution. Let the radius of the pool be r. By looking at Figure 3.5, we can see that the garden has side length $40 + (40 - 2r)$. The pool has area πr^2, and the land area of the garden is 3300, so

$$(80 - 2r)^2 - \pi r^2 = 3300.$$

This equation simplifies to $(4 - \pi)r^2 - 320r + 3100 = 0$. Solve, using the quadratic formula. The roots are approximately 9.95325 and 362.83014. The second answer is absurd, for the side length of the garden would be negative. The diameter of the pool is therefore about 19.9.

Comment. This problem comes from Li Zhi's textbook *Yigu yuandan* (New Steps in Calculation), published in 1259. Li Zhi chose the dimensions so that the quadratic factors nicely if π is taken to be 3. He got 20 as the diameter of the pool.

III-10. A mathematician moonlighting as a baker has made an acute-angled triangular cake, frosted on top but not on the sides. The box for the cake fits perfectly, but only if the cake is put in upside down. Show how to cut the cake into some pieces so that the cake can be put in the box right side up. Try to find more than one solution.

Solution. Three different methods are illustrated in Figure 3.6. In the leftmost picture, the circumcenter of the triangle, that is, the point where the perpendicular bisectors of the sides meet, is joined to the vertices, and the cake is cut along these lines. The cuts divide the cake into three isosceles triangles. To turn the cake into its mirror image, leave the bottom triangle in the picture alone and interchange the left and right triangles.

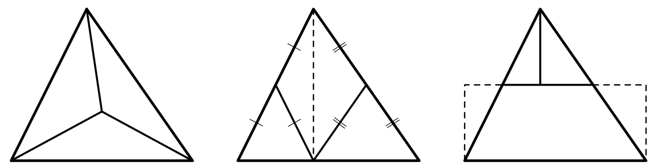

Figure 3.6: The Triangular Cake

The method only works when the circumcenter is in the triangle. That's true in particular if the angles are acute. The next two methods work for any triangle.

Another way: In the middle picture of Figure 3.6, the triangle has been placed with a longest edge at the bottom. Drop a perpendicular (the dashed line) from the top vertex to the bottom edge and let it meet the bottom at P. Draw two lines through P at angles that cut off two isosceles triangles. A bit of angle-chasing shows that these lines bisect the two remaining sides.

We have cut the original triangle into two isosceles triangles and a kite-shaped figure. It is easy to rearrange these three pieces to make the mirror image of the original triangle.

Another way: In the rightmost picture of Figure 3.6, the triangle is placed so that neither angle at the bottom is obtuse. Draw a line parallel to the bottom and halfway up the triangle, and drop a perpendicular from the top vertex to that line. Cut the triangle along these lines. The picture shows how to rearrange the pieces to make a rectangle.

If we had started with the mirror image of the original triangle and used the same cutting procedure, we would arrive at the same rectangle. So, by reversing the process, we can turn the rectangle into the mirror image of the original triangle.

Comment. There is a related general result called the *Bolyai-Gerwien Theorem.* Let $\mathcal{A}$ and $\mathcal{B}$ be polygons with the same area. Then $\mathcal{A}$ can be cut up into a finite number of polygonal pieces that can be reassembled, without turning over any piece, to make $\mathcal{B}$. So for example if the cake box has a square top with the same top area as the cake, then the triangular cake can be cut into pieces that fit into the box perfectly.

III-11. Find the distance from the point $(9, 1)$ to the line with equation $2x + y = 4$.

Solution. The distance from a point P to a line ℓ is the length of PQ, where Q is the point on ℓ nearest to P. A quick sketch shows that if P is not on ℓ and Q is the point on ℓ nearest to P, then PQ is perpendicular to ℓ.

In this problem, $P = (9, 1)$. Let $Q = (u, v)$. Since Q lies on the line $2x+y = 4$, we have $v = 4-2u$. The line $2x+y = 4$ has slope -2, so any line perpendicular to it has slope $1/2$. It follows that PQ has slope $1/2$, so

$$\frac{4-2u-1}{u-9} = \frac{1}{2} \qquad \text{and therefore} \qquad u = 3 \quad \text{and} \quad v = -2.$$

Finally, compute the distance between $(9,1)$ and $(3,-2)$. This is $\sqrt{45}$.

Another way: Let (u,v) travel over the line $2x+y=4$. So $v=4-2u$. We want to choose (u,v) so that the distance from $(9,1)$ to (u,v) is a minimum, or equivalently minimize the square of the distance, namely $(9-u)^2+(1-v)^2$.

Substitute $4-2u$ for v in $(9-u)^2+(1-v)^2$ and simplify. We minimize

$$5u^2-30u+90, \qquad \text{that is,} \qquad 5\left[(u-3)^2+9\right].$$

Since $(u-3)^2$ can't be negative, the minimum occurs when $u=3$, where the function has value 45. The required distance is therefore $\sqrt{45}$.

Comment. Let ℓ be the line with equation $ax+by=c$, and let P be the point with coordinates (p,q). Using the same argument as in the concrete example above, we can show that the distance from P to ℓ is

$$\frac{|ap+bq-c|}{\sqrt{a^2+b^2}}.$$

III-12. The point $(4,6)$ is the midpoint of a chord of the circle with equation $(x-1)^2+(y-2)^2=169$. Find an equation of the line that contains the chord.

Solution. The circle has center $(1,2)$, so the line that joins the center of the circle to the midpoint of the chord has slope $(6-2)/(4-1)$, that is, $4/3$. This line is perpendicular to the chord, and therefore the chord itself has slope $-3/4$.

The line that contains the chord passes through $(4,6)$ and has slope $-3/4$, so its equation can be written as $y-6=(-3/4)(x-4)$, or more attractively as $3x+4y=36$.

III-13. The center of a parallelogram is at $(1,\pi)$ and one of its sides AB lies on the line $x+2y=9$. Find an equation of the line that contains the side of the parallelogram which is opposite AB.

Solution. Let ℓ be the line that passes through $(1,\pi)$ and is parallel to $x+2y=9$. Then ℓ has an equation of the shape $x+2y=a$. Since ℓ goes through $(1,\pi)$ we find that $a=1+2\pi$. So the x-intercept of ℓ is $1+2\pi$.

The line $x+2y=9$ has x-intercept equal to 9. The line which contains the side of the parallelogram opposite AB has equation of the shape $x+2y=b$, where b is its x-intercept.

The x-intercept of ℓ lies midway between 9 and b, and therefore

$$\frac{9+b}{2}=1+2\pi.$$

It follows that $b=4\pi-7$ and the required line has equation $x+2y=4\pi-7$.

III-14. The midpoints of the sides of a triangle have coordinates $(1,1)$, $(5,-3)$, and $(7,6)$. Find the coordinates of the vertices of the triangle.

Solution. Let the vertices be A, B, and C, with $(1,1)$ the midpoint of AB and $(5,-3)$ the midpoint of BC. Let the x-coordinates of A, B, and C be r, s, and t. Then

$$\frac{r+s}{2}=1, \qquad \frac{s+t}{2}=5, \qquad \text{and} \qquad \frac{t+r}{2}=7.$$

Add up the left-hand sides, add up the right-hand sides. We get $r+s+t=13$. Since $s+t=10$, we conclude that $r=3$. Similarly, $s=-1$ and $t=11$.

Find the y-coordinates of A, B, and C in the same way. The vertices of the triangle turn out to be $(3,10)$, $(-1,-8)$, and $(11,2)$.

Comment. The computation has been presented purely algebraically, but there is a rich set of ideas connected with it. Let ABC be a triangle, with $A=(r,u)$, $B=(s,v)$, and $C=(t,w)$. Then the coordinates of the *centroid* of $\triangle ABC$ are $(r+s+t)/3$ and $(u+v+w)/3$. A geometer would define the centroid as the point where the medians of the triangle meet. A physicist would define it as the center of mass of the triangle.

The triangle obtained by joining the midpoints of the edges of $\triangle ABC$ has the same centroid as $\triangle ABC$. In the case that we studied, the midpoints are $(1,1)$, $(5,-3)$, and $(7,6)$, so the centroid of the triangle of midpoints is $(13/3,4)$. Thus $(r+s+t)/3=13/3$. The equation $r+s+t=13$, which we obtained by algebraic manipulation, can therefore be obtained from purely geometric (or physical) considerations.

III-15. Find equations for all lines that pass through $(1,4)$ and that form, together with the x-axis and the y-axis, a triangle of area 9.

Solution. An informal sketch shows that there should be four such lines. Two of the triangles so formed are in the first quadrant, and one each in the second and the fourth. We could reason geometrically, but in this case algebra is faster.

Suppose that line ℓ passes through $(1,4)$ and has slope m. Then ℓ has equation $y=mx+(4-m)$. The y-intercept of ℓ is $4-m$, while the x-intercept is $(m-4)/m$.

The triangle formed by ℓ and the coordinate axes has area $(m-4)^2/2|m|$. This area should be 9, and therefore $(m-4)^2=18|m|$.

Suppose first that m is negative. Then $|m|=-m$ and we obtain the equation $m^2+10m+16=0$, which has the solutions $m=-2$ and $m=-8$. Now suppose that m is positive. We obtain $m^2-26m+16=0$, which has the solutions $m=13\pm\sqrt{153}$.

III-16. The three square fields of Figure 3.7 have area 25, 26, and 85 acres. Find the area of the triangular field.

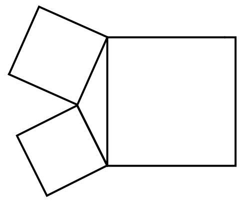

Figure 3.7: The Four Fields

Solution. For the record, an acre is 160 square rods, and a rod is 5.5 yards. To honour the whimsical nature of Imperial measures, let's call the side of a 1 acre field a whimsy. So the triangular field has sides of length $\sqrt{25}$, $\sqrt{26}$, and $\sqrt{85}$ whimsies.

Let θ be the angle opposite the $\sqrt{25}$ whimsy side. By the Cosine Law

$$25 = 26 + 85 - 2\sqrt{26}\sqrt{85}\,\cos\theta$$

so $\cos\theta = 43/(\sqrt{26}\sqrt{85})$, and therefore $\sin\theta = 19/(\sqrt{26}\sqrt{85})$.

The area of the triangular field is

$$\frac{\sqrt{26}\sqrt{85}\sin\theta}{2}.$$

Substitute the computed value of $\sin\theta$ and simplify. The area is 9.5 acres.

III-17. A rectangular sheet of paper is 24 cm long and 10 cm wide. The paper is folded so that two diagonally opposite corners coincide. Find the length of the crease.

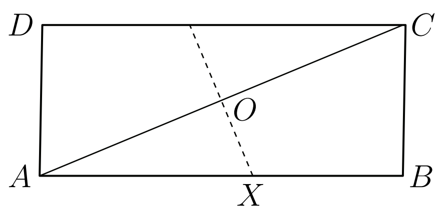

Figure 3.8: Folding a Rectangle

Solution. Fold the paper and then unfold. Let the rectangle be labelled as in Figure 3.8, with $AB = 24$ and $BC = 10$. The diagonal AC and the fold line are perpendicular bisectors of each other. Note that $\triangle AOX$ is similar to $\triangle ABC$, and therefore

$$\frac{OX}{AO} = \frac{BC}{AB} = \frac{10}{24}.$$

By the Pythagorean Theorem, $AC = \sqrt{24^2 + 10^2} = 26$, and therefore $AO = 13$. Thus $OX = (13)(10)/24$, and therefore the length of the crease is $65/6$.

Another way: Instead of using the geometry, we can compute in more or less clumsy ways. We show for example how to solve the problem by using coordinates.

Take the center O of the rectangle as the origin, and let $X = (x, -5)$. We have $A = (-12, -5)$ and $C = (12, 5)$. Since $XA = XC$, the distance formula gives

$$(12 + x)^2 = (12 - x)^2 + 10^2.$$

Simplify and solve. We obtain $x = 100/48$. Since $OX = \sqrt{x^2 + 5^2}$, we can now calculate OX and double the result to find the length of the crease.

III-18. The area of a right-angled triangle is $16\,\text{cm}^2$, and the hypotenuse is $15\,\text{cm}$. Find the perimeter of the triangle.

Solution. Let the legs be a and b. Then $ab = 32$ and $a^2+b^2 = 225$. So $a^2+2ab+b^2 = 289$, and therefore $a + b = 17$. If follows that the perimeter is 32. There are more complicated (and less symmetrical) ways to solve the problem.

Comment. The first surviving problem of this type occurs in Gerbert's *Geometry.* His solution also exploits the symmetry. Gerbert (938–1003) became Pope Sylvester II.

III-19. The vertices of a regular 12-sided polygon lie on a circle of radius 1. Find the sum of the squares of the distances from one vertex P to all the other vertices of the polygon. Generalize.

Solution. Choose as origin the center of the circle. By choosing the coordinate axes appropriately, we can take the vertices to be $P = (1, 0)$, $(\sqrt{3}/2, \pm 1/2)$, $(1/2, \pm\sqrt{3}/2)$, and so on. Now we can use the distance formula to calculate. There is a lot of symmetry, so the calculation doesn't take long. But we can prove a stronger result in an easier way.

We work with a regular $2n$-sided polygon inscribed in a circle of radius 1. Let P be *any* point on the circle, not necessarily a vertex. The vertices of the polygon can be divided into diametrically opposite pairs. If $\{A, B\}$ is such a pair, and P is not A or B, then $\angle APB$ is a right angle, since $\angle APB$ is the angle subtended by a diameter of the circle.

By the Pythagorean Theorem, $(PA)^2 + (PB)^2 = (AB)^2 = 4$. If $P = A$ or $P = B$, we also have $(PA)^2 + (PB)^2 = 4$. Thus each of the n pairs makes a contribution of 4 to the sum of the squares, for a total of $4n$. For the 12-gon, the sum is 24.

Comment. In fact the sum of the squares of the distances is $2m$ for any regular m-gon inscribed in a circle of radius 1, even when m is odd, but the proof is more messy. Let O be the center of the circle, let V_1, V_2, ..., V_m be the vertices, and let θ_i be the angle between OP and OV_i. By the Cosine Law,

$$(PV_i)^2 = (OP)^2 + (OV_i)^2 - 2(OP)(OV_i)\cos\theta_i = 2 - 2\cos\theta_i.$$

Add up from $i = 1$ to $i = m$. The sum of the $\cos\theta_i$ is 0 (to show this takes some work), so the sum of the squares of the distances is $2m$.

III-20. A right-angled triangle has legs of length 3 and 4. Outward facing semicircles are drawn with these legs as diameter. An inward facing semicircle is drawn with the hypotenuse as diameter. Find the area of the shaded region in Figure 3.9.

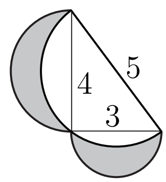

Figure 3.9: A Theorem of Hippocrates

Solution. We could find the area of each shaded piece. This is not the best way to handle the problem, so we merely record the answers. The shaded piece on the left has area

$$2\pi + 3 - \left(\frac{5}{2}\right)^2 \arcsin\frac{4}{5}$$

and the one at the bottom has area

$$\frac{9\pi}{8} + 3 - \left(\frac{5}{2}\right)^2 \arcsin\frac{3}{5}.$$

Here $\arcsin x$ means the angle (in radians, between $-\pi/2$ and $\pi/2$) whose sine is x. Now we could compute using the calculator, but there is no need for one, for the right triangle in the picture shows that $\arcsin(4/5) + \arcsin(3/5) = \pi/2$. Now add the two areas and simplify. The π's disappear and the result is 6.

There is a much less complicated approach. Look at the entire region $\mathcal{R}$ covered by Figure 3.9. We can think of $\mathcal{R}$ as the semicircle "on 4," plus the semicircle "on 3," plus the triangle. We can also think of $\mathcal{R}$ as the semicircle "on 5" plus the shaded region.

The two small semicircles have combined area equal to that of the big semicircle. We can show this by an easy computation, but it is also a consequence of a natural generalization of the Pythagorean Theorem: if similar figures are erected on the two legs and the hypotenuse of a right angled triangle, then the sum of the areas of the figures on the two legs is equal to the area of the figure on the hypotenuse.

It follows by cancellation that the shaded region has the same area as the triangle, namely 6.

Comment. Hippocrates of Chios (circa -450) did the same calculation for the isosceles right triangle with legs 1, concluding that each "lune" has area $1/4$. Thus a natural region with curvy sides can have a nice area. Hippocrates was probably interested in the problem of "squaring the circle," that is, constructing a square with the same area as a given circle. It has been known since the late nineteenth century that this can't be done with straightedge and compass.

III-21. Show how to divide a given triangle $\mathcal{T}$ into 2100 triangles each similar to $\mathcal{T}$. The small triangles won't all be the same size.

Solution. If we divide each side of a triangle into k equal parts, and through the division points draw lines parallel to the sides, the lines divide the given triangle into k^2 small triangles each similar to the original.

Take for instance $k = 45$. So we have divided $\mathcal{T}$ into 2025 congruent triangles each similar to $\mathcal{T}$. We need to make 75 more triangles. That can be done by taking 25 of the 2025 triangles and dividing each into 4 parts by joining the midpoints of the sides (this is the case $k = 2$). There many other ways to do the job.

III-22. A six-sided polygon is inscribed in a circle. All its angles are equal. Show that its sides need not be equal. What can be said about seven-sided equal-angled inscribed polygons? Generalize.

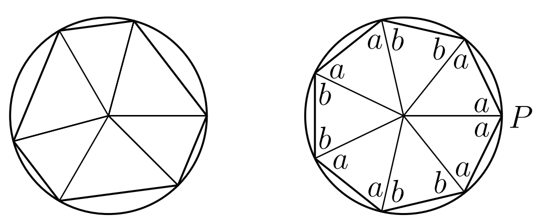

Figure 3.10: Equal-angled Hexagons and Heptagons

Solution. Starting at the center of the left circle in Figure 3.10, and moving counterclockwise, draw isosceles triangles with apex angles that alternate between x and y, where $x + y = 120°$. In the picture, $x = 75°$ and $y = 45°$.

A bit of angle-chasing shows that for every choice of x and y we get an equal-angled inscribed hexagon. (This is in fact the only way to get such hexagons.) If $x \neq y$, then the sides are not all equal. By letting $x + y = 360°/n$ with $x \neq y$, we obtain in the same way an equal-angled inscribed $2n$-gon with unequal sides.

Now suppose that an equal-angled heptagon has been inscribed in the right circle of Figure 3.10. Start at P, and let the first angle we meet be a degrees. Then as we travel counterclockwise, the next angle must be a, and the one after that b, where $a+b=900/7$. The one after that must also be b. Go on all the way around. When we get back to P, we can see that if $a \neq b$ there is a conflict. It follows that the heptagon must be regular. Exactly the same argument works for inscribed equal-angled n-gons where n is odd: every such n-gon has equal sides.

III-23. Find the equation of the line obtained by reflecting the line $y = -3x+3$ in the line $y = x+1$.

Solution. Reflection in $y = x$ is familiar: in equations it simply interchanges the roles of x and y. So we pull both lines down by 1, reflect, then push back up.

When we pull $y = -3x+3$ down by 1, we get the line $y = -3x+2$. The reflection of this line in $y = x$ has equation $x = -3y+2$, or equivalently $y = -x/3+2/3$. When we push this up by 1, we obtain $y = -x/3+5/3$.

Comment. The type of strategy we used is sometimes called *Transform, Solve, Transform Back.* It is an important device throughout mathematics. For other examples in this chapter see III-64 and III-79.

There are other reasonable approaches to the problem. We could focus on what is left *invariant* (unchanged) by the reflection. For example, the point of intersection of the two lines is invariant.

III-24. In Figure 3.11, $ABCD$ is a 1×1 square, and $AX = CY = a$. Find the length of PQ.

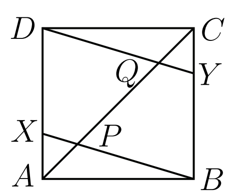

Figure 3.11: Lines in a Square

Solution. We can use coordinate geometry. The calculations won't be difficult, so there is no need to be clever about the choice of origin and axes.

Let $A = (0,0)$ and $B = (1,0)$. Then $X = (0,a)$. The line XB has equation $y = -ax+a$. Since P is the intersection point of the lines $y = -ax+a$ and $y = x$, we find that the x and y coordinates of P are each $a/(1+a)$.

By the distance formula, $AP = \sqrt{2}a/(1+a)$, and by symmetry, $AP = CQ$. Finally, since $AC = \sqrt{2}$,

$$PQ = \sqrt{2} - \frac{2\sqrt{2}a}{1+a} = \frac{\sqrt{2}(1-a)}{1+a}.$$

Another way: Figure 3.12 is obtained from 3.11 by extending AD to Z, with $DZ = a$. For clarity, the part of 3.11 "below" AC has been cut away. Now we

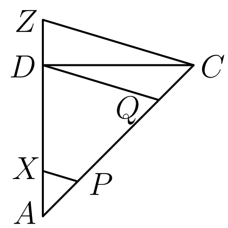

Figure 3.12: Solution to Lines in a Square

can read off the answer. By similarity, $PQ/XD = AC/AZ$. Since $XD = 1 - a$, $AC = \sqrt{2}$, and $AZ = 1 + a$, it follows that $PQ = \sqrt{2}(1 - a)/(1 + a)$. The same idea works if we start with a parallelogram instead of a square.

III-25. Two lines meet at O. Let A, B, and C be any three points on the first line, and P, Q, and R be three points on the second line such that AQ is parallel to BP and BR is parallel to CQ. Show that AR is parallel to CP. Please see Figure 3.13.

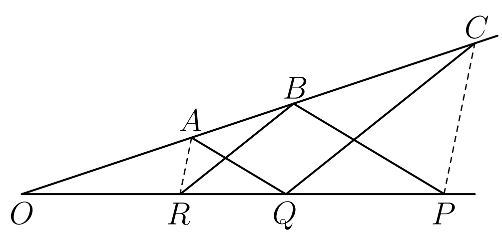

Figure 3.13: A Special Case of a Theorem of Pappus

Solution. Note that $\triangle OAQ$ is similar to $\triangle OBP$, so $OA/OB = OQ/OP$, and therefore $OA \cdot OP = OB \cdot OQ$. Also, $\triangle OBR$ is similar to $\triangle OCQ$, so $OB/OC = OR/OQ$, and therefore $OB \cdot OQ = OC \cdot OR$.

We conclude that $OA \cdot OP = OC \cdot OR$, and therefore $OA/OC = OR/OP$. It follows that $\triangle OAR$ is similar to $\triangle OCP$, and therefore AR is parallel to CP.

Comment. A more general theorem was proved by Pappus of Alexandria, who probably lived in the third century. Much later, Pappus' Theorem was recognized as one of the key theorems of Projective Geometry.

III-26. Show how to split a square into 6 squares; 7 squares; 8 squares; 101 squares.

Solution. Figure 3.14 shows how to split a square into 6, 7, or 8 squares. The decompositions are essentially unique. When $n \geq 9$, there are several ways to split

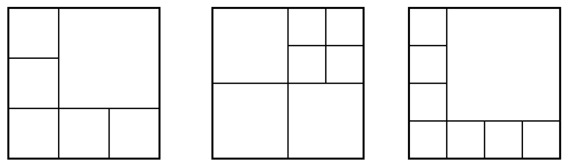

Figure 3.14: Splitting Squares

a square into n squares. Suppose first that n is even, say $n = 2k$. We can imitate the decompositions given for $n = 6$ and $n = 8$: divide the "L" on the South and West into $2k - 1$ equal squares, leaving a single large square on the North-east.

If n is odd, let $n = 2k + 3$. Divide the square into $2k$ squares as described above, then divide the large square on the North-east into 4 equal squares.

Another way: Given a decomposition into m squares, we can get a decomposition into $m+3$ squares by splitting one of the squares into 4 equal squares—gain 4, lose 1. Any $n \geq 9$ differs from one of 6, 7, or 8 by a multiple of 3. So by taking the picture for 6, 7, or 8, and splitting squares into 4 parts repeatedly, we can produce a decomposition into n squares.

III-27. The vertices of a quadrilateral, in counterclockwise order, are A, B, C, and D. Let the midpoints of AB, BC, CD, and DA be P, Q, R, and S. Show that $PQRS$ is a parallelogram (it is called the *Varignon* parallelogram of $ABCD$).

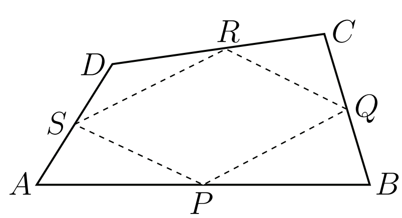

Figure 3.15: The Varignon Parallelogram

Solution. In Figure 3.15, draw the diagonal AC. Since P bisects side AB of $\triangle ABC$, while Q bisects CB, the line PQ is parallel to AC. A similar argument using $\triangle ADC$

shows that SR is also parallel to AC. Since PQ and SR are each parallel to AC, they are parallel to each other.

Now draw the diagonal BD, and argue that since QR and PS are each parallel to BD, they are parallel to each other. Since the sides of $PQRS$ are parallel in pairs, $PQRS$ is a parallelogram.

III-28. In Figure 3.16, lines parallel to the three sides of triangle $\mathcal{T}$ are drawn through an interior point of $\mathcal{T}$. The three small triangles in the picture have areas a, b, and c as shown. Find the area of $\mathcal{T}$.

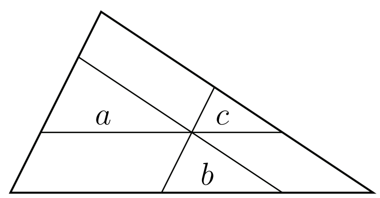

Figure 3.16: Splitting a Triangle

Solution. Because the lines drawn are parallel to the sides, all triangles in Figure 3.16 are similar. Let α, β, and γ be the base lengths of the triangles of area a, b, and c. Because the triangles are similar, there is a constant k such that

$$a = k\alpha^2, \qquad b = k\beta^2, \qquad \text{and} \qquad c = k\gamma^2.$$

The base of $\mathcal{T}$ is made up of three segments, whose lengths, from left to right, are α, β, and γ. It follows that $\mathcal{T}$ has area $k(\alpha + \beta + \gamma)^2$. But

$$\alpha = \sqrt{a/k}, \qquad \beta = \sqrt{b/k}, \qquad \text{and} \qquad \gamma = \sqrt{c/k},$$

so the area of $\mathcal{T}$ is $(\sqrt{a} + \sqrt{b} + \sqrt{c})^2$.

Comment. We can choose the unit of length independently of the unit of area. The solution looks nicer if we choose the unit of area so as to make $k = 1$.

III-29. Quadrilateral $ABCD$ is split into two triangles of equal area by its diagonal AC, and also by its diagonal BD. Explain why the quadrilateral must be a parallelogram.

Solution. Let the diagonals meet at O. Figure 3.17 is inaccurate, since $\triangle ABC$ and $\triangle CDA$ obviously have different area. The picture has been drawn in this way so that we will make no unwarranted assumptions.

Triangles ABC and CDA have the same base AC. Since their areas are the same, their heights h are the same. Triangles ABO and ADO have the same base

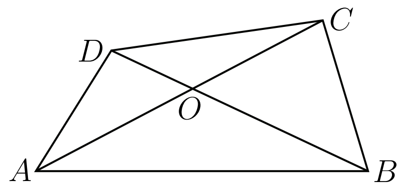

Figure 3.17: Splitting a Quadrilateral

AO, and the same height, namely h, so they have the same area. Similarly, triangles BCO and CDO have the same area.

The same argument, starting from the fact that $\triangle ABD$ and $\triangle BCD$ have the same area, shows that $\triangle ABO$ and $\triangle BCO$ have the same area. It follows that the diagonals split $ABCD$ into four triangles of equal area.

If $\triangle AOB$ and $\triangle BOC$ are viewed as triangles with bases AO and CO, they have equal heights and equal areas, so $AO = CO$. Similarly, $BO = DO$. Since $\angle AOB = \angle COD$, triangles AOB and COD are congruent. It follows that $\angle CAB = \angle ACD$, and therefore AB and DC are parallel. In the same way, we can show that BC and AD are parallel, and therefore $ABCD$ is a parallelogram.

III-30. The corners of a rectangle, taken counterclockwise, are P, Q, R, and S. Suppose that the distances of point X from P, Q, R and S are p, q, r, and s. Show that $p^2 + r^2 = q^2 + s^2$.

Solution. Place P at the origin, and Q on the positive x-axis. Let $PQ = a$ and $QR = b$. Then the coordinates of the corners of the rectangle are $(0, 0)$, $(a, 0)$, (a, b), and $(0, b)$. Let the coordinates of X be (x, y). Then $x^2+y^2 = p^2$, $(x-a)^2+y^2 = q^2$, $(x - a)^2 + (y - b)^2 = r^2$, and $x^2 + (y - b)^2 = s^2$. We need to show that

$$(x^2 + y^2) + ((x - a)^2 + (y - b)^2) = ((x - a)^2 + y^2) + (x^2 + (y - b)^2).$$

This algebraic identity, like most such identities, is easily verified by expanding both sides.

Comment. It is prettier to place the origin at the center of the rectangle and to let the sides be $2a$ and $2b$. Essentially the same proof can be given without coordinates by using the Pythagorean Theorem instead of the distance formula.

But we haven't solved the problem! The solution took it for granted that X lies in the plane of the rectangle. Suppose it isn't. Let X' be the projection of X onto the plane, and let $d = XX'$. Let p', q', r' and s' be the distances of X' from P, Q, R, and S. By the Pythagorean Theorem, $p^2 = d^2 + (p')^2$, $q^2 = d^2 + (q')^2$, and so on. Since $(p')^2 + (r')^2 = (q')^2 + (s')^2$ we obtain the desired result by adding $2d^2$ to both sides.

III-31. In the trapezoid $ABCD$ of Figure 3.18, AB is parallel to CD and the diagonals meet at O. Given that $\triangle AOB$ has area 4 and $\triangle COD$ has area 3, find the area of $ABCD$.

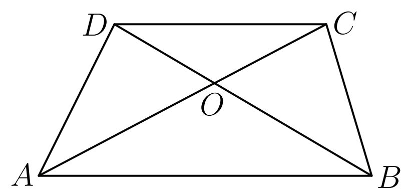

Figure 3.18: The Area of a Trapezoid

Solution. The triangles AOB and COD are similar. Since their areas are in the ratio $3 : 4$, the sides BO and DO are in the ratio $\sqrt{3} : 2$.

If we think of $\triangle AOB$ and AOD as having bases BO and DO, they have the same height, and therefore $\triangle AOD$ has area $4(\sqrt{3}/2)$. Similarly, we can show that $\triangle BOC$ has area $4(\sqrt{3}/2)$. Alternately, note that since $\triangle CAD$ and $\triangle DBC$ have the same area, so do $\triangle AOD$ and $\triangle BOC$. Add up the areas of the four triangles. The result is $7 + 4\sqrt{3}$.

Comment. Suppose that $\triangle AOB$ has area p, and $\triangle COD$ has area q. The above argument shows that the area of the trapezoid is $p+q+2p\sqrt{q/p}$, or more attractively $(\sqrt{p} + \sqrt{q})^2$.

III-32. Five points are picked inside or on an equilateral triangle of side 2 meters. Show that no matter how the points are chosen, two of these points are no more than 1 meter apart.

Solution. Divide the original triangle into four equilateral triangles of side 1 by joining the midpoints of the sides. Since there are only four triangles, and we have picked five points, at least two of these points must lie on or inside the same small triangle. The distance between these two points can't be greater than 1.

Comment. Using the same idea, we can show that if 5 points are picked in a square of side 2 then two of these points are no more than $\sqrt{2}$ apart, and that if 7 points are picked in a regular hexagon of side 1 then two of them are no more than 1 apart.

III-33. Is there a triangle whose three altitudes have length 5, 9, and 12? Is there a triangle with altitudes 6, 10, and 13?

Solution. There is no triangle with altitudes 5, 9, and 12. For suppose to the contrary that there were such a triangle, and it had area $k/2$. Let a, b, and c be the lengths of the bases for which the altitudes are 5, 9, and 12 respectively. Then $5a/2 = 9b/2 = 12c/2 = k/2$ and therefore

$$a = \frac{k}{5}, \qquad b = \frac{k}{9}, \qquad \text{and} \qquad c = \frac{k}{12}.$$

Note that $k/9 + k/12 < k/5$, and therefore $b + c < a$. This is impossible, for the sum of any two sides of a triangle is greater than the third side. We conclude that there is no triangle with the specified altitudes.

The situation is different for altitudes 6, 10, and 13. It is easy to verify that the sum of any two of $1/6$, $1/10$, and $1/13$ is greater than the third. So there is a triangle $\mathcal{T}$ with sides $1/6$, $1/10$, and $1/13$. Let the area of $\mathcal{T}$ be $k/2$.

Since the sides of $\mathcal{T}$ are $1/6$, $1/10$, and $1/13$, and its area is $k/2$, the corresponding altitudes are $6k$, $10k$, and $13k$. Now rescale $\mathcal{T}$ by multiplying all sides by $1/k$. The altitudes are rescaled by the same factor, so they become 6, 10, and 13.

Comment. The area of $\mathcal{T}$ can be computed explicitly by using *Heron's Formula*—the area of the triangle with sides a, b, and c is $\sqrt{s(s-a)(s-b)(s-c)}$, where s is the semiperimeter $(a + b + c)/2$. The area is needed if we want to find the sides of the triangle that has altitudes 6, 10, and 13. But the area is not needed if we merely want to show that there *is* such a triangle.

III-34. Triangle ABC has a right angle at C. The inscribed circle of $\triangle ABC$ meets the hypotenuse at W, where $AW = p$ and $BW = q$. Find a formula for the area of $\triangle ABC$ in terms of p and q.

Solution. Let the inscribed circle meet BC at U and AC at V. A glance at Figure 3.19 shows that $AV = p$ and $BU = q$. Let $r = CU = CV$. (It so happens that r is the radius of the inscribed circle, but that will play no role.) Then the right triangle has legs $p + r$ and $q + r$. By the Pythagorean Theorem,

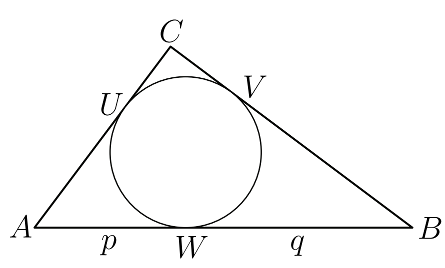

Figure 3.19: The Area of a Triangle

$$(p + r)^2 + (q + r)^2 = (p + q)^2.$$

Rewrite the above equation as $r^2 + pr + qr = pq$, then add pq to both sides to make the left side factor. We obtain $(p + r)(q + r) = 2pq$. The expression on the left is twice the area of $\triangle ABC$, so the triangle has area pq.

Comment. Things turn out almost as nicely if $\angle ACB$ is 120° instead of 90°. The Cosine Law gives

$$(p + r)^2 + (q + r)^2 + (p + r)(q + r) = (p + q)^2.$$

This equation can be rewritten as $(p + r)(q + r) = 4pq/3$. The area of $\triangle ABC$ is $(1/2)(p + r)(q + r)\sin 120°$, that is, $pq/\sqrt{3}$.

More generally, suppose that $\angle ACB = \theta$. Then

$$(p + r)^2 + (q + r)^2 - 2(p + r)(q + r)\cos\theta = (p + q)^2.$$

Expand the three squares, do some cancellation, divide by 2, and add pq to both sides. We get

$$(p + r)(q + r) - (p + r)(q + r)\cos\theta = 2pq.$$

The above equation can be rewritten as

$$\frac{(p + r)(q + r)\sin\theta}{2} = pq\frac{\sin\theta}{1 - \cos\theta}.$$

The expression on the left is the area of $\triangle ABC$. Now use the trigonometric identities

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2) \quad \text{and} \quad \cos\theta = 1 - 2\sin^2(\theta/2).$$

The area of $\triangle ABC$ is therefore $pq\cot(\theta/2)$.

III-35. The face diagonals of a box have length 5, 6, and 7. Find the volume of the box.

Solution. Let the faces with diagonal 5 have side a and b, and let c be the third side of the box. By the Pythagorean Theorem,

$$a^2 + b^2 = 25, \qquad b^2 + c^2 = 36, \qquad \text{and} \qquad c^2 + a^2 = 49.$$

To solve this system, we use a trick motivated by the symmetry. Adding up gives $2a^2 + 2b^2 + 2c^2 = 110$. So $c^2 = (a^2 + b^2 + c^2) - (a^2 + b^2) = 30$. Similarly, $b^2 = 19$ and $a^2 = 6$. The volume is $\sqrt{6 \cdot 19 \cdot 30}$.

III-36. Rectangle $\mathcal{R}$ has area 10.5 square meters. The circle that passes through the four corners of $\mathcal{R}$ has area 7π square meters. Find the sides of $\mathcal{R}$.

Solution. Let the sides be a and b. Then

$$ab = 10.5 \qquad \text{and} \qquad \pi\left(\frac{\sqrt{a^2+b^2}}{2}\right)^2 = 7\pi,$$

or more simply $2ab = 21$ and $a^2 + b^2 = 28$, so $a^2 + 2ab + b^2 = 49$. By subtracting $2ab$ instead of adding, we get in the same way $(a-b)^2 = 7$. Thus

$$a + b = 7 \qquad \text{and} \qquad a - b = \pm\sqrt{7}.$$

Solve for a and b by adding and subtracting. The sides are $(7+\sqrt{7})/2$ and $(7-\sqrt{7})/2$.

Comment. Another in the endless string of problems involving symmetric functions! Finding the perimeter would have been even simpler.

The equations can be solved almost as easily by substituting $10.5/a$ for b in $a^2 + b^2 = 28$—that's what most students do. But "breaking symmetry" early is often unwise, so it is useful to train oneself to hold on to symmetries as long as possible.

III-37. When a fine saw blade makes a straight cut, it turns to sawdust a strip of width 0.01 units, called the *kerf.* The blade was used to cut a 10×10 sheet of plywood into 100 equal squares. Calculate the area turned to sawdust, to 4 places after the decimal point.

Solution. Make the cuts and then squeeze to close the small gaps created by the saw. There are 9 cuts in each direction, each of width 0.01. After cutting and squeezing we have a 9.91×9.91 square, with area 98.2081. The area turned to sawdust is therefore 1.7919.

Another way: There are 9 cuts in each direction, each of length 10 and width 0.01, giving 1.8 as the area turned to sawdust. No, that's not quite right, for each of the 0.01×0.01 squares where the saw cuts meet has been counted twice. These 81 squares have combined area 0.0081, so the area turned to sawdust is 1.7919.

Comment. In the second solution, we obtained an incorrect answer of 1.8, then corrected it to 1.7919. But 1.8 is close enough for all practical purposes. This example illustrates the fact that if ϵ is close to 0, then ϵ^2 is often negligibly small in comparison with ϵ. (In the plywood problem, $\epsilon = 0.01$.) The fact that ϵ^2 is negligible compared with ϵ lies at the heart of the calculus.

III-38. A bamboo is 10 *ch'ih* tall. It is broken, and the top touches the ground 3 *ch'ih* from the root. How high up is the break?

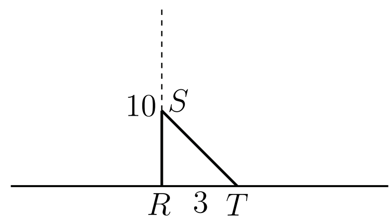

Figure 3.20: The Broken Bamboo

Solution. Let the bottom ("root") of the bamboo stalk be at R, the place of the break at S, and let the old top of the bamboo meet the ground at T. Assume that the bamboo grows vertically on level ground. So $\angle SRT$ is a right angle. Please see Figure 3.20. Let the height of the break, namely RS, be x. Then $ST = 10 - x$. By the Pythagorean Theorem, $x^2 + 3^2 = (10 - x)^2$. Simplify and solve: $x = 91/20$.

Another way: We can use two variables. Let $z = ST$. Then $z + x = 10$ and $z^2 - x^2 = 9$, so $z - x = 9/10$. It follows that $2x = 91/10$, so $x = 91/20$.

Comment. This problem is taken from the *Jiuzhang suanshu*—see I-14. The broken bamboo theme appears often in old Chinese texts. The same question occurs in the work of Brahmagupta, a seventh-century Indian astronomer and mathematician. His bamboo is 18 units long, and the top meets the ground 6 units from the root. It is not known whether the problem was transmitted from one civilization to the other.

There are no recorded sloping ground broken bamboo problems. If the top of the bamboo heads downhill, such a problem can be solved by using the Cosine Law instead of the Pythagorean Theorem.

III-39. The line ℓ is tangent to the parabola with equation $y = x^2$, and passes through $(2, 1)$. Find an equation for ℓ.

Solution. A quick sketch shows that there should be two such tangent lines. Let m be the slope of ℓ. Since ℓ passes through $(2, 1)$, it has equation $y = m(x - 2) + 1$.

In the equation $y = x^2$, replace y by $m(x - 2) + 1$. A tangent line meets the parabola in a single point. Equivalently, the equation $x^2 - mx + 2m - 1 = 0$ has exactly one solution. This is the case precisely if the discriminant $m^2 - 4(2m - 1)$ is equal to 0. Solve the resulting quadratic equation for m. The solutions are $m = 4 \pm 2\sqrt{3}$.

III-40. A square town has a gate at the midpoint of each of its sides. Twenty *chang* North of the northern gate is a tree which is just visible from the point one gets to by walking 14 *chang* South from the southern gate, then 1775 *chang* West. Find the length of a side of the town.

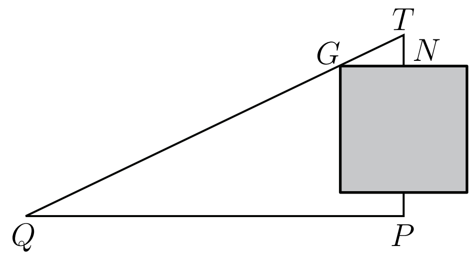

Figure 3.21: The Town with Four Gates

Solution. Please see Figure 3.21, which for clarity is not drawn to scale. Let the side of the town have length $2x$, let N be the middle of the northern gate, and T the location of the tree. Let P be the point 14 *chang* South of the southern gate, and Q the point 1775 *chang* West of P. Finally, let G be the point where the line QT just grazes the Northwest corner of the town.

Triangles TPQ and TNG are similar, and therefore $(34 + 2x)/1775 = 20/x$. Rewrite this as $2x^2 + 34x - 35500 = 0$ and solve: $2x = 250$.

Comment. This problem is taken from the *Jiuzhang suanshu* (see I-14). The numbers were obviously chosen by working backwards from the intended answer. Mathematics may change, but teachers of mathematics do not.

III-41. In a lake that swarmed with red geese, the tip of a lotus bud was seen half a *hasta* above the surface of the water. Forced by the wind, it gradually advanced, and was submerged at a distance of two *hasta*. Compute quickly, mathematician, the depth of the lake.

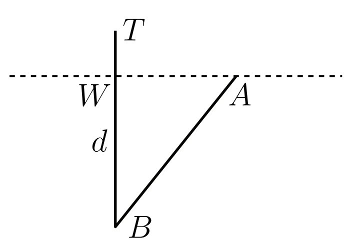

Figure 3.22: The Lotus Plant

Solution. Let the bottom of the lotus be at B, the tip at T, and suppose that initially the plant breaks the surface of the water at W. Then BW is the depth of the lake; call this d. (Please see Figure 3.22.)

When the lotus tip advanced, it met the water at the point labelled A, where $WA = 2$. And since AB is the height of the lotus plant, $AB = d + 1/2$. (We are meant to assume that the lotus stalk behaves like a rigid rod hinged at B.)

By the Pythagorean Theorem, $d^2 + 2^2 = (d + 1/2)^2$. Simplify and solve: $d = 15/4$.

Comment. This comes from Bhāskara's *Līlāvatī* which was written in verse, in keeping with an Indian custom of the time that it would be interesting to revive.

It is said that Bhāskara had a daughter named Līlāvatī whose husband died shortly after their wedding day, and that Bhāskara composed mathematical problems to help her through her grief. In another version of the story, Bhāskara found bad omens in Līlāvatī's horoscope and called off her wedding. Long ago, before universities and institutes, mathematicians could earn a respectable living as astrologers.

III-42. Let $\mathcal{H}$ be a regular hexagon. Join midpoints of consecutive edges of $\mathcal{H}$ to make the regular hexagon $\mathcal{K}$. Find the ratio of the area of $\mathcal{K}$ to the area of $\mathcal{H}$.

Solution. Choose the unit of length so that the edges of $\mathcal{H}$ have length 1, and let s be the edge length of $\mathcal{K}$.

Let A and B be consecutive vertices of $\mathcal{K}$, and let P be the vertex of $\mathcal{H}$ that lies between A and B. Then $\angle APB = 120°$, and sides PA and PB of $\triangle APB$ have length 1/2. By the Cosine Law

$$s^2 = (1/2)^2 + (1/2)^2 - 2(1/2)(1/2)(-1/2) = 3/4.$$

If lengths in a figure are scaled by the factor s, then the area of the figure is scaled by the factor s^2. It follows that $\mathcal{K}$ has three-quarters the area of $\mathcal{H}$.

Another way: The next solution uses less machinery, only Figure 3.23. In the picture, O is the (common) center of the hexagons, A and B are consecutive vertices of $\mathcal{K}$, and P is the vertex of $\mathcal{H}$ between A and B.

Draw lines from O to the vertices of $\mathcal{K}$. These lines divide $\mathcal{K}$ into 6 triangles, and $\mathcal{H}$ into 6 kite-shaped figures, so it is enough to compare the areas of OAB and $OAPB$. Let X be the center of the equilateral $\triangle OAB$, and join X to O, A, and B, splitting $\triangle OAB$ into three triangles each congruent to $\triangle APB$. Since $\triangle OAB$ is made up of three triangles congruent to $\triangle APB$ while the kite $OAPB$ is made up of four such triangles, the area of $\triangle OAB$ is three-quarters the area of the kite.

III-43. Two bamboos stand 27.8 *hasta* apart. One is 15 *hasta* high, and the other is 10 *hasta* high. Straight strings run from the top of each bamboo to the base of the other. How far above the ground do the two strings meet?

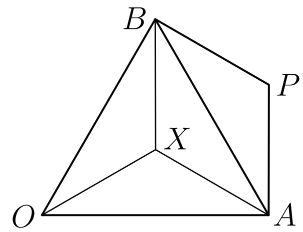

Figure 3.23: Joining the Edge Bisectors of a Hexagon

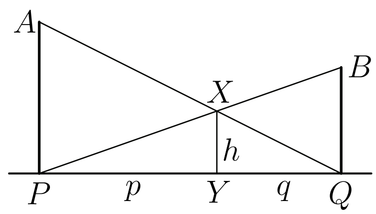

Figure 3.24: Two Bamboos

Solution. Please see Figure 3.24. Let A be the top of the 15 *hasta* bamboo and P its base. Let B and Q be the corresponding points for the other bamboo. For convenience and generality, write a for 15 and b for 10.

Let the strings meet at X and let Y be the point where the perpendicular from X meets the ground. Let p be the length of PY and q the length of QY. Finally, let h be the height of X off the ground.

Triangles APQ and XYQ are similar and therefore

$$\frac{h}{a} = \frac{q}{p+q} \qquad \text{and by symmetry} \qquad \frac{h}{b} = \frac{p}{p+q}.$$

Add up. The numbers p and q magically disappear, and some manipulation gives $h = ab/(a+b)$.

Comment. Note that the answer is independent of the distance between the bamboos. This question is adapted from a problem in Bhāskara's *Līlāvatī*—see III-41. Bhāskara didn't specify the distance between the bamboos.

III-44. In Figure 3.25, $\triangle ABC$ has area 27 square units, P is a point on the line segment AB, and Q is a point on BC. Line segment AP is 4 units long, $PB = 6$, $BQ = 8$, and $QC = 10$. Find the area of $\triangle PBQ$.

Solution. Draw the line segment AQ. View $\triangle ABQ$ as having base BQ, and $\triangle ABC$ as having base BC. The two triangles then have equal heights, so their areas are in the proportion $8 : 18$, and therefore $\triangle ABC$ has area 12.

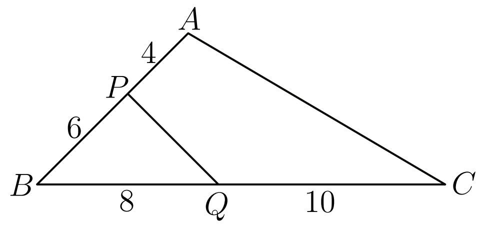

Figure 3.25: The Area of $\triangle PBQ$

Now view $\triangle PBQ$ and $\triangle ABQ$ as having bases 6 and 10 and the same height. So their areas are in the proportion $6 : 10$, and therefore $\triangle PBQ$ has area 7.2.

III-45. Find the area of an equiangular hexagon whose sides are alternately a and b.

Solution. The angles between the edges of any hexagon add up to 720°. Since the hexagon is equiangular, these angles are all 120°. Let the vertices of the hexagon, taken sequentially, be A, B, C, D, E, and F. The line segments AC, CE, and EA divide the hexagon into four parts, three congruent triangles and an inner equilateral triangle.

By the Cosine Law, the inner equilateral triangle has sides $\sqrt{a^2 + ab + b^2}$, and therefore area $(\sqrt{3}/4)(a^2 + ab + b^2)$. Each of the three outer triangles has area $(ab/2)\sin 120°$, that is, $ab\sqrt{3}/4$. So altogether the area of the hexagon is $(\sqrt{3}/4)(a^2 + 4ab + b^2)$.

III-46. A circle of radius 5 is internally tangent at P to a circle $\mathcal{C}$ of radius 9. The diameter of $\mathcal{C}$ that passes through P meets $\mathcal{C}$ again at Q. A tangent line drawn through Q to the smaller circle meets $\mathcal{C}$ at R. Find the length of QR. Please see Figure 3.26.

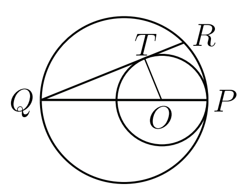

Figure 3.26: Internally Tangent Circles

Solution. Let O be the center of the smaller circle, and let T be the point of tangency of QR with this circle. Then $\angle QTO$ is a right angle. Also, $\angle QRP$ is a right angle. It follows that $\triangle QTO$ and $\triangle QRP$ are similar. Thus $QR/QT = QP/QO$.

But $QO = 18 - 5 = 13$, and therefore by the Pythagorean Theorem $QT = 12$. It follows that $QR = 12 \cdot (18/13)$.

Comment. In almost every problem about circles, it is useful to connect the center of a circle with some point.

III-47. A paper drinking cup is cone-shaped. When there is water in the cup to a depth of 4 inches, the cup contains 16 cubic inches of water. How much water is in the cup when the water is 6 inches deep?

Solution. The volume of a cone of height h and base radius r is $\pi r^2 h/3$. Let the radius of the "base" of the cone of water when the water is 4 inches deep be r. Then $4\pi r^2/3 = 16$, so $r = \sqrt{48/4\pi}$.

Now let R be the radius of the base of the cone of water when water is 6 inches deep. By a similar triangles argument, $R/r = 6/4$, and therefore $R = (6/4)\sqrt{48/4\pi}$. The volume of the larger cone is $6\pi R^2/3$. Now calculate (without a calculator). There is a lot of cancellation, and the answer turns out to be 54 cubic inches.

Another way: The second cone is simply a scaled up version of the first, with all lengths multiplied by 6/4. Scaling lengths by a scale factor s scales the volume by the factor s^3. Thus the new volume is $16(6/4)^3$.

Comment. The second solution is much more attractive than the first. It is essentially immediate, and relies on fundamental principles rather than manipulation of formulas.

The first solution only deals with a special case: the problem didn't say that the cone is right circular, but the solution relied on the volume formula for such cones. The second solution works for all cones, and even generalized cones. (A generalized cone is the surface obtained by joining all points on a simple closed plane curve to some point not in the plane of that curve.) So the second proof applies, for example, to a pyramidal drinking cup.

A slightly more complicated problem involves two identical cups. The water in the first is 6 inches deep and the second cup is empty. Water is poured into the second cup from the first to a depth of 4 inches, and the first cup is placed upright again. How deep is the water that remains in the first cup? Analysis using the volume formula is unpleasant, but a scaling analysis shows that the depth is $\sqrt[3]{6^3 - 4^3}$, about 5.3 inches, perhaps surprisingly high.

III-48. A quadrilateral is divided into four triangles by its diagonals. As we travel counterclockwise, the first three triangles that we meet have areas 8, 9, and 10. Find the area of the fourth triangle.

Solution. Let the vertices be P, Q, R, and S, and let the diagonals meet at M. Suppose that $\triangle PMQ$ has area 8, $\triangle QMR$ has area 9, and $\triangle RMS$ has area 10. Let a be the area of $\triangle SMP$.

View the first two triangles as having bases MP and MR. Then they have the same height, and therefore $MP/MR = 8/9$. The same argument applied to $\triangle RMS$ and $\triangle SMP$ shows that $MP/MR = a/10$. Thus $a = 80/9$.

Another way: The angles around M all have the same sine; call that common sine s. Recall that if x and y are two sides of a triangle, and the angle between these sides has sine s, then the triangle has area $(xy)(s/2)$. Since the first three triangles have area 8, 9, and 10,

$$(MP)(MQ)(s/2) = 8, \quad (MQ)(MR)(s/2) = 9, \quad \text{and} \quad (MR)(MS)(s/2) = 10.$$

It follows from these equations that $(MS)(MP)(s/2) = 80/9$.

III-49. On a spherical asteroid far far away, miners have bored a 10 kilometer straight tunnel that leads from the surface back again to the surface. When they are 1 km into the tunnel, they are 100 meters below the surface of the asteroid. How far below the surface are they halfway into the tunnel?

Solution. We first find the radius r of the asteroid. Let P be the start of the tunnel, Q the end, and W the point 1 kilometer into the tunnel from P. Let S and T be endpoints of the diameter of the asteroid that passes through W, with S the one closest to W. Then $WP = 1$, $WQ = 9$, $WS = 0.1$, and $WT = 2r - 0.1$. (Please see Figure 3.27.) Note that $\triangle WSP$ and $\triangle WQT$ are similar. This follows from the

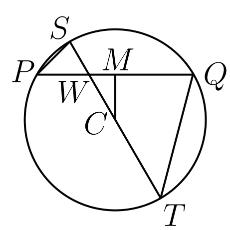

Figure 3.27: The Tunnel

fact that $\angle WSP = \angle WQT$—they are both angles subtended by the chord PT. So $0.1/1 = 9/(2r - 0.1)$, and therefore $r = 45.05$.

Finally, let C be the center of the asteroid, and M the midpoint of PQ. By the Pythagorean Theorem, $CM = \sqrt{(45.05)^2 - 5^2}$. So at the midway point the miners are $r - CM$, about 278 meters, below the surface.

III-50. The circle in Figure 3.28 has radius 1. Find the area of the square $ABCD$, where A and B are on the circle and side CD is tangent to the circle.

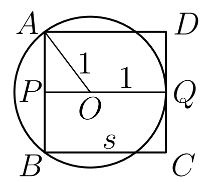

Figure 3.28: Circle and Square

Solution. Let O be the center of the circle. Draw the diameter through O that is perpendicular to CD. Let this diameter meet AB at P and CD at Q, and let s be the side length of the square.

Note that OP has length $s-1$, while AP has length $s/2$. Thus by the Pythagorean Theorem $(s-1)^2+(s/2)^2=1$. Simplify, solve for s, and square the result. The area is 2.56.

III-51. Let ABC be a triangle, with $CA=5$, $CB=12$, and $AB=13$. Let M be the midpoint of side AB. Find the length of CM.

Solution. Note that $5^2+12^2=13^2$, so by the converse of the Pythagorean Theorem $\triangle ABC$ is right-angled at C. The circle with center M and radius $13/2$ goes through A and B.

If AB is a diameter of a circle, and P is any point on the circle other than A and B, then $\angle APB=90°$. What is useful here is that the converse holds: if $\angle APB=90°$, then P lies on the circle. So C lies on the circle, and therefore $CM=13/2$.

Comment. The same argument works for any right-angled triangle. But one can say more. Let ABC be *any* triangle, let M be the midpoint of AB. The line segment CM is called a *median* of the triangle. Let $m=CM$, $a=BC$, $b=AC$, and $c=AB$. We express m in terms of a, b,, and c.

Let $\theta=\angle AMC$, and let $\phi=\angle BMC$. By the Cosine Law

$$b^2=\left(\frac{c}{2}\right)^2+m^2-cm\cos\theta \qquad \text{and} \qquad a^2=\left(\frac{c}{2}\right)^2+m^2-cm\cos\phi.$$

But $\cos\phi=-\cos\theta$. Adding gets rid of the cosine terms and gives the result

$$m^2=\frac{a^2+b^2}{2}-\frac{c^2}{4}.$$

The idea also works if M divides AB in the ratio $p:q$ instead of $1:1$.

The formula above is sometimes called the *Theorem of Apollonius,* after the great mathematician Apollonius of Perga, born around −260. Apollonius did brilliant work on conics. The result about the size of the median is minor, and may not be due to him.

III-52. Suppose that PA has slope 2, PB has slope 3, and $PA = PB$. What can be said about the slope of AB?

Solution. Move the axes so that the origin ends up at P. The coordinates of A then have shape $(a, 2a)$ and the coordinates of B have shape $(b, 3b)$. The condition $PA = PB$ translates as $5a^2 = 10b^2$. Without loss of generality, we can take $b = 1$, since the answer doesn't change under change of scale, or under rotation through $180°$ about P. Thus $a = \pm\sqrt{2}$.

The slope of AB is therefore $(3 \mp 2\sqrt{2})/(1 \mp \sqrt{2})$. That can be simplified to $1 \mp \sqrt{2}$.

III-53. A *regular octagon* is an eight-sided polygon with all sides equal and all angles equal. Find, exactly, the area of a regular octagon with sides 1.

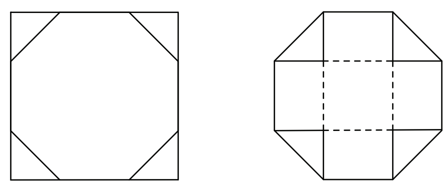

Figure 3.29: The Area of a Regular Octagon

Solution. Two approaches are illustrated in Figure 3.29. Think of the octagon as a square with the corners knocked off. Each sliced off corner is an isosceles right triangle with hypotenuse 1, so each leg has length $1/\sqrt{2}$. Thus the square has side $1 + \sqrt{2}$. The corner triangles have combined area 1, either by a computation or (better!) because they can be assembled into a 1×1 square. So the area of the octagon is $(1 + \sqrt{2})^2 - 1$.

Another way: Start with the same square and fold the corners inwards. The octagon can now be viewed as four triangles with combined area 1, together with a cross made up of a central 1×1 square and four arms that are $1 \times (1/\sqrt{2})$ rectangles.

Another way: Join the vertices to the center O of the octagon, splitting the octagon into eight triangles. Let OAB be such a triangle. Then $AB = 1$, $\angle AOB = 45°$, and $\angle OAB = \angle OBA = 67.5°$.

If h is the perpendicular distance from O to AB, then $\triangle OAB$ has area $h/4$. But $h = (1/2)\tan 67.5°$, and now we can use the calculator. To find the *exact* answer, compute $\tan 67.5°$ by using the fact that $\tan 2\theta = 2\tan\theta/(1 - \tan^2\theta)$. If $\theta = 67.5°$, then since $\tan 2\theta = -1$, $\tan^2\theta - 2\tan\theta - 1 = 0$. Now solve for $\tan\theta$.

A slower way is to use the more familiar identity $\cos 2\theta = 2\cos^2\theta - 1$. Or else OA can be calculated using the Cosine Law: if $OA = r$, then $1^2 = 2r^2 - 2r^2 \cos 45°$. Then h can be found by using the Pythagorean Theorem. Ugly!

III-54. Draw the circle of radius 1 with center the origin O. Let $A = (1, 0)$, and B the point on the top half of the circle such that $\angle AOB = 60°$. Put two red dots distance 1 apart on the edge of a ruler. Slide/swing the ruler so that it passes through B and one red dot falls on the negative x-axis while the other falls on the circle. Let the ruler meet the x-axis at X. Find $\angle BXO$, exactly.

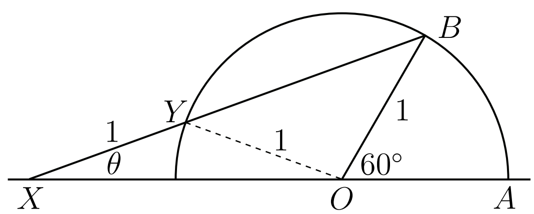

Figure 3.30: Trisecting Angles with a Marked Ruler

Solution. Let Y be the red dot where the ruler meets the circle. Because OB and OY are radii, each has length 1. And $XY = 1$ because the red dots ended up at X and Y. Let $\theta = \angle BXO$. We find θ by doing some angle-chasing in Figure 3.30.

Since $\triangle XOY$ is isosceles, $\angle XOY = \theta$. Thus $\angle OYX = 180° - 2\theta$, and therefore $\angle OYB = 2\theta$. Since $\triangle BYO$ is isosceles, $\angle OBY = 2\theta$, and $\angle BOX = 180° - \theta - 2\theta$. But $\angle BOX = 180° - 60°$, so $3\theta = 60°$ and $\theta = 20°$.

Comment. The instructions "Slide/swing the ruler ..." need to be demonstrated with a real marked straightedge, since it is an unorthodox use of the ruler. Exactly the same method can be used to trisect any acute angle, not just the 60° angle. The above construction, which was known to Archimedes and probably earlier, shows that the general angle can be trisected with compass and *marked* straightedge.

Since antiquity there have been attempts to trisect the general angle with compass and *unmarked* straightedge. Finally, in 1837, Wantzel *proved* that it can't be done—Gauss may have proved this in 1800. Wantzel's result didn't deter the armies of angle-trisectors who kept submitting fallacious compass and straightedge trisections.

III-55. Triangle ABC has perimeter 18, with $AB = 8$. The bisector of $\angle ACB$ meets AB at D, where $AD = 3$. Find the length of AC.

Solution. View $\triangle ADC$ and $\triangle BDC$ as having bases AD and BD. Then they have the same height, so their areas are in the same proportion as their bases, namely $3 : 5$.

Let $AC = x$ and $BC = y$. Reflect $\triangle BDC$ in line CD. Then B is reflected to some point B' on the line AC. View $\triangle ADC$ and $\triangle B'DC$ as having bases CA and CB' and the same height. So their areas are in the proportion $x : y$.

We conclude that $x/y = 3/5$. But $x + y = 10$, and therefore $x = 10(3/8) = 3.75$.

III-56. Reinforcing poles run from six equally spaced points around the base of a cone-shaped tent to the apex of the tent. The angle between neighbouring poles is 30°. Find the ratio of the height of the tent to the radius of its base.

Solution. Let the base have radius 1; we compute the height h. Let A be the apex of the tent, and let S and T be points where neighbouring poles meet the base. Then S and T are consecutive vertices of a hexagon inscribed in a circle of radius 1, so $ST = 1$.

The length p of a pole can be found by using the Cosine Law on $\angle SAT$. We have $1 = p^2 + p^2 - 2p^2(\sqrt{3}/2)$, and therefore $p^2 = 1/(2 - \sqrt{3}) = 2 + \sqrt{3}$. But by the Pythagorean Theorem $h^2 + 1^2 = p^2$, so $h = \sqrt{1 + \sqrt{3}}$. The ratio of the height to the radius is about 1.65.

Comment. The tent, with various numbers of reinforcing poles, can be used to generate accessible questions that require some thought about the geometry. The numbers here permit an "exact" answer, which is not really appropriate in this applied context. It may be better to have 7 poles, and a 26° angle between poles.

School texts have analogous questions about pyramids. Unfortunately the ideas used are forgotten, or not learned, and what remains is canned formulas to be used on tests. Since we seldom bump into pyramids, the formulas have little practical importance, but ideas always do.

III-57. A triangle ABC with $AB = 22$, $AC = 23$, and $BC = 24$ is cut out of paper. When the triangle is folded along a line parallel to BC, 64% of the area of $\triangle ABC$ sticks out beyond BC. Find the length of the fold line.

Solution. The triangle that sticks out is similar to $\triangle ABC$. Because it has area 0.64 times the area of $\triangle ABC$, its sides are $\sqrt{0.64}$ times the corresponding sides of $\triangle ABC$. Unfold, and label points as in Figure 3.31. Since the line segment RS

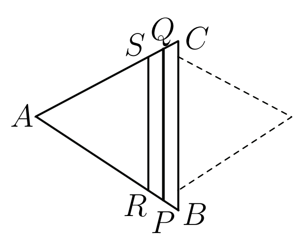

Figure 3.31: The Folded Triangle

has length 0.8 times BC, and PQ is equidistant from RS and BC, it follows that

$\triangle APQ$ has 0.9 times the linear dimensions of $\triangle ABC$. In particular the fold line PQ has length 21.6.

III-58. Circles $\mathcal{A}$ and $\mathcal{B}$ have centers A and B. Each circle has radius 1, and they are tangent to each other at O. A third circle $\mathcal{C}$ of of radius 1 is tangent to AB at O. Find the area of the region which is inside $\mathcal{C}$ but outside $\mathcal{A}$ and $\mathcal{B}$.

Solution. Circle $\mathcal{C}$ has area π. Remove from this twice the area of overlap between $\mathcal{C}$ and $\mathcal{A}$. These two circles meet at O and at another point X.

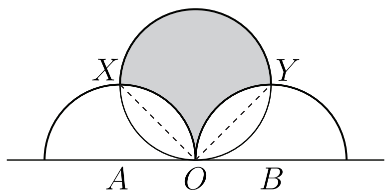

Figure 3.32: Three Circles

To find the area in common between $\mathcal{A}$ and $\mathcal{C}$, first draw the line segment XO. That splits the region of overlap into two equal parts. Either part can be viewed as a quarter-circle with a triangle removed, and so has area $\pi/4 - 1/2$. Thus the region inside $\mathcal{C}$ but outside $\mathcal{A}$ and $\mathcal{B}$ has area $\pi - 4(\pi/4 - 1/2)$, that is, 2. Interesting answer! No π, despite all the curvy bits?

Another way: Using a razor knife, cut out rectangle $ABYX$. Then cut out the semicircle XOY. That leaves two pieces, each with two straight sides and a side which is a quarter-circle. The three pieces can be reassembled to make the shaded region, so the area of the shaded region is the same as the area of the rectangle, namely 2.

III-59. A right-angled triangle has legs a and b. Outward facing squares are erected on the three sides, then string is placed around the whole figure and drawn tight as in Figure 3.33. Express the area enclosed by the string

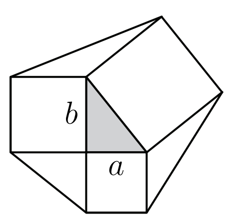

Figure 3.33: A Triangle, Squares, and String

in terms of a and b.

Solution. The hexagon enclosed by the string is made up of three squares and four triangles. The areas of the squares are easy: a^2, b^2, and, by the Pythagorean Theorem, $a^2 + b^2$. The only remaining concern is with the areas of two of the outer triangles. Let the vertices of the triangle be A, B, and C, with the usual conventions.

Look at the outer triangle that has A as a vertex. Its area can be computed by using a formula. But it is more attractive to rotate the triangle clockwise about A through 90° until one of its edges coincides with AB. The rotated triangle has base b and height a, just like $\triangle ABC$, so it has area $ab/2$. Similarly, the outer triangle that has B as a vertex has area $ab/2$. The other two triangles are right-angled with legs a and b, so they also have area $ab/2$. Add up: The string encloses an area of $2(a^2 + ab + b^2)$ square units.

III-60. Two parallel lines have x-intercepts that differ by 6 and y-intercepts that differ by 8. Find the distance between the lines.

Solution. Choose a point P on one of the lines. Move the x-axis up or down until it passes through P, and move the y-axis left or right until it passes through P. Suppose that the other line meets the new x-axis at Q, and the new y-axis at R. Then $PQ = 6$ and $PR = 8$, for although x-intercepts change, their difference does not, and the same remark holds for the y-intercepts.

Now drop a perpendicular from P to QR and use similar triangles. Or else note that by the Pythagorean Theorem, $QR = 10$, so if h is the perpendicular distance from P to QR then $\triangle PQR$ has area $5h$. But this triangle has area 24, and therefore $h = 24/5$.

Comment. It is often strategically useful to move a figure to a better position with respect to the coordinate axes, or equivalently to move the axes. In the problem above, though the intercepts change, the distances between them do not, so moving the axes does not change the geometry.

III-61. One side of a triangle has length 12 and another has length 5. The angles of the triangle are all less than 90°. What can be said about the third side?

Solution. Since $5^2 + 12^2 = 13^2$, the converse of the Pythagorean Theorem shows that the triangle with sides 5, 12, and 13 is right-angled. Imagine that there is a hinge between the sides 5 and 12, and that the third side z can grow or shrink.

Increase z beyond 13. The angle between the sides 5 and 12 then grows beyond 90°. So if all angles are to be acute, then z must be less than 13.

Start again with $z = 13$, and let z shrink. For a while, all angles are acute. They remain acute until the triangle becomes right-angled again, with hypotenuse 12. If the third side is z_0 at that time, then $5^2 + z_0^2 = 12^2$, so $z_0 = \sqrt{119}$. Thus all angles are acute if and only if $\sqrt{119} < z < 13$.

III-62. Let $\mathcal{T}$ be a right-angled triangle. An outward facing square $\mathcal{S}$ is erected on the hypotenuse of $\mathcal{T}$. Show that if line ℓ bisects the right angle of $\mathcal{T}$, then ℓ splits $\mathcal{S}$ into two equal parts.

Solution. Let $\mathcal{T}$ have its right angle at P, and let a and b be the legs of $\mathcal{T}$. In Figure 3.34, triangle $\mathcal{T}$ has been embedded in an $(a+b) \times (a+b)$ square $PQRS$, in order to make everything as symmetrical as possible.

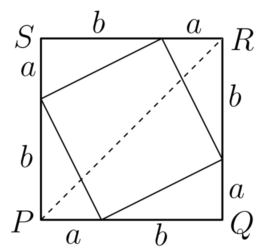

Figure 3.34: Bisecting the Square on the Hypotenuse

The dashed line PR bisects the right angle at P, so it is line ℓ of the problem. Use a razor knife to cut the square $PQRS$ into two triangles along the dashed line, turn $\triangle PRS$ through 180°, and place it on top of $\triangle RPQ$. The symmetry of the diagram shows that now all the lines drawn in $\triangle PRS$ coincide with the corresponding lines on $\triangle RPQ$, and therefore the square on the hypotenuse is indeed bisected. In the absence of a razor knife, note that there is mirror symmetry across a point mirror at the center of $PQRS$.

Another way: At the other extreme, it is possible to grind things out in an ugly but mechanical way. In Figure 3.34, choose P as the origin, and let the other two vertices of $\mathcal{T}$ have coordinates $(a, 0)$ and $(0, b)$. Then the hypotenuse lies along the line $x/a + y/b = 1$. Line ℓ has equation $y = x$, so ℓ meets the hypotenuse at a point whose coordinates are easy to calculate—they happen to be $ab/(a+b)$.

Now look at the line that passes through $(a+b, a)$ and $(b, a+b)$. This is the side of the square on the hypotenuse which is parallel to the hypotenuse itself. It is not hard to find where ℓ meets this line. Then one can show that the square on the hypotenuse is bisected by computing two distances.

III-63. A farmer grows Christmas trees in an 80 meter by 80 meter field. Any two trees must be at least 2 meters apart, and the trees must be planted in rows parallel to one of the edges of the field. About how many trees can be planted?

Solution. The rows should *not* be 2 meters apart. It is significantly more efficient to arrange the trees to form equilateral triangles with side 2 meters. The height of an equilateral triangle of side 2 is $2\sin 60°$, so the rows can be placed $\sqrt{3}$ meters apart. The farmer can put in 47 rows, since $80/\sqrt{3}$ is about 46.2. A little playing with a sketch shows that 24 of the rows can have 41 trees each and the rest 40, giving 1904 trees. Putting rows 2 meters apart only allows 1681 trees.

Comment. Questions about optimal packing can be difficult. For example, *Kepler's Problem* about the best way to pack space with equal spheres was only solved very recently, four hundred years after being posed! Optimal packing is important in the design of good *error-correcting codes* for the reliable transmission of information over "noisy" channels.

III-64. Find the slopes of the straight lines that pass through $(0, 5)$ and are tangent to the ellipse $x^2/36 + y^2/9 = 1$.

Solution. The standard approach uses the differential calculus, but there are other ways to solve the problem. Let the equation of the line be $y = mx + b$. The y-intercept b is equal to 5. To find m, we use the fact that $y = mx + 5$ meets the ellipse at exactly one point.

Substitute $mx + 5$ for y in the equation of the ellipse and simplify. This yields

$$(4m^2 + 1)x^2 + (40m)x + 64 = 0,$$

which has one solution if and only if the discriminant is 0, that is, if and only if

$$(40m)^2 - (4)(4m^2 + 1)(64) = 0.$$

Solve for m. It turns out that $m = \pm 2/3$. (The problem is symmetrical about the y-axis, so slopes *must* come in $\pm$ pairs.)

Another way: There is a nicer geometric solution. Imagine that the tangent line has been drawn. Stretch all distances in the up-down direction by a factor of 2. The equation of the ellipse is transformed to $x^2 + y^2 = 36$.

The tangent line now meets the y-axis at $y = 10$. Suppose that the point of tangency is given by $x = a$, $y = b$. The equation $x^2 + y^2 = 36$ is the equation of a circle, so the line joining the center of the circle to the point of tangency is perpendicular to the tangent line.

By using similar triangles, or the fact that slopes of perpendicular lines have product -1, we find that

$$\frac{b - 10}{a} = -\frac{a}{b}, \qquad \text{that is,} \qquad a^2 + b^2 = 10b.$$

But $a^2 + b^2 = 36$, so $b = 36/10$, and therefore $a = \pm 24/5$.

Now undo the stretching by multiplying all lengths in the y direction by $1/2$. The real y-coordinate of the point of tangency is therefore $9/5$, while the x-coordinate is unchanged. The possible slopes of the tangent line turn out to be $\pm 2/3$.

Comment. The strategy used in the second solution is sometimes called *Transform, Solve, Transform Back.* The slightly unfamiliar ellipse is transformed into a circle by suitable stretching. Note that tangency is preserved by the stretching, but slope is not. The tangent problem is solved for the circle by exploiting special features of the geometry of the circle, and then the stretching is undone to go back to the world of the ellipse. For other examples of the strategy in this chapter, see 3 and III-79.

The fact that an ellipse is just a stretched circle has a number of other applications. For example, it can be used to argue that if a and b are positive, the area enclosed by the ellipse $x^2/a^2 + y^2/b^2 = 1$ is πab.

III-65. A right-angled triangle has area 1 and perimeter 8. Find the length of the hypotenuse.

Solution. It is no harder to find the hypotenuse if the area is A and the perimeter p. We could also ask what combinations of A and p are possible. That is more difficult: see X-33.

Let x and y be the legs of the triangle. Then

$$A = \frac{xy}{2} \qquad \text{and} \qquad p = x + y + \sqrt{x^2 + y^2}.$$

We can find x and y and then $\sqrt{x^2 + y^2}$, but it is better to preserve symmetry. Let h be the hypotenuse. Then

$$(x + y)^2 = h^2 + 2xy = h^2 + 4A.$$

But

$$(x + y)^2 = (p - h)^2 = p^2 - 2ph + h^2.$$

Thus $h^2 + 4A = p^2 - 2ph + h^2$, and $h = (p^2 - 4A)/2p$.

III-66. A 3 cm wide ruler is placed across a 15 cm by 15 cm sheet of paper. One edge of the ruler passes through a corner of the sheet, and the *other edge* of the ruler passes through the diagonally opposite corner. How much of the sheet is covered by the ruler?

Solution. Let the corners of the sheet, taken counterclockwise, be A, B, C, and D, and suppose that the edges of the ruler pass through A and C. Let e be the edge of the ruler that passes through C. We may assume that e meets the line segment AB, say at P. (If the sheet were not square, there would be two cases to consider.) Drop a perpendicular from A to e, meeting e at R. Then $AR = 3$. (Please see Figure 3.35.)

Let $AP = x$. By the Pythagorean Theorem, $PC = \sqrt{(15 - x)^2 + 15^2}$. By similarity

$$\frac{x}{3} = \frac{\sqrt{(15 - x)^2 + 15^2}}{15}.$$

Square both sides, simplify, and solve: $x = 15/4$.

The region covered by the ruler is a parallelogram with base 15/4 and height 15, so its area is 1/4 of the area of the sheet.

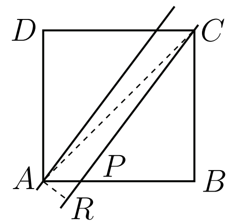

Figure 3.35: The Area Covered by the Ruler

Another way: Use again Figure 3.35. Since AC is the diagonal of a square of side 15, $AC = 15\sqrt{2}$. Let $\theta = \angle ACR$. Then $\sin\theta = 3/(15\sqrt{2})$.

Now the area of $\triangle PBC$ can be found to great accuracy. The calculator shows that θ is close to $8.1301°$. Since $\angle ACB = 45°$, it follows that $\angle PCB$ is close to $36.8700°$, and therefore 18.7500 is an excellent approximation to PC. To find the area covered by the ruler, multiply by 3.

Comment. The same methods work for rectangular sheets of paper, and can be extended to parallelograms.

A calculator was used in the second solution, but the idea can be pushed through without one. Let $\phi = \angle PCB$. Then $\phi = 45° - \theta$. By the formula for the cosine of the difference of two angles,

$$\cos\phi = \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta.$$

But $\sin\theta = 1/(5\sqrt{2})$ and therefore $\cos\theta = 7/(5\sqrt{2})$, so $\cos\phi = 2/5$.

III-67. An 8.5 by 11 sheet of note paper is folded, and one corner lands on the diagonally opposite corner. Find the area covered by the folded sheet.

Solution. There is nothing particularly interesting about 8.5 and 11, so let the width of the paper be w and the height h, with $h > w$. Label points as in Figure 3.36. The region covered by the folded sheet can be viewed as $\triangle CDQ$ together with

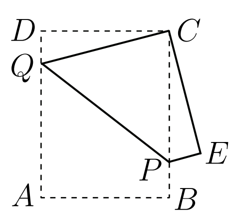

Figure 3.36: The Area of a Folded Sheet

quadrilateral $PECQ$. To find the area of $\triangle CDQ$, it is enough to find DQ. So let

$DQ = x$. Then $QC = QA = h - x$. By the Pythagorean Theorem, $(h-x)^2 = b^2 + x^2$, and therefore $x = (h^2 - b^2)/2h$. So $\triangle CDQ$ has area $b(h^2 - b^2)/4h$.

The quadrilateral $PECQ$ is congruent to $PBAQ$. But the area of $PBAQ$ is just half the area of the sheet of paper, that is, $bh/2$. (To see that PQ bisects the sheet of paper, draw AC and use symmetry.)

Finally, add and simplify. The region covered by the folded sheet has area $b(3h^2 - b^2)/4h$. If $b = 8.5$ and $h = 11$, the area is about 56.17.

III-68. Let ABC be a triangle, and let ℓ be the line that passes through A and B. Unless $\angle ACB$ is a right angle, there are two points P, Q on ℓ such that $\angle CPA = \angle CQB = \angle ACB$. Show that

$$(AC)^2 + (BC)^2 = (AB)(AP + BQ).$$

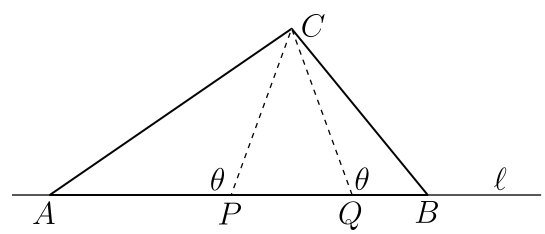

Figure 3.37: ibn Qurra's Generalization of the Pythagorean Theorem

Solution. Let $\theta = \angle ACB$. Figure 3.37 has been drawn with θ obtuse. Because $\triangle ACB$ and $\triangle APC$ are similar, $AC/AP = AB/AC$. Similarly, $BC/BQ = AB/BC$. Thus

$$(AC)^2 = (AB)(AP) \qquad \text{and} \qquad (BC)^2 = (AB)(BQ).$$

Add to obtain the desired result.

Comment. If θ is acute, then the positions of P and Q are roughly speaking interchanged. Note that P and Q need not lie between A and B.

If θ is a right angle, then P and Q coincide, so $(AB)(AP + QB) = (AB)^2$ and we obtain the Pythagorean Theorem. This generalization of the Pythagorean Theorem was probably discovered by the ninth-century mathematician Thābit ibn Qurra.

III-69. Triangle PQR has area 4 square units. Point P has coordinates $(2, 1)$, and Q has coordinates $(3, -2)$. The third vertex R lies on the x-axis. Find the coordinates of R.

Solution. Figure 3.38 makes it clear that there are two possibilities for R. Let X be the point where PQ meets the x-axis. To find the x-coordinate of X, find the equation of line PQ, and set $y = 0$. More efficiently, note that XP has the same slope as PQ, so if $X = (x, 0)$ then $-1/(x-2) = -3$, and therefore $x = 7/3$. Triangle

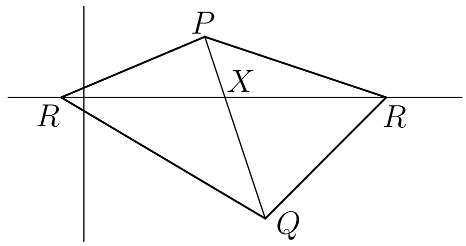

Figure 3.38: The Third Vertex of a Triangle

PQR can be cut up into $\triangle PXR$, which has base XR and height 1, and $\triangle QXR$, with base XR and height 2. So $\triangle PQR$ has area $(3/2)(XR)$. But that area is 4, and therefore $XR = 8/3$. It follows that the x-coordinate of R is $7/3 \pm 8/3$.

III-70. Let $\mathcal{S}$ be a square of area 2. The variable point P travels in the plane of $\mathcal{S}$ in such a way that the sum of the squares of the distances from P to the vertices of $\mathcal{S}$ is 6. Identify the path followed by P.

Solution. Place the origin at the center of the square, and the axes so that the vertices of $\mathcal{S}$ are $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. Let $P = (x, y)$. The condition on P translates to

$$(x-1)^2 + y^2 + x^2 + (y-1)^2 + (x+1)^2 + y^2 + x^2 + (y+1)^2 = 6.$$

Expand and simplify. We get $x^2 + y^2 = 1/2$, the equation of the circle which has the same center as the square and which is internally tangent to the square.

Comment. The problem can also be solved geometrically, but there is no good reason to do so. The problem has obvious symmetry about the center of the square, so if we guess that the answer is a circle, it is easy to see *which* circle it is. But it is not clear that the answer should be a circle. If for example "sum of squares of distances" is replaced by "sum of distances,"' then the geometry may feel similar, but the curve is definitely not a circle.

Let A_1, A_2, ..., A_n be points in the plane, and let a_1, a_2, ..., a_n, and k be real numbers. A point P moves in the plane of the A_i in such a way that

$$a_1(PA_1)^2 + a_2(PA_2)^2 + \cdots + a_n(PA_n)^2 = k.$$

In general, the path traced by P is a circle, but in degenerate cases the path may be empty, or a single point, or a line. Analytic geometry makes the proof easy.

Remarkably enough, this result was proved around −200 by the great mathematician Apollonius, more than 1800 years before the birth of analytic geometry. It is one of 147 theorems in the lost work *Plane Loci.* We only know about these theorems because Pappus (300?) commented on some of them.

III-71. Show that the circles

$$(x-a)^2+(y-b)^2=2a^2 \quad \text{and} \quad (x-b)^2+(y+a)^2=2b^2$$

meet at right angles.

Solution. Let P and Q be the centers of the circles. So $P=(a,b)$ and $Q=(b,-a)$. Let R be an intersection point of the circles. What is *meant* by the angle between the two circles at R? A sensible interpretation is that it is the angle between the tangent lines at R. The tangent line to a circle at R is perpendicular to the diameter through R. It is therefore enough to show that $\angle PRQ=90°$. Note that

$$(PQ)^2=(a-b)^2+(b+a)^2=2a^2+2b^2=(RP)^2+(RQ)^2.$$

By the converse of the Pythagorean Theorem $\angle PRQ=90°$. Thus *if* the circles meet, they do so at right angles. To check that they meet, we can compute the intersection points explicitly. It is better to note that PQ is greater than either radius, but less than the sum of the radii, so the circles do meet.

III-72. The rectangle $ABCD$ in Figure 3.39 has $AB=3$ and $BC=4$. Find the area of rectangle $BPQD$.

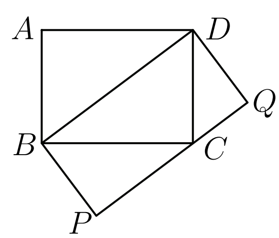

Figure 3.39: A Rectangle on the Diagonal of a Rectangle

Solution. By the Pythagorean Theorem, $BD=5$. Let $h=BP$. Note that $\triangle BPC$ is similar to $\triangle DCB$. It follows that $h/4=3/5$. So $h=12/5$, and therefore the area is 12.

Is it a coincidence that the area of $ABCD$ is also 12? In general let $AB=m$ and $BC=n$. Then $BD=\sqrt{m^2+n^2}$. The similarity argument of the preceding paragraph shows that $h/n=m/\sqrt{m^2+n^2}$. It follows that $h=mn/\sqrt{m^2+n^2}$, and therefore the area of $BPQD$ is mn, which is also the area of $ABCD$.

Another way: The result can be proved without computation. Use a razor knife to cut out $\triangle CQD$, and slide the triangle to the left until CD coincides with BA. Since $\angle DCQ = \angle ABD$, it follows that CQ now runs along BD.

Cut out $\triangle BPC$ and slide it upwards until BC coincides with AD. Now $\triangle ABD$ is covered without overlap by the cut out pieces. Thus $BPQD$ can be dissected into three pieces that can be reassembled to form $ABCD$, and therefore the two rectangles have equal areas.

Comment. More complicated-looking problems that use the same ideas can be generated by starting with a parallelogram $ABCD$.

III-73. The circle C_0 has center $(0, 1)$ and radius 1. Let $\mathcal{F}$ be the family of all circles that are tangent both to C_0 and to the x-axis, and whose centers do not lie on the y-axis. What kind of curve is swept out by the centers of the circles in $\mathcal{F}$?

Solution. An informal sketch shows that the centers lie on a curve that looks roughly like a parabola. Many curves that look like parabolas are not, but this one will turn out to be a parabola.

Let C be a circle with center (x, y), where $x \neq 0$. A sketch shows that if $y \leq 0$, then C can't be simultaneously tangent to the x-axis and to C_0. So we may assume that $y > 0$. Then C is tangent to the x-axis if and only if it has radius y.

Suppose now that C has center (x, y) and radius y, and that $x \neq 0$. A sketch shows that C is tangent to C_0 if and only if the distance from the center of C to the center of C_0 is $1 + y$, or equivalently

$$\sqrt{(x-0)^2 + (y-1)^2} = 1 + y.$$

Square both sides and simplify. We get $x^2 = 4y$, the equation of a standard parabola. Not quite. Since $y > 0$, the vertex is missing.

Another way: Let C_0 be a circle with center A and radius r, and let ℓ_0 be a line tangent to C_0 at the point O. Let $\mathcal{S}$ be the family of circles that are tangent to both ℓ_0 and C_0, but not at O. We want to identify the curve swept out by the centers of circles in $\mathcal{S}$.

Let ℓ be the line which is at distance r from ℓ_0, but on the other side of ℓ_0 from C_0. Then P is the center of a circle tangent to ℓ_0 and to C_0 (but not at O) if and only if P is equidistant from A and ℓ, and $P \neq O$.

The classical definition of a parabola is that it is the curve swept out by a particle that travels so as to remain equidistant from a fixed point, the *focus,* and a fixed line, the *directrix.* So without coordinates, and in a somewhat more general setting, we can conclude that our curve is a parabola.

III-74. A farmer wants to divide the trapezoidal field of Figure 3.40 equally between two children, using a line PQ parallel to the southern boundary AB of the field. Given that AB is 50 meters long, $BC = 90$, $CD = 32$, and BC is perpendicular to AB, how should the division be done?

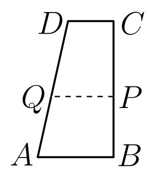

Figure 3.40: Dividing the Trapezoidal Field

Solution. Drop a perpendicular from the northwest corner of the field to the southern edge. That divides the field into a triangle $\mathcal{T}$ with area $9 \cdot 90$ and a rectangle $\mathcal{R}$ with area $32 \cdot 90$. Each child should therefore receive 1845 square meters.

The division will be completely specified once the length w of CP is known. By a scaling argument, the part of $\mathcal{T}$ above PQ has area $(w/90)^2(9 \cdot 90)$, that is, $w^2/10$. The part of $\mathcal{R}$ above PQ has area $32w$, and therefore

$$\frac{w^2}{10} + 32w = 1845.$$

The positive root of the above equation is about 49.88. So the division point P should be 49.88 meters away from C along the line segment CB.

Another way: A similar calculation can be done using coordinates. Since the first quadrant is the most comfortable part of the plane, it would be nice to have the northern boundary at the bottom, and the eastern boundary along the y-axis. So rotate the map through 180°.

The corners are then $(0, 0)$, $(32, 0)$, $(50, 90)$, and $(90, 0)$. The line joining $(32, 0)$ and $(50, 90)$ has equation $y = 5(x - 32)$. Let the division line be $y = w$. This line meets $y = 5(x - 32)$ at $x = 32 + w/5$.

By the standard formula for the area of a trapezoid, the area of the part of the field below $y = w$ is $(32 + 32 + w/5)(w/2)$. Set this equal to 1845, half the area of the field, and simplify. That gives the equation $w^2 + 320w - 18450 = 0$, which has positive root close to 49.88. The division line is about 49.88 meters south of the northern edge of the farm.

Another way: The last argument is harder but more geometric. Extend AD and BC until they meet at a point O. Let k be the area of $\triangle ODC$. There is no need to calculate k, indeed there is good reason not to: this approach works even if AB and BD are not perpendicular.

By similarity, $\triangle OAB$ has area $(50/32)^2 k$, so the field has area $((50/32)^2 - 1)k$. Divide by 2 and add k to find the area of $\triangle OQP$. The ratio of the areas of $\triangle OQP$ and $\triangle ODC$ is therefore $((50/32)^2 + 1)/2$.

We conclude that OP is OC multiplied by the scale factor s, where $s = \sqrt{((50/32)^2 + 1)/2}$. But OB is OC multiplied by the scale factor $50/32$, and therefore

$$CP : 90 = s - 1 : \frac{50}{32} - 1.$$

Calculate: CP is about 49.88. The farmer should walk 49.88 meters from C toward B.

Comments. 1. This problem is adapted from Qin Jiushao's thirteenth century work *Shushu jiuzhang* (Nine Sections of Mathematics). In the original, the unit of measure is not the meter, the division is into three equal parts, and the children are sons.

In the coordinate geometry solution, it was convenient to turn the map upside down. If we had used the diagram that accompanies Qin Jiushao's original problem, there would be no need to do that: in old Chinese maps, North is at the bottom.

2. The first solution can be modified to show that the assumption that BC is perpendicular to AB is unnecessary. Let θ be the angle between AB and BC. Instead of dropping a perpendicular from D to AB, draw a line through D parallel to BC. Suppose that P is reached by walking distance w from C along CB. Every area obtained in the first solution should be multiplied by $\sin\theta$, which then cancels out!

3. It is said that one of the Ptolemies asked Euclid for an easy way to solve geometrical problems, and Euclid replied "There is no royal road to geometry." Exactly the same story is told of Alexander the Great and Menaechmus. Both stories are almost certainly apocryphal—it can be unhealthy to dis a king. And in fact there *is* a royal road to geometry, the method of coordinates developed by Fermat, Descartes, and others in the seventeenth century.

III-75. The sum of the squares of the diagonals of a parallelogram is 40. Find the possible values of the sum of the squares of the four sides.

Solution. Suppose that parallelogram $ABCD$ has sides p and q, and let $\angle ABC = \theta$. The square of the diagonal AC is $p^2 + q^2 - 2pq\cos\theta$, by the Cosine Law. But $\angle DAB = \pi - \theta$, and therefore the square of the diagonal BD is $p^2 + q^2 + 2pq\cos\theta$. Add. The sum of the squares of AC and BD is $2p^2 + 2q^2$, exactly the same as the sum of the squares of the sides.

Comment. Let two sticks AC and BD be pinned together into a rough cross at the *middle* of AC (the pin needn't be at the middle of BD). As the angle between the sticks is varied, the lengths of AB, BC, CD, DA change, but the sum of their squares does not. To prove this, let O be where the sticks meet, do four Cosine Law calculations using the angles at O, and add.

III-76. One meter from a wall and parallel to it is a one meter high fence. How high up the wall can a six-meter ladder reach if the bottom of the ladder must be outside the fence?

Solution. When the highest point is reached, the ladder touches the fence. Let y be the height achieved, and x the distance from the wall to the foot of the ladder. A similar triangles argument shows that $(y-1)/1 = 1/(x-1)$, that is, $xy = x + y$. By the Pythagorean Theorem, $x^2 + y^2 = 36$.

If only an approximate answer is desired, note that $x = y/(y-1)$. Substitute for x in $x^2 + y^2 = 36$. The resulting equation in y can be solved approximately with the *Solve* button on the calculator, or even by graphing.

For an exact solution, add $2xy - 2(x+y)$, namely 0, to both sides of $x^2+y^2 = 36$ and simplify. The resulting equation

$$(x+y)^2 - 2(x+y) - 36 = 0$$

has the solution $x + y = 1 + \sqrt{37}$.

For simplicity, write a for $1 + \sqrt{37}$. Then $x^2 - 2xy + y^2 = 36 - 2a$, so $x - y = -\sqrt{36 - 2a}$. (The negative sign is used because $x < y$.) Finally,

$$y = \frac{\sqrt{36 + 2a} + \sqrt{36 - 2a}}{2}.$$

The expression $\sqrt{36 + 2a}$ is equal to a, but the formula for y looks nicer the way it is written. The height reached is about 5.88.

Comment. This is an old standard. If the distance of the fence from the wall is different from the height of the fence, the analysis starts in the same way, but the system of equations is more difficult to solve exactly. There *is* a formula for the roots of fourth degree equations, due to the sixteenth century mathematician Ferrari, and improved by, among others, Euler, but the formula is cumbersome. In the days before calculators, problems that didn't have simple closed form answers were avoided.

III-77. The isosceles triangle ABC has two sides of length 10, while BC has length 16. The point P inside the triangle is at perpendicular distance 2 from each of AB and BC. Find the perpendicular distance from P to AC

Solution. There are complicated ways of proceeding, but there is also an elegant, symmetrical way. Together, the triangles APB, BPC, and CPA make up, without overlap, all of $\triangle ABC$.

Using the Pythagorean Theorem, we find that the perpendicular distance from A to BC is 6, and therefore $\triangle ABC$ has area 48. Since $\triangle APB$ has base 10 and height 2, it has area 10. Similarly, $\triangle BPC$ has area 16. So $\triangle CPA$ has area 22. Since $\triangle CPA$ has base 10, the perpendicular distance from P to AC is 44/10.

Comment. The same sort of area argument shows that if $\triangle ABC$ is equilateral and P is a point inside the triangle, then the sum of the perpendicular distances from P to the sides of the triangle is independent of the location of P. This is sometimes called *Viviani's Theorem.*

III-78. The quadrilateral $\mathcal{Q}$ has diagonals of length 3 and 4 that meet inside $\mathcal{Q}$ at an angle of 60°. Find the area of $\mathcal{Q}$.

Solution. There are infinitely many quadrilaterals that meet the specifications. The wording of the problem seems to imply that they all have the same area. So we could "cheat" by making the diagonals bisect each other. But maybe the problem was carelessly worded.

Let the vertices of $\mathcal{Q}$, taken counterclockwise, be A, B, C, and D, and let the diagonals meet at P. Let $a = AP$, $b = BP$, $c = CP$, and $d = DP$, and suppose that $AC = 4$ and $BD = 3$. Then $a + c = 4$ and $b + d = 3$.

The area of $\triangle APB$ is $(ab/2)\sin 60°$. Since $\sin 120° = \sin 60°$, the area of $\triangle BPC$ is $(bc/2)\sin 60°$. There are similar expressions for the other two triangles around P.

The sum of the areas of the triangles is $(ab+bc+cd+da)\sqrt{3}/4$. But $ab+bc+cd+da = (a+c)(b+d) = 12$, so $\mathcal{Q}$ has area $3\sqrt{3}$.

Comment. Take two sticks, pinned and glued so as to form an angle θ, and make a kite with these sticks as cross-bracing. Then the area of the kite depends only on the lengths of the sticks and the angle θ, and not on the location of the pin.

III-79. Without using the calculus, find the slope of the tangent line to the curve $y = 1/x$ at the point $(2, 1/2)$.

Solution. The line with slope m through $(2, 1/2)$ has equation

$$y - \frac{1}{2} = m(x - 2).$$

To find the x-coordinate of the point(s) where this line meets the curve $y = 1/x$, substitute $1/x$ for y. After a while, we get the equation

$$2mx^2 + (1 - 4m)x - 2 = 0.$$

A sketch shows that $m \neq 0$, and that there are two distinct roots except when there is tangency. We can solve the equation using the quadratic formula, or more simply note that the quadratic factors as

$$(2mx + 1)(x - 2).$$

The two roots coincide if $m = -1/4$, so the tangent line has slope $-1/4$.

Another way: Stretch the graph of $y = 1/x$ in the y-direction by a factor of 4. So the point $(2, 1/2)$ turns into the point $(2, 2)$, and the hyperbola becomes $xy = 4$. By symmetry, the tangent line to this hyperbola at $(2, 2)$ has slope -1. Finally, divide y-coordinates by 4 to undo the stretching. This divides slopes by 4, so the tangent line has slope $-1/4$.

Comments. 1. The second solution is another instance of the problem-solving technique called *Transform, Solve, Transform Back.* For other examples in this chapter, see 3 and III-64.

2. See III-39 for a calculation of the slopes of tangent lines to $y = x^2$. A more challenging but still accessible problem is to find the tangent line to $y = x^3$ or $y = x^4$ at $x = a$.

3. By the middle of the seventeenth century (and before the conventional beginning of calculus) both Fermat and Descartes had generalized the first method and had found ways to calculate slopes of tangent lines to curves $P(x, y) = 0$, where $P(x, y)$ is any polynomial.

III-80. Triangle ABC has $AB = 6$, $AC = 12$, and $\angle BAC = 100°$. Let the bisector of $\angle BAC$ meet BC at D. Find the length of AD, rounded to 4 decimal places.

Solution. Recall that if a triangle has two legs of length p and q, and the angle between them is θ, then the area of the triangle is $(pq/2)\sin\theta$. Thus the area of $\triangle ABC$ is $36 \sin 100°$.

Let t be the length of AD. The area of $\triangle ABC$ is $3t \sin 50°$, and the area of $\triangle ACD$ is $6t \sin 50°$. Thus

$$36 \sin 100° = 3t \sin 50° + 6t \sin 50°.$$

Solve for t. It is marginally helpful to use the fact that $\sin 100° = 2 \sin 50° \cos 50°$. So $t = 8 \cos 50°$. The calculator says that this is about 5.1423. (Essentially the same idea is used in III-55.) If $AB = p$, $AC = q$, and $\angle BAC = 2\theta$, then $AD = pq \cos\theta/(p + q)$.

III-81. The point X is inside a rectangle, at distances 4, 7, and 8 from three of the corners. How far is P from the fourth corner?

Solution. It may look as if there isn't enough information to solve the problem, since the size of the rectangle is not given. There is in fact not enough information, but the size of the rectangle isn't needed!

In Problem III-30, it is shown by introducing coordinates that if P, Q, R, and S are consecutive vertices of a rectangle, and X is any point, and p, q, r, and s are the distances of X from P, Q, R, and S, then $p^2 + r^2 = q^2 + s^2$.

In the current problem, we may assume that $p = 4$. There are now three geometrically distinct cases: (i) $r = 7$; (ii) $r = 8$; and (iii) r is neither 7 nor 8.

Let d be the distance to the fourth point. In case (i), $4^2 + 7^2 = 8^2 + d^2$, so $d = 1$. In case (ii), $d = \sqrt{31}$, while in case (iii), $d = \sqrt{97}$.

III-82. A sphere $\mathcal{S}$ of radius 1 sits on the ground touching a wall perpendicular to the ground. Find the largest sphere that can pass through the gap between $\mathcal{S}$ and the wall.

Solution. Let O be the center of $\mathcal{S}$, and P the point where $\mathcal{S}$ touches the wall. Let ϖ be the plane through P that is perpendicular both to the wall and to the ground. We can confine attention to ϖ.

Let Q be the point where the wall, the ground, and ϖ meet. Then $OQ = \sqrt{2}$. Let C be the location of the center of the small sphere as that center passes though ϖ, and let r be the radius of the small sphere. Then $CQ = r\sqrt{2}$, and $\sqrt{2} = 1 + r + r\sqrt{2}$. It follows that

$$r = \frac{\sqrt{2}-1}{\sqrt{2}+1} = 3 - 2\sqrt{2}.$$

III-83. A circular frying pan is placed, base down, into a cupboard. The top edge of the pan touches the back wall and the left side wall. The diameter of the base of the pan is 4 cm less than the diameter of the outside top. A certain point on the edge of the base is 27 cm from the back wall and 10 cm from the left wall. Find the diameter of the base.

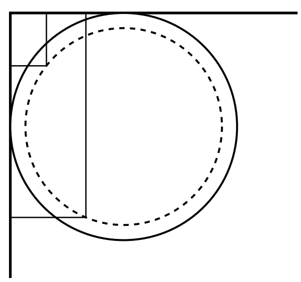

Figure 3.41: The Frying Pan

Solution. The plane of the cupboard shelf meets the left wall and the back wall in two lines, which are respectively chosen as the x and the y axes. Let the outer diameter of the top of the pan be $2r$. Then the center of the base has coordinates (r, r), and the outside edge of the base has equation $(x-r)^2 + (y-r)^2 = (r-2)^2$. A certain point on the circumference of the base has coordinates $(27, 10)$. Substitute in the preceding equation and simplify. We obtain $r^2 - 70r + 825 = 0$. Solve: It turns out that $r = 15$ or $r = 55$. The value $r = 55$ arises if the point on the base is the one nearer the cupboard corner in Figure 3.41. But pans with diameter over 1 meter aren't put into cupboards. The base has diameter 26 cm.

III-84. Triangle ABC has $AB = 11$ and $AC = 13$. The median that joins A to the midpoint of BC has length 10. Find the area of the triangle.

Solution. Let the median meet BC at M, and let $BC = 2d$. Let $\theta = \angle AMB$. View $\triangle ABC$ as having base BC. Its height is then $10\sin\theta$, so its area is $10d\sin\theta$.

The cosine of $\angle AMC$ is $-\cos\theta$, so by the Cosine Law

$$11^2 = 10^2 + d^2 - 20d\cos\theta \quad \text{and} \quad 13^2 = 10^2 + d^2 + 20d\cos\theta.$$

Add and subtract. It turns out that $d^2 = 45$ and $d\cos\theta = 48/40$. Thus

$$d\sin\theta = \sqrt{d^2 - d^2\cos^2\theta} = \frac{33}{5},$$

so the area is 66.

III-85. A 4 meter long cylindrical water tank of radius 1 m lies on its side on level ground. The maximum depth of water in the tank is measured by dipping a stick into the tank. This depth is 0.25 m. How much water is in the tank?

Solution. Draw a circle with center O and radius 1, and a horizontal line 0.75 units below O, as in Figure 3.42. Let P and Q be the points where the line meets the circle.

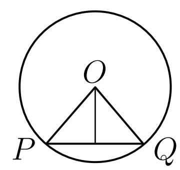

Figure 3.42: The Cylindrical Tank

Let θ be the angle that the (short) arc between P and Q subtends at O. Then θ is twice the angle whose cosine is 0.75. The calculator says that θ is about 82.8°. So the area of the pizza wedge OPQ is about $(82.8/360)\pi$. To find the area of $\triangle OPQ$, use the fact that it is $(1/2)\sin\theta$, or find $(PQ)/2$ using the Pythagorean Theorem.

Subtract the area of the triangle from the area of the pizza wedge, then multiply by 4 to find how much water is in the tank. The answer is about 0.9 cubic meters.

III-86. Alicia is looking through binoculars as a distant ship sails away. Her eyes are 10 meters above sea level, and the ship's smokestack is 40 m above sea level. How far away is the tip of the smokestack at the instant that it disappears from view? Assume that the Earth is a sphere of radius 6400 kilometers.

Solution. In Figure 3.43, E marks Alicia's eyes, and S is the tip of the smokestack at the instant it disappears from view. The line ES is tangent to the surface of the Earth, T is the point of tangency, and C is the center of the Earth. We need to find $ET + TS$.

By the Pythagorean Theorem, $(ET)^2 + (CT)^2 = (EC)^2$. But $EC = 6400 + 0.01$, and therefore

$$(ET)^2 = (2)(6400)(0.01) + (0.01)^2.$$

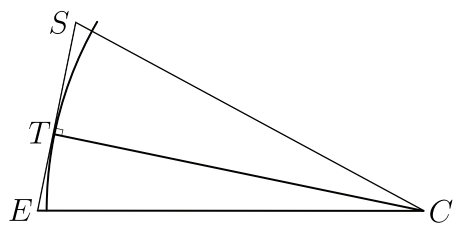

Figure 3.43: The Distance to the Horizon

The second term in the sum is negligible in comparison with the first, so for all practical purposes $ET = \sqrt{(6400)(0.02)}$. Similarly, TS is about $\sqrt{(6400)(0.08)}$. We conclude that ES is about 34 kilometers.

Comment. Discarding the term $(0.01)^2$ in the calculation of ET makes a difference of less than 5 millimeters! There are much larger sources of error, including uncertainties about the size of the Earth, the fact that it is not spherical, and diffraction effects.

III-87. A circular sector has a $60°$ central angle and radius 1. Find the radius of the largest circle that fits into this sector.

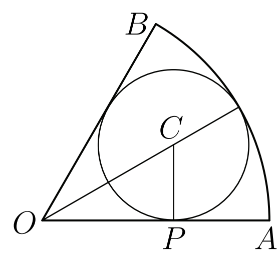

Figure 3.44: The Largest Circle in a Sector

Solution. Label points as in Figure 3.44. The largest circle that fits has center at a point C on the line that bisects $\angle AOB$. Let $OC = x$. The required circle touches the arc that joins A and B, so it has radius $1 - x$. The circle also touches OA and OB.

The radial line CP is perpendicular to OA. Since $\angle COP = 30°$, the length of CP is $x/2$, so the radius of the circle is $x/2$. It follows that $1 - x = x/2$, and therefore $x = 2/3$.

III-88. A circular hole of radius r is cut out of a horizontal plane, and a sphere of radius $R > r$ is placed on the hole. The bottom of the sphere ends up distance h below the level of the plane. Express R in terms of r and h.

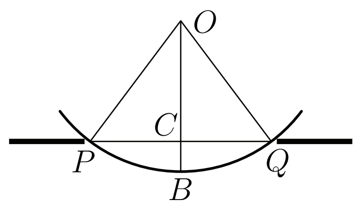

Figure 3.45: The Spherometer

Solution. Draw a plane through the center O of the sphere and the center C of the hole, and label points as in Figure 3.45.

From the fact that $CB = h$, we obtain $OC = R - h$. The Pythagorean Theorem applied to $\triangle OPC$ gives

$$r^2 = R^2 - (R-h)^2 = 2Rh - h^2.$$

Thus $R = (r^2 + h^2)/2h$.

Comment. Imagine a precision instrument built like a tiny three-legged stool. The legs of the stool end in sharp points that form an equilateral triangle whose circumradius r is accurately known. A ratcheted screw goes down from the center of the seat of the stool.

To find the *radius of curvature* R of a convex lens, an optical worker places the points of the instrument, called a *spherometer,* on the lens, and turns the screw until it just meets the lens. The distance h by which the center of the lens sticks up from the plane of the three points is read off. For a picture, just turn Figure 3.45 upside down.

Finally, R is computed by using the formula just obtained. (Many old spherometers gave a direct readout of $1/R$ by using an ingenious linkage. Nowadays that's handled by a microprocessor.)

III-89. A piece of salami is shaped like a cylinder of radius 2 cm. Slice the salami so that the knife makes an angle of 36° with the central axis of the salami. Find the area of cross-section of the slices. Hint: Slice the salami thinly and stack the slices.

Solution. Take a piece of salami of length ℓ, slice a big piece off at 36° to the central axis, and glue it to the other end to make a salami with parallel end faces at 36° to the central axis.

Slice the glued salami very thinly parallel to the new ends, and stack the pieces, lining them up so that the *bottoms* of the slices lie vertically above each other. If a slice has thickness t in a direction parallel to the central axis, then it has perpendicular height $t\cos 36^\circ$. It follows that the stack has height $\ell\cos 36^\circ$.

Let the area of cross-section be A. If the slices are extremely thin, then the stack is almost a generalized cylinder, with an oval base of area A and height $\ell\cos 36^\circ$. Thus $4\pi\ell = A\ell\cos 36^\circ$, and therefore $A = 4\pi\sec 36^\circ$, about 15.5 square centimeters.

Comments. 1. It so happens that $\cos 36^\circ = (\sqrt{5}+1)/4$, but it is unreasonable to seek an "exact" answer to a problem about salami.

2. Around 1640, before the discovery of the integral calculus, Cavalieri used ingenious slicing arguments to find areas and volumes. The idea goes back to Archimedes, who used a slicing argument to find the volume of the part of a sphere on one side of a plane.

3. In the salami problem, the cross-section is elliptical. So the problem can be solved by producing or looking up a formula for the area of an ellipse. That idea can be turned around, and we can use salami slicing to *find* a formula for the area of an ellipse.

III-90. The sides of a triangle are in geometric progression. Find all possible values of the ratio of the largest side to the smallest side.

Solution. The smallest side can be taken to be 1. Then the other two sides are r and r^2 for some $r \geq 1$. Three positive numbers are the sides of a triangle if and only if the sum of any two is greater than the third. In this case, the condition reduces to $r^2 < r+1$.

Note that $r^2 - r - 1 = (r-a)(r-b)$, where $a = (1-\sqrt{5})/2$ and $b = (1+\sqrt{5})/2$. Thus $r^2 - r - 1 < 0$ if and only if $a < r < b$. It follows that a necessary and sufficient condition on r is $1 \leq r < (1+\sqrt{5})/2$. The question asked for conditions on r^2. They are $1 \leq r^2 < (3+\sqrt{5})/2$.

III-91. An old lighthouse is 30 meters high. How tall should it be in order to be visible from twice as far? Assume that Earth is a sphere of radius 6400 kilometers, and that the viewer's eye is at sea level.

Solution. Twice as far as what? We can interpret distance as (i) line of sight distance from the viewer to the top of the lighthouse or (ii) distance along the Earth's surface to the base of the lighthouse—this is the usual interpretation. We can even imagine burrowing through the Earth. We solve the problem under both interpretations.

Compute first under interpretation (i). Let L be the top of the lighthouse, V the position of the viewer, and C the center of the Earth. For a picture, look at Figure 3.43.

Note that $\angle CVL$ is a right angle. It is useful to work with symbols than numbers. Let R be the radius of the Earth, and h the height of a lighthouse. By the Pythagorean Theorem, $(VL)^2 = (R+h)^2 - R^2$. Thus $VL = \sqrt{2Rh + h^2}$. For the old lighthouse, $h = 0.03$. The calculator gives $VL = 19.595941$.

The h^2 term is very small compared to $2Rh$. If we throw h^2 away, we obtain instead $VL = 19.595918$—throwing away h^2 makes no practical difference.

Let h be the height required to double VL. We want $1536.0324 = 2Rh + h^2$. The quadratic formula gives

$$h = \frac{-12800 + \sqrt{(12800)^2 + 6144.1296}}{2}.$$

Plugging into the calculator could produce catastrophic rounding error, since we are subtracting two large nearly equal numbers. But in this case things turn out OK, barely.

It is easier, safer, and essentially as accurate to use the approximate formula $VL = \sqrt{2Rh}$. To double $\sqrt{2Rh}$ we must multiply h by 4, so the new height of the lighthouse should be about 120 meters.

Compute now under interpretation (ii), measuring distances along the surface of the water to the base B of the lighthouse. The cosine of $\angle BCV$ is $6400/6400.03$. The calculator says that $\angle BCV$ is about 0.003061856 radians. For the new lighthouse, this angle must be doubled, so the new cosine should be about 0.99998125. If h is the new height, then $R/(R+h) = 0.99998125$. The calculator gives $h = 0.120001408$. (The last few digits can't be trusted.)

The answer is almost the same as under interpretation (i). Note that the maximum line of sight distance VL to the old lighthouse is 19.595941 and the corresponding water level distance VB is 19.5958784. Locally, the Earth, or at least the surface of a calm sea, is flat.

Comment. The eleventh-century Central Asian Islamic astronomer al-Bīrūnī found the height h of a mountain that rose steeply at the edge of a large plain. He then found the distance d from the mountain to the point in the plain where the top of the mountain disappeared from sight, and used the relationship $d = \sqrt{2Rh}$ to estimate the radius of the Earth.

III-92. Let A, B, C, and D be consecutive vertices of a regular hexagon with edge length 1. A ruler that has two red dots on its edge, distance 1 apart, is placed so that the ruler passes through A, and one dot ends up on BD while the other ends up on BC (extended). Let X be the point where BD meets the ruler. Find the length of AX *exactly.*

Solution. Please see Figure 3.46. There, Y is the point where the ruler meets the line BC, and O is the center of the hexagon. By symmetry, O is the midpoint of AD, and AD is parallel to BC. Since $\triangle OAB$ is equilateral, $AD = 2$.

By properties of parallel lines, $\angle ADX = \angle YBX$, so $\triangle ADX$ and $\triangle YBX$ are similar. Let $a = AX$ and $b = BY$. By similarity, $2/a = b/1$.

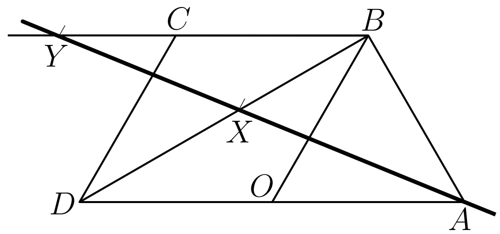

Figure 3.46: Duplicating the Cube with Marked Ruler

Now focus attention on $\triangle ABY$. Since $\angle ABY = 120°$, the Cosine Law yields

$$(a+1)^2 = 1^2 + b^2 + b.$$

When we substitute $2/a$ for b and simplify, we obtain

$$a^4 + 2a^3 - 2a - 4 = 0, \qquad \text{that is,} \qquad (a+2)(a^3 - 2) = 0.$$

Thus $a = \sqrt[3]{2}$.

Comment. The result gives a way of "duplicating the cube" with a compass and a *marked* straightedge. In 1837, Wantzel proved that the cube *cannot* be duplicated with compass and unmarked straightedge.

The task of constructing a cube with volume twice that of a given one—or equivalently, constructing a length $\sqrt[3]{2}$—is a famous problem that goes back to Greek antiquity. The story is told that during a plague, the Delos oracle told the Delians that the plague would end if they constructed a new altar to Apollo with twice the volume of the previous one.

Archimedes (circa −250) probably knew that the cube can be duplicated with compass and marked straightedge. The above construction is a streamlined variant of constructions given by Viète and later by Newton.

III-93. On March 21, Alfonso and Beti were doing a science experiment as the sun set over the coast of Ecuador. Alfonso was standing 50 meters above sea level, and Beti was on a cliff 50 meters above Alfonso.

At the instant that Alfonso saw the sun dip below the horizon, he signalled Beti, who found with a stopwatch that the sun dipped below *her* horizon 22.5 seconds after it dipped below Alfonso's. Find the radius of the Earth.

Solution. The geometry is developed in the solutions of III-86 and III-91 where it is shown that if R is the radius of the Earth, then for someone at reasonable height h above sea level the distance to the horizon is very close to $\sqrt{2Rh}$. Line of sight distance and distance along the surface of the Earth are the same for all practical purposes.

Let R be measured in kilometers. For convenience, let a be the height of Alfonso above sea level, b the height of Beti, and t the elapsed time shown by the stopwatch.

The distance to Alfonso's horizon is about $\sqrt{2aR}$, while the distance to Beti's is $\sqrt{2bR}$. The Earth goes through one revolution every 24 hours, so points at the equator travel at $2\pi R/(24 \cdot 3600)$ kilometers per second. It took Earth t seconds to travel $\sqrt{2bR} - \sqrt{2aR}$, so

$$\frac{(\sqrt{2bR} - \sqrt{2aR})(24 \cdot 3600)}{2\pi R} = t$$

and therefore

$$R = \frac{(\sqrt{2b} - \sqrt{2a})^2 (24 \cdot 3600)^2}{4\pi^2 t^2}.$$

Calculate. To the nearest kilometer, the result is 6408.

Comment. The answer is suspiciously close to the true value. Alfonso and Beti didn't feel like doing all that climbing, so they obtained the 22.5 seconds by working backwards from the known value of R. The horizon method has some built-in inaccuracies, but a ball-park estimate of R could be obtained in this way.

III-94. One corner of a box is O, and the three neighbouring corners are A, B, and C. Let a be the area of $\triangle BOC$, b the area of $\triangle COA$, c the area of $\triangle AOB$, and d the area of $\triangle ABC$. Show that $a^2 + b^2 + c^2 = d^2$.

Solution. Let $p = OA$, $q = OB$, and $r = OC$. By the Pythagorean Theorem, the squares of the sides of $\triangle ABC$ are $q^2 + r^2$, $r^2 + p^2$, and $p^2 + q^2$.

There are elegant ways to find d^2 without breaking symmetry. We do it crudely by finding one of the heights of $\triangle ABC$. Drop a perpendicular from A to the point P on BC. Let $h = AP$ and $x = BD$. The Pythagorean Theorem applied to $\triangle ABP$ gives

$$h^2 + x^2 = p^2 + q^2.$$

A similar calculation with $\triangle ACP$ gives

$$h^2 + (\sqrt{q^2 + r^2} - x)^2 = p^2 + r^2.$$

If we subtract, we get a linear equation for x. It turns out that

$$x = \frac{q^2}{\sqrt{q^2 + r^2}},$$

and therefore

$$h^2 = \frac{q^2 r^2 + r^2 p^2 + p^2 q^2}{q^2 + r^2}.$$

We conclude that $d^2 = (q^2 r^2 + r^2 p^2 + p^2 q^2)/4$. But $qr = 2a$, $pr = 2b$, and $pq = 2c$, so $d^2 = a^2 + b^2 + c^2$.

Comment. The above result is one of the several possible generalizations of the Pythagorean Theorem to three dimensions. Numerical versions of the problem are more accessible.

Chapter 4

Equations and Inequalities

Introduction

Most of these problems read “find the solutions of” such and such an explicitly given equation or system of equations, or find all numbers that satisfy a given inequality. The equations do not involve trigonometric functions, which are saved for Chapter 5.

Many of the equations have obvious symmetries and these symmetries are exploited in the solution. Basic facts about polynomials are often needed, such as the Remainder Theorem (the remainder when $P(x)$ is divided by $(x-a)$ is $P(a)$, and consequently $(x-a)$ divides $P(x)$ if and only if $P(a) = 0$).

Several of the problems are most simply solved by “rationalizing” a numerator, and for others it is useful to know about the discriminant of a quadratic.

Several of the solutions use the relationship between the coefficients of a polynomial and certain symmetric functions of its roots. In particular, one needs to know that if α and β are the roots of $x^2 + px + q = 0$, then since $(x-\alpha)(x-\beta)$ is the same polynomial as x^2+px+q, it follows that $\alpha+\beta = -p$ and $\alpha\beta = q$. Similarly, if α, β, and γ are the roots of $x^3 + px^2 + qx + r = 0$ then $\alpha+\beta+\gamma = -p$, $\beta\gamma+\alpha\gamma+\alpha\beta = q$ and $\alpha\beta\gamma = -r$, and so on for higher degree polynomials.

Problem IV-69 deals with the solutions of the cubic, and could form the beginnings of an exploration. Problem IV-81 deals with the square roots of a complex number. It can made to lead to important ideas, such as de Moivre’s formula.

Problems and Solutions

IV-1. Factor $x(y-z)^3 + y(z-x)^3 + z(x-y)^3$.

Solution. Don't expand! Call our expression $A(x, y, z)$. First think of A as a polynomial $P(x)$ in x, with parameters y and z. Note that $P(y) = 0$, so by the Factor Theorem $x - y$ divides $P(x)$, that is, $x - y$ divides A. Similarly, $y - z$ and $z - x$ divide A. It follows that

$$A(x, y, z) = (x-y)(y-z)(z-x)Q(x, y, z)$$

for some polynomial Q in the variables x, y, and z. All the terms of A have degree 4, and therefore Q is of degree 1, meaning that

$$Q(x, y, z) = k(x + y + z)$$

for some constant k. There are various ways to find k. For example, put $x = 0$, $y = 1$, and $z = 2$. Then

$$(x-y)(y-z)(z-x)k(x+y+z) = 6k \qquad \text{but} \qquad A(0, 1, 2) = 6,$$

so $k = 1$. Or else compare say coefficients of yx^3 (-1 for A, and $-k$ in the factored expression).

Comment. There are many possible questions based on the same idea. The key is to use an expression $A(x, y, z)$ that doesn't change when (x, y, z) is replaced by (z, x, y), that is, when each variable is pushed forward by one position, with wraparound.

IV-2. Solve for x: $a(x^2 + 2) = x(a^2 + 2)$.

Solution. We can manipulate and quickly find the answer. But it can't hurt to look first. It is obvious from the symmetry that one solution is $x = a$. If $a = 0$, that's all there is. If $a \neq 0$, we have a quadratic whose constant term is twice the coefficient of x^2, so the product of the roots is 2 and therefore the other solution is $x = 1/2a$.

IV-3. Find all solutions of the system $x^2 + y^2 = 1$, $x^3 + y^3 = 1$.

Solution. It is tempting to begin to manipulate, maybe by eliminating y. We have $y^2 = 1 - x^2$ and $y^3 = 1 - x^3$, and therefore

$$(1 - x^2)^3 = (1 - x^3)^2.$$

Expand and simplify. After a while we arrive at the answer. It is probably better to keep symmetry and write

$$(x^2 + y^2)^3 = (x^3 + y^3)^2.$$

This simplifies to $3x^2y^2(x^2 + y^2) - 2x^2y^2(xy) = 0$. If neither x nor y is 0, then $2xy = 3$. But then $x^2 - 2xy + y^2 = -1$, which is impossible. Or else we can substitute $3/(2x)$ for y in the equation $x^2 + y^2 = 1$, and after some manipulation arrive at $4x^4 - 4x^2 + 9 = 0$, a quadratic in x^2 with no real solutions.

Another way: The system has the obvious solutions $x = 1$, $y = 0$ and $x = 0$, $y = 1$. If neither x nor y is 1, then from $x^2 + y^2 = 1$ we conclude that $|x| < 1$ and $|y| < 1$. But then $|x^3|$ and $|y^3|$ are "too small." More formally, $|x^3| < x^2$ and $|y^3| < y^2$, and therefore $|x^3| + |y^3| < 1$, so $x^3 + y^3 \neq 1$.

IV-4. Let α, β, and γ be the solutions of $x^3 - 3x + 1 = 0$. Find

$$(2\alpha - 1)(2\beta - 1)(2\gamma - 1).$$

Solution. By sketching $y = x^3 - 3x + 1$, or otherwise, we can see that the given equation has three real roots. If the problem came up in a practical setting, we could use a graphing program, or some other method, to produce good numerical estimates of the roots, and use these estimates to calculate an approximate answer. But we can also find an *exact* answer.

Expand the given expression. We obtain

$$8(\alpha\beta\gamma) - 4(\alpha\beta + \beta\gamma + \gamma\alpha) + 2(\alpha + \beta + \gamma) - 1.$$

Note that

$$x^3 - 3x + 1 = (x - \alpha)(x - \beta)(x - \gamma).$$

By expanding the right-hand side, and comparing terms, we find that $\alpha+\beta+\gamma = 0$, $\alpha\beta + \beta\gamma + \gamma\alpha = -3$, and $\alpha\beta\gamma = -1$. Substitute. We conclude that our expression is equal to 3.

Another way: Let $f(x) = x^3 - 3x + 1$. Then $f(1/2) = -3/8$. But

$$f(1/2) = (1/2 - \alpha)(1/2 - \beta)(1/2 - \gamma).$$

Multiply both sides by -8. We conclude that $(2\alpha - 1)(2\beta - 1)(2\gamma - 1) = 3$.

Comment. We could instead use the ideas of IV-69 or V-37 to find exact expressions for the roots of the cubic, but that approach is much more complicated.

In general, expanding like we did in the first solution isn't a good idea, since it makes things look more complicated. But in this case the terms $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$, and $\alpha\beta\gamma$ that turn up can be read off from the coefficients of $x^3 - 3x + 1$.

IV-5. If x is a real number, the *fractional part* of x is defined to be $x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Denote the fractional part of x by $\{x\}$. How many solutions does the equation $\{x^2\} = \{4x\}$ have in the interval $0 \leq x \leq 10$?

Solution. Let $f(x) = x^2 - 4x$. Then $\{x^2\} = \{4x\}$ if and only if $f(x)$ is an integer. With some effort, we can now use a graph of $y = x^2 - 4x$ to count the number of values of x between 0 and 10 for which $f(x)$ is an integer.

It is easier to *imagine* graphing. Note that $f(x) = (x-2)^2 - 4$. As x increases from 0 to 2, $f(x)$ decreases from 0 to -4, so $f(x)$ takes on integer values at 5 values of x in the interval $0 \le x \le 2$. As x increases from 2 to 10, $f(x)$ increases from -4 to 60, so $f(x)$ takes on integer values at 64 values of x in the interval $2 < x \le 10$. Thus our equation has 69 solutions in the interval $0 \le x \le 10$.

IV-6. Find the real numbers a such that the solutions of

$$x^4 - 2x^3 + x^2 - a^2 = 0$$

are all real.

Solution. Rewrite the equation as

$$\left(x(x-1)\right)^2 = a^2, \qquad \text{or equivalently} \qquad x(x-1) = \pm|a|.$$

The roots of $x^2 - x - |a| = 0$ are real for any a, since the discriminant $1 + 4|a|$ is always positive. The roots of $x^2 - x + |a| = 0$ are real if and only if the discriminant $1 - 4|a|$ is non-negative. That is true when $-1/4 \le a \le 1/4$.

IV-7. Find all real numbers k such that one solution of the equation $x^2 - 2kx + 4 = 0$ is the cube of the other.

Solution. The solutions are $k \pm \sqrt{k^2-4}$. Which is to be the cube of which? Replacing k by $-k$ turns one possibility into the other, so we neeed only look at

$$\left(k + \sqrt{k^2-4}\right)^3 = k - \sqrt{k^2-4}.$$

Should we expand and solve? Maybe it won't turn out as ugly as it's beginning to look. But one should resist the urge to expand, for expanding often destroys structure. Instead, multiply both sides by $k + \sqrt{k^2-4}$ (it is clear that this can't be 0). We obtain

$$\left(k + \sqrt{k^2-4}\right)^4 = 4,$$

so $k + \sqrt{k^2-4} = \lambda$, where λ is one of $\sqrt{2}$, $-\sqrt{2}$, $\sqrt{-2}$, or $-\sqrt{-2}$. But since k is real, the last two are impossible.

Now multiply both sides by $k - \sqrt{k^2-4}$, then divide by λ. We obtain $k - \sqrt{k^2-4} = \pm 2\sqrt{2}$. "Add" $k + \sqrt{k^2-4} = \pm\sqrt{2}$ to this equation. We conclude that $k = \pm 3\sqrt{2}/2$.

Another way: Let the roots be r and r^3. The product of the roots is equal to the constant term of the polynomial. Thus $r^4 = 4$, and therefore $r = \pm\sqrt{2}$ or $r = \pm\sqrt{-2}$.

If $r = \pm\sqrt{2}$, then $r^3 = \pm 2\sqrt{2}$. The sum of the roots is $2k$, and therefore

$$2k = \pm(\sqrt{2} + 2\sqrt{2})$$

so $k = \pm 3\sqrt{2}/2$.

If $r = \pm\sqrt{-2}$, then $r^3 = \mp 2\sqrt{-2}$. But then the sum of the two roots is not a real number, contradicting the fact that k is real.

Comment. There is a version of the first approach that uses trigonometric identities. Let $k = 2\sec\theta$. Since $1 + \tan^2\theta = \sec^2\theta$, our equation can be rewritten

$$8(\sec\theta + \tan\theta)^3 = 2(\sec\theta - \tan\theta).$$

Now we can use trigonometric identities to arrive at the answer. "Trigonometric substitutions" can be a handy way to get rid of square roots.

IV-8. Let $f(x) = x^2 + 2x - 1$. Solve the equation $f(f(x)) = f(x)$.

Solution. Calculating $f(f(x)) - f(x)$ is a strategic error. The original problem has structure. Expanding hides that structure, and produces a fourth degree equation whose roots are not at all obvious.

Rewrite our equation as $(f(x))^2 + 2f(x) - 1 = f(x)$, and use the quadratic formula to solve for $f(x)$. We conclude that the equation holds at x if and only if

$$f(x) = \frac{-1 \pm \sqrt{5}}{2}.$$

Now solve the quadratic equations

$$x^2 + 2x - 1 = \frac{-1 + \sqrt{5}}{2} \quad \text{and} \quad x^2 + 2x - 1 = \frac{-1 - \sqrt{5}}{2}.$$

We can add 2 to both sides, making the left-hand side into a perfect square. Or more clumsily we can use the quadratic formula. The roots are

$$-1 \pm \sqrt{\frac{3 + \sqrt{5}}{2}} \quad \text{and} \quad -1 \pm \sqrt{\frac{3 - \sqrt{5}}{2}}.$$

These expressions simplify considerably if we notice that $(1 \pm \sqrt{5})^2 = 6 \pm 2\sqrt{5}$. But the structure comes out more clearly if we solve the problem

Another way: Suppose that $f(x) = x$. Apply the function f to both sides. We conclude that $f(f(x) = f(x)$. So any solution of $f(x) = x$ is a solution of our equation. The equation $f(x) = x$ can be rewritten as $x^2 + x - 1 = 0$, whose roots are $(-1 \pm \sqrt{5})/2$.

To find the other two roots of $f(f(x) = f(x)$, note that $f(x) = (x+1)^2 - 2$, and therefore $f(-x-2) = f(x)$ for any x. Thus if $f(x) = -x - 2$ then $f(f(x)) = f(x)$. Now solve the equation $x^2 + 2x - 1 = -x - 2$. The roots are $(-3 \pm \sqrt{5})/2$.

Comment. Both techniques work for *any* quadratic polynomial $f(x)$. If f is any function, a solution of $f(x) = x$ is called a *fixed point* of f. Fixed points are useful in many areas of mathematics.

IV-9. Find all real numbers x that satisfy the inequalities

$$x < x^3 < x^4 < x^2.$$

Solution. There are solutions! We can use a graphing program to graph the curves $y = x$, $y = x^3$, $y = x^4$, and $y = x^2$, and read off the answer from the screen. But the curves are so simple that we can probably learn more from a hand-drawn sketch. We can also solve the problem without pictures as follows.

Since x^2 is non-negative, $x^4 < x^2$ precisely if $0 < x^2 < 1$. But if $0 < x < 1$, then $x > x^3$. So the only possibilities left are $-1 < x < 0$. They work.

IV-10. Given that $\log_a x = 4$ and $\log_b x = -5$, find $\log_{ab} x$.

Solution. The equation $\log_a x = 4$ is equivalent to $x = a^4$. Since x must be positive, it follows that $a = x^{1/4}$. Similarly, $b = x^{-1/5}$, and therefore $ab = x^{1/4}x^{-1/5} = x^{1/20}$. Thus $x = (ab)^{20}$, and therefore $\log_{ab} x = 20$.
Another way: We can use the change of base formula

$$\log_q x = \frac{\log_p x}{\log_p q}.$$

Convert all logarithms to some convenient common base p. Then

$$\log_a x = \frac{\log_p x}{\log_p a} = 4 \qquad \text{and} \qquad \log_b x = \frac{\log_p x}{\log_p b} = -5.$$

Rewrite the equations as

$$\frac{\log_p a}{\log_p x} = \frac{1}{4} \qquad \text{and} \qquad \frac{\log_p b}{\log_p x} = -\frac{1}{5}.$$

Add, and use the fact that $\log_p a + \log_p b = \log_p ab$. We conclude that

$$\frac{\log_p ab}{\log_p x} = \frac{1}{20}.$$

Using again the change of base formula, we get $\log_{ab} x = 20$.

Comment. The second approach is unattractive. It depends on a change of base formula that has to be established or remembered, and it is longer and much uglier. The first approach is more conceptual, for it only uses the all-important relationship between exponential functions and logarithms.

IV-11. Find a function f such that $f(xy) + f(x)f(y) + x + y$ is identically equal to 2.

Solution. Suppose that $f(xy)+f(x)f(y)+x+y = 2$ for all x and y. Set $x = y = 0$; then $f^2(0) + f(0) - 2 = 0$. The quadratic formula (or factoring) yields $f(0) = -2$ or $f(0) = 1$.

Suppose first that $f(0) = -2$. Set $y = 0$, leaving x free to roam. Then $-2 - 2f(x) + x = 2$, and therefore $f(x) = x/2 - 2$ for all x. Let's check whether this works by substituting back in the original equation. It doesn't.

So suppose now that $f(0) = 1$. Again set $y = 0$. Then $1 + f(x) + x = 2$, and therefore $f(x) = 1 - x$ for all x. This works, for $(1 - xy) + (1 - x)(1 - y) + x + y$ is identically equal to 2.

IV-12. Find three numbers in geometric progression whose sum is 2 and the sum of whose squares is 8.

Solution. It is natural to let the numbers be a, ar, and ar^2. That works just fine, but it is more attractive to let them be x, y, and z, with y the middle one. Since the numbers are in geometric progression, $xz = y^2$.

We were told that

$$x + y + z = 2 \qquad \text{and} \qquad x^2 + y^2 + z^2 = 8.$$

Square both sides of the first equation. We obtain

$$x^2 + y^2 + z^2 + 2(xy + xz + yz) = 4.$$

Now use the second equation to conclude that $xy + xz + yz = -2$. Since $xz = y^2$, we can rewrite this as $y(x + y + z) = -2$. It follows that $y = -1$.

Since $y = -1$, the first equation yields $x + z = 3$. Note that $xz = y^2 = 1$. Thus x and z are the roots of $u^2 - 3u + 1 = 0$. These roots are $(3 \pm \sqrt{5})/2$, and x can be either one of them, and z is the other.

Comment. The same problem, with sum 20 and sum of squares 140, can be found in Isaac Newton's *Universal Arithmetic* (late seventeenth century).

IV-13. Let $P(x) = (x - 1)(x)(x + 1) - (a - 1)(a)(a + 1)$. Find all real numbers a such that $P(x) = 0$ has three distinct real solutions.

Solution. The equation has the obvious solution $x = a$. To find the other solutions, note that

$$\begin{aligned} P(x) &= (x^3 - x) - (a^3 - a) = (x^3 - a^3) - (x - a) \\ &= (x - a)(x^2 + ax + a^2 - 1). \end{aligned}$$

We want the quadratic equation $x^2 + ax + a^2 - 1 = 0$ to have distinct real solutions. That happens when the discriminant $a^2 - 4(a^2 - 1)$ is positive, that is, for $-2/\sqrt{3} < a < 2/\sqrt{3}$.

All roots should be distinct, so we also need

$$\frac{-a \pm \sqrt{4-3a^2}}{2} \neq a.$$

There is equality only when $\pm\sqrt{4-3a^2} = 3a$. This equation reduces to $12a^2 = 4$, which has the solutions $a = \pm 1/\sqrt{3}$. It follows that $P(x) = 0$ has distinct real roots for any a in the interval $-2/\sqrt{3} < a < 2/\sqrt{3}$ except for $a = \pm 1/\sqrt{3}$.

Comment. The product of the roots of $P(x) = 0$ is $(a-1)(a)(a+1)$, so the roots other than a have product $(a-1)(a+1)$. Also, in $P(x)$ the coefficient of x^2 is 0, so the sum of the roots is 0, and therefore the sum of the roots other than a is $-a$. It follows that the roots other than a satisfy the equation $x^2 + ax + a^2 - 1 = 0$. In the main solution this result was obtained by factoring.

IV-14. A cardboard box of the usual shape has sides of area $720\,\text{cm}^2$, $1000\,\text{cm}^2$, and $1250\,\text{cm}^2$. Find the volume of the box.

Solution. Let the edge lengths be a, b, and c. We know their products in pairs. With suitable labelling $bc = 720$, $ca = 1000$, and $ab = 1250$. Multiply. We get $(abc)^2 = 720 \cdot 1000 \cdot 1250$, and therefore $abc = 30000$.

Comment. The calculation exploits the fact that volume is symmetric in a, b, and c. In general, one should be reluctant to “break symmetry.” Most students set up the equations, use elimination to find say a, then calculate b and c, and finally find the product. This approach works, but it is inelegant and therefore inefficient.

In a problems workshop, a student who had done well in mathematics competitions was still working on the problem, fingers flying over the calculator, when some others had finished. It turned out that he expected a, b, and c to be integers, and was using “guess and test” to find them!

IV-15. For what k does the system

$$x^2 - y^2 = 0; \qquad (x-k)^2 + y^2 = 1$$

have (i) no solutions; (ii) one solution; (iii) two solutions; (iv) three solutions; (v) four solutions?

Solution. If we rewrite the first equation as $(x-y)(x+y) = 0$, we can see that it is the equation of a “curve” made up of the familiar lines $y = x$ and $y = -x$. The second equation is the equation of the circle with center $(k, 0)$ and radius 1. We take advantage of symmetry by looking at the case $k \geq 0$ and then reflecting across the y-axis. Any rough sketch shows that there are four solutions if $k = 0$. As k increases, the circle moves to the right. When the circle passes through the origin ($k = 1$), the number of solutions drops to three. It jumps immediately back to four, stays at four until the circle becomes tangent to $y = x$, when the number of

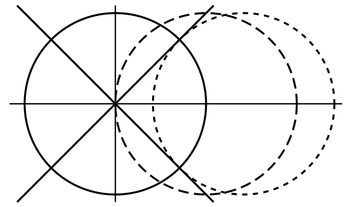

Figure 4.1: The Curves $x^2 - y^2 = 0$ and $(x-k)^2 + y^2 = 1$

solutions drops to two. By the Pythagorean Theorem, that happens when $k = \sqrt{2}$. And after that there are no solutions.

So there are no solutions when $|k| > \sqrt{2}$, two solutions when $|k| = \sqrt{2}$, three when $|k| = 1$, and four when $|k| < 1$ and also when $1 < |k| < \sqrt{2}$.

Another way: The problem has strong geometric content, so it ought to be done geometrically. But it can be handled using only algebra. Use the two given equations to eliminate y^2. We obtain $2x^2 - 2kx + k^2 - 1 = 0$. This has no roots when the discriminant $8 - 4k^2$ is negative, that is, when $|k| > \sqrt{2}$. If $|k| = \sqrt{2}$, then $x = 1/\sqrt{2}$ and $y = \pm 1/\sqrt{2}$, so the original problem has two solutions. When $|k| < \sqrt{2}$, there are two possible values of x, and hence four solutions, except if one value of x is 0 ($k = \pm 1$), when there are only three solutions.

IV-16. Find a function f such that $f(f(f(f(x))))$ is identically equal to $4x + 1$.

Solution. Look for a solution of the form $f(x) = ax + b$. Then

$$f(f(x)) = a(ax+b) = a^2x + ab + b.$$

Keep on computing. After a while we get

$$f(f(f(f(x)))) = a^4x + a^3b + a^2b + ab + b.$$

We need to have $a^4 = 4$. That's easy to arrange: just let $a = \sqrt{2}$. With this choice of a, we need $(2\sqrt{2} + 2 + \sqrt{2} + 1)b = 1$, so let $b = (\sqrt{2} - 1)/3$.

Comment. The problem and solution may look like "algebra," but the motivation is geometric. The transformation that takes x to $4x+1$ can be thought of as scaling by a factor of 4, then shifting by 1. We want to express this transformation as the result of applying a certain transformation 4 times. It is natural to try to scale by some factor a and shift by some b. The scale factor $a = \sqrt[4]{4}$ jumps out, although $-\sqrt[4]{4}$ is also reasonable (scaling by $\sqrt[4]{4}$ followed by a reflection).

IV-17. Find all real numbers x and y such that

$$\frac{1}{x}+\frac{1}{y}=3 \qquad \text{and} \qquad \frac{1}{x^2}+\frac{1}{y^2}=11.$$

Solution. The fractions are intimidating, so let $u=1/x$ and $v=1/y$. The equations become $u+v=3$ and $u^2+v^2=11$. If we substitute $3-u$ for v in $u^2+v^2=11$, we obtain a quadratic equation for u, and the rest is routine.

More symmetrically, let $u=1.5+t$ and $v=1.5-t$. Then $u^2+v^2=11$ becomes $4.5+2t^2=11$, and the rest is easy.

Or else use the fact that

$$u^2-2uv+v^2=2(u^2+v^2)-(u+v)^2=13$$

to conclude that $u-v=\pm\sqrt{13}$. Thus

$$u=\frac{3\pm\sqrt{13}}{2} \qquad \text{and} \qquad v=\frac{3\mp\sqrt{13}}{2}.$$

Invert and simplify: $x=(-3\pm\sqrt{13})/2$ and $y=(-3\mp\sqrt{13})/2$.

IV-18. Find all real numbers x and y such that

$$3x^2+6xy+2y^2=4 \qquad \text{and} \qquad 5xy+3y^2=8.$$

Solution. We can use a computer program to graph the curves (they are hyperbolas) with these equations and read off the solutions, at least approximately. But there is an algebraic approach that gives an exact answer.

There seem to be no interesting symmetries to exploit, so we get rid of the constant term to get a *homogeneous* equation. If the given equations hold, then

$$2(3x^2+6xy+2y^2)-(5xy+3y^2)=0,$$

or equivalently $y^2+7xy+6x^2=0$. By the quadratic formula, or by factoring, we find that $y=-x$ or $y=-6x$.

Replace y by $-x$ in the equation $5xy+3y^2=8$. We get $-2x^2=8$, which has no real solutions. Replace y by $-6x$ in $5xy+3y^2=8$. We get $78x^2=8$. That gives the (possible) solutions $x=2/\sqrt{39}$, $y=-6/\sqrt{39}$ and $x=-2/\sqrt{39}$, $y=6/\sqrt{39}$.

We should check that these answers really work, for two reasons, one practical—errors in arithmetic are always possible—and the other logical. We have only shown that *if* (x,y) is a solution, then (x,y) can't be anything other than our two candidates. In general, that doesn't imply that the candidates satisfy the equations. (In this problem it is easy to show that the implications are reversible: if $y^2+7xy+6x^2=0$ and $5xy+3y^2=8$, then $3x^2+6xy+2y^2=4$.)

IV-19. Find all solutions of the inequality

$$\frac{1}{x-1}+\frac{1}{x+1}\geq\frac{1}{3}.$$

Solution. We will—carefully— clear denominators by multiplying both sides of the inequality by $3(x-1)(x+1)$,

Suppose first that $-1 < x < 1$. Then $(x-1)(x+1) < 0$ and we obtain the inequality $3(x+1)+3(x-1) \le (x-1)(x+1)$, which simplifies to $x^2-6x-1 \ge 0$. The polynomial x^2-6x-1 is 0 at $x = 3\pm\sqrt{10}$ and it is negative only between these roots. Thus in the interval $(-1,1)$ the inequality holds only when $-1 < x < 3-\sqrt{10}$.

Suppose now that $|x| > 1$. Then $(x-1)(x+1) > 0$, and we obtain the inequality $x^2-6x-1 \le 0$. So when $|x| > 1$ the inequality only holds when $1 < x \le 3+\sqrt{10}$.

Comment. It is all too easy to make errors when handling inequalities. A graphing calculator or a graphing program can be useful. Graph the curve $y = 1/(x-1) + 1/(x+1)$ and determine approximately where this curve lies above $y = 1/3$. Then find the boundary points exactly by solving a quadratic equation.

IV-20. Find all real numbers x such that

$$\frac{x^2}{x+1} + \frac{x+1}{x^2} = \frac{25}{12}.$$

Solution. The second term is the reciprocal of the first, so let $w = x^2/(x+1)$. Then $w + 1/w = 25/12$, or equivalently $12w^2 - 25w + 12 = 0$. Solve using the quadratic formula or in some other way. We get $w = 3/4$ or $w = 4/3$.

Rewrite $x^2/(x+1) = 3/4$ as $4x^2 - 3x - 3 = 0$. This has the solutions $x = \left(3 \pm \sqrt{57}\right)/8$. The equation $x^2/(x+1) = 4/3$ has solutions $x = -2/3$ and $x = 2$.

IV-21. Let $f(x) = |x-1| - |x-2| + |x-4|$. Find all c such that the equation $f(x) = c$ has exactly three solutions.

Solution. Sketch the curve $y = f(x)$. If $x \le 1$, then $|x-1| = 1-x$, $|x-2| = 2-x$, and $|x-4| = 4-x$, and therefore $f(x) = -x+3$. If $1 \le x \le 2$, then $|x-1| = x-1$, $|x-2| = 2-x$, and $|x-4| = 4-x$, and therefore $f(x) = x+1$. Similarly, if $2 \le x \le 4$, then $f(x) = -x+5$, and if $x \ge 4$ then $f(x) = x-3$. Figure 4.2 shows the graph of $y = f(x)$. Now read off the answer from the picture. The equation

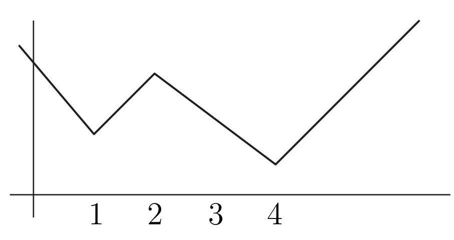

Figure 4.2: An Absolute Value Graph

$f(x) = c$ has three solutions if $c = f(1) = 2$ and if $c = f(2) = 3$.

IV-22. Let a, b, and c be all different. Solve the equation

$$\frac{a^2(x-b)(x-c)}{(a-b)(b-c)} + \frac{b^2(x-c)(x-a)}{(b-c)(b-a)} + \frac{c^2(x-a)(x-b)}{(c-a)(c-b)} = x^2.$$

Solution. We could simplify. That takes effort and a little luck, for there are many $-$ signs, and many chances to err. Instead, *imagine* simplifying. We would get something of the shape $px^2 + qx + r = 0$.

By looking at the equation as given, we can see that $x = a$, $x = b$, and $x = c$ are solutions. But a quadratic equation has no more than two solutions, and a linear equation has only one, so the equation must simplify to $0x^2 + 0x + 0 = 0$. The equation holds for *any* x.

IV-23. Find all real numbers x such that $(x-1)^4 + (x-3)^4 = 56$.

Solution. Bring out the symmetry by letting $y = x - 2$. The equation becomes $(y+1)^4 + (y-1)^4 = 56$.

Expand. Note that $(y+1)^4 = y^4 + 4y^3 + 6y^2 + 4y + 1$, while $(y-1)^4 = y^4 - 4y^3 + 6y^2 - 4y + 1$. Add and simplify. We get $y^4 + 6y^2 - 27 = 0$. This is a quadratic in y^2. Solve, either with the quadratic formula or by noting that the equation can be rewritten as $(y^2-3)(y^2+9) = 0$. Since y^2 can't be negative, $y = \pm\sqrt{3}$, and therefore $x = 2 \pm \sqrt{3}$.

IV-24. Find all solutions of the system

$$x^2(x^2+1) = y^2(y^2+1) = z^2(z^2+1) = 4xyz.$$

Solution. If, for example, $|x| > |y|$, then $x^2(x^2+1) > y^2(y^2+1)$. So x, y, and z must all have the same absolute value.

Look first for non-negative solutions. Then $x = y = z = t$ for some $t \geq 0$, where $t^2(t^2+1) = 4t^3$. This has the obvious solution $t = 0$. If $t \neq 0$, then $t^2 - 4t + 1 = 0$, and therefore $t = 2 \pm \sqrt{3}$.

If one of x, y, or z is negative, then exactly two must be, since $4xyz$ must be positive. Thus each positive solution (t, t, t) gives birth to the three additional solutions $(t, -t, -t)$, $(-t, t, -t)$, and $(-t, -t, t)$. There are therefore exactly 9 solutions.

IV-25. Find all polynomials $P(x)$ such that $(x-4)P(2x)$ is identically equal to $(8x-4)P(x)$.

Solution. The condition holds if $P(x)$ is identically 0. Now look for other solutions.

Let $P(x)$ have degree n. So the leading term of $P(x)$ has shape ax^n for some non-zero constant a. The leading term of $(x-4)P(2x)$ is then $(2^n a)x^{n+1}$, while the leading term of $(8x-4)P(x)$ is $(8a)x^{n+1}$. Thus $2^n a = 8a$ and therefore $n = 3$.

We identify $P(x)$ by studying the roots of $P(x) = 0$. Note that $(x-4)P(2x) = 0$ when $x = 4$. Since $(x-4)P(2x)$ is identically equal to $(8x-4)P(x)$, it follows that

$P(4) = 0$. But then $P(2x) = 0$ when $x = 2$, and therefore $P(2) = 0$. And because $P(2x) = 0$ when $x = 1$, it follows that $P(1) = 0$. Since $P(x)$ is a cubic, there are no other roots.

So $P(x)$ must be of the form $a(x-1)(x-2)(x-4)$ for some constant a. Finally, we must check that if $P(x) = a(x-1)(x-2)(x-4)$, then $(x-4)P(2x)$ is identically equal to $(8x-4)P(x)$. This is easy.

IV-26. Show that if x and y are real and not both 0, then

$$\text{(a)} \quad x^2 + xy + y^2 > 0; \qquad \text{(b)} \quad x^4 + x^3y + x^2y^2 + xy^3 + y^4 > 0.$$

Solution. (a) By completing the square, we get

$$x^2 + xy + y^2 = \left(x + \frac{y}{2}\right)^2 + \frac{3y^2}{4}.$$

Since $x^2 + xy + y^2$ is a sum of two squares, it is positive unless both squares are equal to 0. But if $x + y/2 = 0$ and $y = 0$, then x must be 0. We conclude that $x^2 + xy + y^2$ is positive unless x and y are both 0.

(b) The result is obvious if x and y are both ≥ 0, and also if they are both ≤ 0. Because our expression is symmetric in x and y, we may without loss of generality assume that $|x| \geq |y|$. We may also assume that $x > 0$. For if $x = 0$, then $y = 0$. And if $x < 0$, we can replace x and y by $-x$ and $-y$ without changing the value of $x^4 + x^3y + x^2y^2 + xy^3 + y^4$.

Group the terms as follows:

$$(x^4 + x^3y) + (x^2y^2 + xy^3) + y^4.$$

The first group is equal to $x^3(x + y)$. Since x is positive and $|x| \geq |y|$, we have $x + y \geq 0$, so the first group is ≥ 0, and is > 0 unless $y = -x$. The same idea works for the second group. So our expression is always ≥ 0, and is only 0 if $y = 0$ and $y = -x$, that is, if $x = y = 0$.

Another way: Without loss of generality we may take $x \geq y$. If $x = y$ the result is obvious, so suppose that $x > y$. Then $x^5 > y^5$. Now use the identity

$$(x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4) = x^5 - y^5.$$

Since $x^5 - y^5$ and $x - y$ are positive, so is $x^4 + x^3y + x^2y^2 + xy^3 + y^4$.

Comment. The arguments used in part (b) also work for part (a). The same arguments show that if n is a positive integer, then

$$x^{2n} + x^{2n-1}y + x^{2n-2}y^2 + \cdots + xy^{2n-1} + y^{2n} > 0$$

unless $x = y = 0$.

IV-27. Find all real numbers x and y such that

$$x + y = 10; \qquad \text{and} \qquad \left(\frac{x}{y}\right)^2 - \left(\frac{y}{x}\right)^2 = 2.$$

Solution. Let $x/y = w$. The second equation can be rewritten as $w^2 - 1/w^2 = 2$, or equivalently $w^4 - 2w^2 - 1 = 0$. This is a quadratic in w^2. Since w^2 can't be negative, we conclude that $w^2 = 1 + \sqrt{2}$, and therefore $w = \pm\sqrt{1+\sqrt{2}}$.

Divide both sides of $x + y = 10$ by y. So $x/y + 1 = 10/y$ and therefore

$$y = \frac{10}{1 \pm \sqrt{1+\sqrt{2}}}.$$

Finally, multiply by $\pm\sqrt{1+\sqrt{2}}$ to find x.

Comment. This is one of the sixty-nine solved problems in the *Algebra* of abū Kāmil (ninth century). Many of abū Kāmil's problems were borrowed without attribution by Leonardo of Pisa ("Fibonacci," early thirteenth century). But then again, abū Kāmil borrowed freely from al-Khwārizmī, the mathematician whose book title *al-jabr wa-l-muqābala* mutated into the word *algebra*.

The fact that one of the equations in the system is $x + y = 10$ continues a tradition that goes back to Diophantus of Alexandria. What algebraists *really* meant to do is to use $x + y = a$, where a is a fixed but arbitrary number (a *parameter*). But appropriate notation didn't exist, so they had to pretend that they were dealing with specific numbers. That went on for many centuries and only ended around 1600 with the work of Viète.

IV-28. Is there a function f such that $f(x) + xf(1 - x)$ is identically equal to 1?

Solution. Suppose that $f(x) + xf(1 - x) = 1$ for all x. Substitute $1 - x$ everywhere for x. Since $1 - (1 - x) = x$, we get $f(1 - x) + (1 - x)f(x) = 1$. that is, $f(1 - x) = 1 - (1 - x)f(x)$.

Substitute $1 - (1 - x)f(x)$ for $f(1 - x)$ in the main equation. We obtain $f(x) + x(1 - (1 - x)f(x)) = 1$. Now solve for $f(x)$ and simplify:

$$f(x) = \frac{1 - x}{1 - x + x^2}.$$

Note that $f(x)$ is defined for all x, since $1 - x + x^2$ is never 0. And

$$f(1 - x) = \frac{1 - (1 - x)}{1 - (1 - x) - (1 - x)^2} = \frac{x}{1 - x + x^2},$$

so $f(x) + xf(1 - x)$ is indeed identically equal to 1.

IV-29. Let $P(x) = x^{11} + cx^2 - cx - 1$. Find all c such that $P(x)$ is divisible by $(x-1)^2$.

Solution. We could divide $P(x)$ by $x^2 - 2x + 1$, using ordinary polynomial division. The remainder turns out to be $(11+c)x - (11+c)$. We want this remainder to be identically 0. That forces $c = -11$. The division process isn't hard, but there is a simpler way.

Since $P(1) = 0$, by the Factor Theorem $P(x)$ is divisible by $x - 1$ for any c. We can find an explicit expression for the quotient $Q(x)$.

$$P(x) = (x^{11} - 1) + cx(x-1) = (x-1)\left((x^{10} + x^9 + \cdots + x + 1) + cx\right)$$

and therefore $Q(x) = (x^{10} + x^9 + \cdots + x + 1) + cx$. But $Q(x)$ is divisible by $x - 1$ if and only if $Q(1) = 0$, that is, if and only if $11 + c = 0$. Thus -11 is the only c for which $P(x)$ is divisible by $(x-1)^2$.

Comment. Let $P(x)$ be any polynomial. It can be shown that $(x-a)^2$ divides $P(x)$ if and only if $P(a) = 0$ and $P'(a) = 0$, where $P'(x)$ is the derivative of $P(x)$. In our problem, $a = 1$, $P(1) = 0$, and $P'(x) = 11x^{10} + 2cx - c$. From $P'(1) = 0$ we conclude that $c = -11$.

IV-30. Solve the equation $x^4 + 1 = 4x(x^2 + 1)$.

Solution. We use a trick based on the fact that $x^2 + 1/x^2$ is (almost) the square of $x + 1/x$. Since 0 is not a solution, we divide both sides of our equation by x^2 and obtain the equivalent equation

$$x^2 + \frac{1}{x^2} = 4\left(x + \frac{1}{x}\right).$$

Let $w = x + 1/x$. Then $x^2 + 1/x^2 = w^2 - 2$, and the equation can be written as $w^2 - 2 = 4w$. By the quadratic formula, $w = 2 \pm \sqrt{6}$.

Now solve $x + 1/x = 2 \pm \sqrt{6}$, or equivalently

$$x^2 - (2+\sqrt{6})x + 1 = 0 \qquad \text{and} \qquad x^2 - (2-\sqrt{6})x + 1 = 0.$$

(The solutions of the second equation aren't real.)

IV-31. If x is a real number, the *fractional part* of x is defined to be $x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Denote the fractional part of x by $\{x\}$. Find the largest real number $x < 1$ such that

$$\{x\} + \left\{\frac{1}{x}\right\} = 1.$$

Solution. First look for a solution in the interval $1/2 < x < 1$. Then $1/x$ lies strictly between 1 and 2, and therefore $\{1/x\} = 1/x - 1$. So we want to solve $x + 1/x - 1 = 1$. This equation simplifies to $(x-1)^2 = 0$, which has no solution in our interval.

Now let x travel downward from $1/2$ toward $1/3$. Then $\{1/x\}$ climbs from 0 toward 1, while $\{x\}$ falls from $1/2$ to $1/3$. So they must meet somewhere between $1/2$ and $1/3$.

Let x be the solution. Then $\{x\} = x$ and $\{1/x\} = 1/x - 2$, so $x + 1/x - 2 = 1$, or equivalently $x^2 - 3x + 1 = 0$, and therefore $x = (3 - \sqrt{5})/2$.

IV-32. Find all real numbers x and y such that

$$x^2 + xy + x = 1 \qquad \text{and} \qquad y^2 + xy + y = 2.$$

Solution. Rewrite the equations as

$$x(x + y + 1) = 1 \qquad \text{and} \qquad y(x + y + 1) = 2,$$

then "divide" the second equation by the first. We conclude that $y = 2x$. If we substitute for $2x$ for y in the first equation, we find that $3x^2 + x - 1 = 0$, and therefore $x = (-1 \pm \sqrt{13})/6$. So $x = (-1 + \sqrt{13})/6$ and $y = (-1 + \sqrt{13})/3$ or $x = (-1 - \sqrt{13})/6$ and $y = (-1 - \sqrt{13})/3$

IV-33. For what values of the parameter c does the equation

$$(x - 32)^{10} + (x - 34)^{10} = c$$

have a real solution?

Solution. The symmetry comes out more clearly if we let $w = x - 33$. The equation becomes

$$(w - 1)^{10} + (w + 1)^{10} = c.$$

Imagine expanding each term on the left-hand side. When we expand $(w + 1)^{10}$, we get an expression of the form

$$w^{10} + a_1 w^9 + \cdots + a_9 w + 1,$$

where the constants a_1, a_2, and so on are positive. The a_i can be computed by using the Binomial Theorem, but there is no need to do that.

When we expand $(w - 1)^{10}$, we get

$$w^{10} - a_1 w^9 + \cdots - a_9 w + 1.$$

Add. The result is

$$2w^{10} + 2a_2 w^8 + \cdots + 2a_8 w^2 + 2.$$

This expression is 2 when $w = 0$, greater than 2 when $w \neq 0$, and can be made arbitrarily large by making $|w|$ sufficiently large. So the original equation has a real solution for any $c \geq 2$.

Comment. If we plot $y = (x-32)^{10} + (x-34)^{10}$ using the *right* viewing window, we can spot the minimum value of y. A viewing window centered on $x = 0$ is useless.

IV-34. A circle meets the parabola $y = x^2$ at four points. The x-coordinates of three of the points are 2, 3, and 4. Find the x-coordinate of the fourth point.

Solution. A natural approach is to suppose that the circle has center (a, b) and radius r, and calculate these three numbers. The circle has equation

$$(x-a)^2 + (y-b)^2 = r^2.$$

The circle passes through $(2, 4)$, $(3, 9)$, and $(4, 16)$ and therefore

$$(2-a)^2 + (4-b)^2 = r^2; \quad (3-a)^2 + (9-b)^2 = r^2; \quad (4-a)^2 + (16-b)^2 = r^2.$$

If we "subtract" the second equation from the first and simplify, we get $a+5b = 35$. If we subtract the third from the second, we get $a + 7b = 91$. Solve. It turns out that $b = 28$, $a = -105$, and therefore $r^2 = 12025$.

The circle turns out to have equation

$$x^2 + y^2 + 210x - 56y - 216 = 0.$$

Substitute x^2 for y. We obtain

$$x^4 - 55x^2 + 210x - 216 = 0.$$

Three of the roots of this equation are known, so it is easy to find the fourth. A complicated way of doing it is to divide the polynomial successively by $x-2$, $x-3$, and $x-4$. It is much easier to note that the coefficient of x^3 is 0, and therefore the sum of the roots is 0. Since three of them are known, and their sum is 9, the fourth must be -9.

Another way: Almost all of the calculations were unnecessary! The circle has equation of the shape $(x-a)^2 + (y-b)^2 = r^2$. We don't need to know a, b and r. *Imagine* substituting x^2 for y in the above equation. We get an equation of the form $P(x) = 0$, where $P(x)$ is a polynomial of degree 4 and the coefficient of x^3 is 0. It follows that the sum of the roots of $P(x)$ is 0, so if we know three roots we can easily find the fourth.

IV-35. Find all solutions of the system

$$\begin{aligned} u + v + w &= 1 \\ \frac{1}{u} + \frac{1}{v} + \frac{1}{w} &= 1. \end{aligned}$$

Solution. Note that none of u, v, or w can be 0. As long as we remember that, we can replace the second equation by

$$vw + uw + uv = uvw.$$

Whatever u, v, and w may be, they are the roots of the cubic equation

$$(x-u)(x-v)(x-w) = 0,$$

which expanded becomes

$$x^3 - (u+v+w)x^2 + (vw+uw+uv)x - uvw = 0.$$

Let $p = uvw$. So u, v, and w are the roots of

$$x^3 - x^2 + px - p = 0.$$

Note that 1 is a root of this equation. So at least one of u, v, or w is 1. For now suppose that $u = 1$. Substituting in the original equations, we obtain $v + w = 0$, and $1/v + 1/w = 0$. The solutions of this system are $v = t$, $w = -t$, where t is any non-zero real.

It follows that the solutions (u, v, w) of the original system are all triples of the form $(1, t, -t)$, $(-t, 1, t)$, or $(t, -t, 1)$, where $t \neq 0$.

IV-36. Let $P = (0, 1)$, and $Q = (-b, c)$. The line segment PQ is a diameter of a certain circle. Where does that circle meet the x-axis?

Solution. The center of the circle is the midpoint of the segment PQ, namely the point with coordinates $(-b/2, (c+1)/2)$. The square of the radius of the circle is $\left(b^2 + (c-1)^2\right)/4$. Thus the circle has equation

$$\left(x + \frac{b}{2}\right)^2 + \left(y - \frac{c+1}{2}\right)^2 = \frac{b^2 + (c-1)^2}{4}.$$

The circle crosses the x-axis at the point (or points) where $y = 0$. In the equation of the circle, multiply through by 4, set y equal to 0, simplify, and solve for x. We obtain

$$(2x + b)^2 = b^2 - 4c, \qquad \text{that is,} \qquad x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

If $b^2 - 4c$ is negative, there is no x-intercept. If $b^2 - 4c = 0$, there is only one, while if $b^2 - 4c > 0$ there are two.

Comment. The problem shows how to solve $x^2 + bx + c = 0$ geometrically. For *given* the numbers b, and c, we can *construct* the points P and Q with straightedge and compass, and then construct the circle of the problem. We end up with a straightedge and compass construction of the roots of the equation.

IV-37. Find all real numbers x such that $4^x = 2^{x^2}/16$.

Solution. Take the logarithms of both sides—any base will do, but 2 is best. We have $\log_2 4^x = \log_2 2^{2x} = 2x$, and $\log_2(2^{x^2}/16) = x^2 - 4$. So our equation holds if and only if $2x = x^2 - 4$, that is, when $x = 1 \pm \sqrt{5}$.

IV-38. Find a formula $F(x, y)$ that uses only basic arithmetical operations and square root such that $F(x, y)$ is always the larger of x and y. How about a formula $G(x, y, z)$ such that $G(x, y, z)$ is the largest of x, y, and z? Use the convention that $\sqrt{u}$ is the non-negative number whose square is u.

Solution. Let $F(x, y) = \bigl(x + y + \sqrt{(x-y)^2}\bigr)/2$. If $x \geq y$, then $\sqrt{(x-y)^2} = x - y$, so $F(x, y) = x$. If $x \leq y$, then $\sqrt{(x-y)^2} = y - x$, and $F(x, y) = y$. With three variables, let $G(x, y, z) = F(F(x, y), z)$—not symmetric, but easy!

Comments. 1. In many secondary school books, $\sqrt{9} = \pm 3$ is considered to be correct; in university calculus courses $\sqrt{9} = 3$; and in a few advanced mathematics courses again $\sqrt{9} = \pm 3$. It all depends on what dialect of Mathematics one speaks. The great mathematician Euler wrote $\sqrt{9} = \pm 3$. My calculator claims that $\sqrt{9} = 3$.
2. There is no particular virtue in expressing the function "the greater of x and y" by a formula. The formula given in the solution is less informative than the forthright $\max(x, y)$. And what is a formula? If the trigonometric functions had not been given names, there would be no "formula" for the cosine function, but that function would still exist.

There is a widespread superstition that all functions are expressible by formulas. But many important functions are either given by different formulas over different parts of their domain, or are not expressible by a formula at all. The fact that "function" and "formula" are not the same thing was only recognized in the nineteenth century, and led to significant progress. But then again, once upon a time there was no algebraic notation, and hence no formulas. In early seventeenth century Europe, formulas were a great forward leap.

IV-39. The numbers x_1, x_2, …, x_{99} satisfy the equations $x_1 + x_2 = 1$, $x_2 + x_3 = 2$, …, $x_{98} + x_{99} = 98$, $x_{99} + x_1 = 99$. Find x_1.

Solution. From the first equation, $x_2 = 1 - x_1$. But $x_3 = 2 - x_2$, so $x_3 = 2 - (1 - x_1) = 1 + x_1$. Similarly, $x_4 = 3 - x_3 = 2 - x_1$, $x_5 = 2 + x_1$, and so on up to $x_{99} = 49 + x_1$. The final equation gives $x_1 = 50 - x_1$, so $x_1 = 25$.

We aren't quite done. The problem said that the numbers x_1, x_2, … satisfy certain equations. We showed that *if* there are such numbers, then $x_1 = 25$. Are there really such numbers? Should we trust the poser of the problem or check for ourselves? The answer is clear—we check. Let $x_1 = 25$, $x_2 = -24$, $x_3 = 26$, $x_4 = -23$, $x_5 = 27$, $x_6 = -22$, and so on. Then all the equations up to the last one are satisfied. But $x_{99} = 74$, so the last equation also holds.
Another way: "Add up" the equations:

$$2(x_1 + x_2 + \cdots + x_{99}) = 1 + 2 + \cdots + 99.$$

Now add together the 2nd, 4th, 6th, ..., up to the 98th equation, and multiply by 2:

$$2(x_2 + x_3 + \cdots + x_{99}) = 2(2 + 4 + \cdots + 98).$$

Subtract, grouping terms for convenience. We get

$$2x_1 = (1 - 2) + (3 - 4) + \cdots + (97 - 98) + 99 = -49 + 99 = 50.$$

Again, we should check that there really is a solution.

Comments. 1. If n is odd, the system $x_1 + x_2 = a_1$, $x_2 + x_3 = a_2$, ..., $x_n + x_1 = a_n$ always has a solution, and

$$2x_1 = a_n - a_{n-1} + a_{n-2} - \cdots + a_1.$$

If n is even, then either of the methods we used gives

$$x_1 = a_n - a_{n-1} + \cdots + a_2 - a_1 + x_1,$$

so there is no solution unless $a_n - a_{n-1} + \cdots + a_2 - a_1 = 0$. If this last sum is 0, there are solutions, and x_1 can be chosen arbitrarily.

2. This type of system comes up repeatedly in the early Indian literature. Here is an instance from the *Bakhshālī Treatise*, which was found in 1881 in what was then northwest India, and dates back to possibly as long ago as the year 200.

Five persons each possess a certain amount of wealth. The riches of the first and second taken together amount to 16. The riches of the second and third taken together are 17. The riches of the third and fourth are 18, the riches of the fourth and fifth are 19 and the riches of the fifth and first together amount to 20. Tell me what is the amount of each.

IV-40. Let a and b be different non-zero numbers. Suppose that the vertical line through $(a, 0)$ meets the parabola $y = x^2$ at A, and the vertical line through $(b, 0)$ meets the parabola at B. Where does the line AB meet the y-axis?

Solution. This is the sort of problem that yields quickly to computation. We can find the coordinates of A and B, then find the equation of the line AB, then calculate the y-intercept.

More simply, let the y-intercept P be $(0, y)$. The line segments PA and AB have the same slope, so $(a^2 - y)/a = (b^2 - a^2)/(b - a) = b + a$ and therefore $y = -ab$.

IV-41. Find the remainder when the polynomial x^{100} is divided by $x^2 - 5x + 6$.

Solution. We could actually divide. That's not quite as bad as it sounds. The first few terms of the quotient are $x^{98}+5x^{97}+19x^{96}+65x^{95}$. With a little luck we spot the pattern $5 = 3^2 - 2^2$, $19 = 3^3 - 2^3$, $65 = 3^4 - 2^4$. The pattern continues, but with this approach it isn't easy to prove that it does.

Instead, just *imagine* dividing. If we divided, we would obtain a quotient $Q(x)$ and a remainder of the shape $ax + b$, where a and b are constants. Since $x^2 - 5x + 6 = (x-2)(x-3)$, the relation

$$x^{100} = (x-2)(x-3)Q(x) + ax + b.$$

holds for all x. Put $x = 2$. The nice thing about this choice is that we can conclude that $2^{100} = 2a + b$ without computing $Q(2)$. In the same way, we find that $3^{100} = 3a + b$. Solve: $a = 3^{100} - 2^{100}$ and $b = 3 \cdot 2^{100} - 2 \cdot 3^{100}$.

Another way: Without loss of generality, we may assume that the remainder has shape $c(x-2) + d(x-3)$. So

$$x^{100} = (x-2)(x-3)Q(x) + c(x-2) + d(x-3).$$

Set $x = 2$. We get $d = -2^{100}$. Similarly, $c = 3^{100}$.

Comment. Using the same methods, we can compute the remainder when any polynomial $F(x)$ is divided by $(x-p)(x-q)$, where $p \neq q$. The second method produces a simple explicit general formula. The ideas generalize.

The quotient when x^{100} is divided by $(x-p)(x-q)$ turns out to be

$$x^{98} + (p+q)x^{97} + (p^2+pq+q^2)x^{96} + \cdots + (p^{98} + p^{97}q + \cdots + q^{98}).$$

IV-42. Someone used the incorrect equation $\log ab = (\log a)(\log b)$ in a calculation and oddly enough ended up with the right answer. Find all possible values of a and b.

Solution. We interpret the question to mean: for which (positive) a and b does the given equation hold? Since in fact $\log ab = \log a + \log b$, it follows that a and b must satisfy the equation $(\log a)(\log b) = \log a + \log b$.

For clarity, let $u = \log a$ and $v = \log b$. Thus $uv = u + v$. Rewrite this as $(u-1)(v-1) = 1$. This says that $u-1$ and $v-1$ are reciprocals: if $u - 1 = s$ then $v - 1 = 1/s$. The solutions are therefore $u = 1 + s$, $v = 1 + 1/s$, where s is any non-zero number. Hence the possible values of a and b are $a = 10^{1+s}$, $b = 10^{1+1/s}$, where s ranges over the non-zero reals.

IV-43. The equation $x^2 + 2(k-1)x + 9 = 0$ has exactly one solution x. Determine the possible values of k.

Solution. A quadratic equation has exactly one solution precisely if the discriminant is 0. But the given equation is not necessarily a quadratic—the coefficient of x^2 can be 0. If $k = 0$, the equation is linear, with a unique solution. If $k \neq 0$, then we have a quadratic and can apply the discriminant test.

The discriminant is $4(k-1)^2 - 36k^2$. When we set this equal to 0 and simplify, we get $8k^2 - 2k - 1 = 0$. Solve, using the quadratic formula (also, the polynomial happens to factor nicely). The solutions are $-1/4$ and $1/2$, and therefore the possible k are 0, $-1/4$, and $1/2$.

Comment. It's easy to miss the solution $k = 0$. It is surprising how invisible 0 can be—or maybe not, after all it is nothing. This invisibility also comes up with an equation like $x^2 + 7x = 3x^2 + x$. We cancel the x with relief, since that simplifies the equation, sometimes forgetting that we are throwing away a root.

IV-44. Find the intersection points of the curves $y = |x|/2$ and $y = ||x| - 5|$.

Solution. Each curve is symmetric about the y-axis, so it is enough to deal with $x \geq 0$. For such x, we are looking at the curves $y = x/2$ and $y = |x-5|$. (Symmetry cut the work in half, and as a bonus got rid of two absolute values.) If we sketch these curves, as in the right half of Figure 4.3, or tell a machine to do so, we can see where the intersections are. To compute the intersection points, note first that

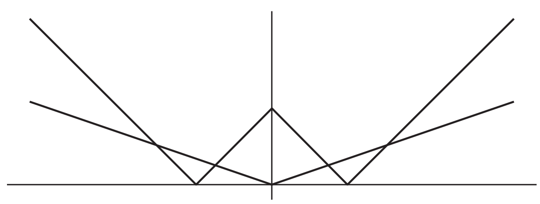

Figure 4.3: The Curves $y = |x|/2$ and $y = ||x| - 5|$

if $x \geq 5$, then $|x - 5| = x - 5$. Put $x/2 = x - 5$. Then $x = 10$, which yields the point $(10, 5)$. If $0 \leq x < 5$, then $|x - 5| = 5 - x$. Put $x/2 = 5 - x$. Then $x = 10/3$, which yields the point $(10/3, 5/3)$. For the other intersection points, reflect in the y-axis.

IV-45. Find all solutions of the system

$$y + z = \frac{5}{x+y+z}; \quad z + x = \frac{6}{x+y+z}; \quad x + y = \frac{7}{x+y+z}.$$

Solution. If we "divide" the first equation by the second, we obtain a linear equation linking x, y, and z. If we do the same thing with the next two equations, we obtain another linear equation. Use these equations to express two of the variables, say x and y, in terms of the third. Finally, substitute into the first equation and solve for

z. The work is tedious and doesn't exploit the symmetry enough. Here is a simpler way.

"Add" the three equations and simplify. We obtain $2(x+y+z)^2 = 18$ and conclude that $x+y+z = \pm 3$. Work first with $x+y+z=3$. From the first equation, we find that $y+z = 5/3$, and therefore $x = 4/3$. Similarly, $y = 1$ and $z = 2/3$.

We can work in the same way with $x+y+z = -3$. But there is no need to compute, for if (x,y,z) is a solution of the system then so is $(-x,-y,-z)$. So the other solution is $x = -4/3$, $y = -1$, and $z = -2/3$.

IV-46. Show that for any a the solutions of

$$ax^3 = x^2 + x + 1$$

are not all real. Hint. Let $x = 1/u$.

Solution. Let $P(x) = 1 + x + x^2 - ax^3$. Since 0 is not a root of $P(x)$, put $x = 1/u$, and multiply through by u^3. Our equation becomes $Q(u) = 0$, where $Q(u) = u^3 + u^2 + u - a$. The roots of $Q(u)$ are just the reciprocals of the roots of $P(x)$, so they are all real if and only if the roots of $P(x)$ are.

Let the roots of $Q(u)$ be p, q, and r. Then $Q(u) = (u-p)(u-q)(u-r)$. Expand and compare coefficients. We get

$$p + q + r = -1 \qquad \text{and} \qquad pq + qr + pr = 1.$$

Since

$$(p+q+r)^2 = p^2 + q^2 + r^2 + 2(pq + qr + pr)$$

we conclude that $p^2 + q^2 + r^2 = -1$. That's impossible if p, q, and r are real.

Comment. The same method shows that if

$$P(x) = 1 + px + qx^2 + \text{ higher order terms}$$

and $p^2 \le 2q$ then not all the roots of $P(x)$ are real.

IV-47. Find all four solutions of

$$(x)(x-2)(x-4)(x-6) = 8 \cdot 6 \cdot 4 \cdot 2.$$

Solution. The solution $x = 8$ is obvious; $x = -2$ is less apparent, but jumps out if we put $x = -y$. Let

$$P(x) = (x)(x-2)(x-4)(x-6) - 8 \cdot 6 \cdot 4 \cdot 2.$$

Since 8 is a solution of $P(x) = 0$, the Factor Theorem shows that $(x-8)$ divides $P(x)$. Do the division (it can be done without multiplying $P(x)$ out) and let the quotient be $Q(x)$. But $(x+2)$ divides $Q(x)$ because -2 is a solution of $Q(x) = 0$.

Divide. The quotient is a quadratic polynomial whose roots can be found in the usual way. Maybe we can save some time by dividing only once, by the product $(x-8)(x+2)$.

Another way: The sum of the roots of the original equation is the negative of the coefficient of x^3, that is, 12. The original product of the roots is $-(8 \cdot 6 \cdot 4 \cdot 2)$. Imagine dividing $P(x)$ by $(x-8)(x+2)$. We get a quadratic polynomial with sum of roots $12-(8-2)$ and product of roots $6 \cdot 4$. So the quadratic is $x^2-6x+24$, and its roots are $3 \pm \sqrt{-15}$.

Another way: Exploit the symmetry by letting $y = x-3$. The equation becomes $(y^2-9)(y^2-1) = 384$. Exploit the symmetry a little more by letting $z = y^2-5$. The equation becomes $z^2 = 400$. Find z, then y, then x.

IV-48. Let ϵ be a given positive number. Solve the inequality

$$\sqrt{x+1} - \sqrt{x} < \epsilon.$$

Solution. Let $f(x) = \sqrt{x+1} - \sqrt{x}$. Note that $f(x)$ is only defined when $x \geq 0$. Multiply "top" and "bottom" by $\sqrt{x+1} + \sqrt{x}$. We conclude that

$$f(x) = \frac{1}{\sqrt{x+1} + \sqrt{x}}.$$

The function $\sqrt{x+1} + \sqrt{x}$ has value 1 at $x = 0$, and grows steadily without bound as x becomes large. So if $\epsilon > 1$, then our inequality holds for all x, and if $\epsilon = 1$ then the inequality holds for all positive x.

Suppose now that $0 < \epsilon < 1$. Since $f(x)$ decreases steadily from 1 towards 0, $f(x)$ takes on the value ϵ only once. To find where $f(x) < \epsilon$ we first solve the equation $f(x) = \epsilon$.

Rewrite $\sqrt{x+1} - \sqrt{x} = \epsilon$ as $\sqrt{x+1} = \sqrt{x} + \epsilon$, square both sides and simplify. We get $1 - \epsilon^2 = 2\epsilon\sqrt{x}$. Square again. We conclude that $x = (1-\epsilon^2)^2/4\epsilon^2$. Since if $\epsilon < 1$ there is a solution, and we have found only one expression for x, that expression must be the solution.

There is a better way to solve $\sqrt{x+1} - \sqrt{x} = \epsilon$. Rationalize the numerator and flip. We get $\sqrt{x+1} + \sqrt{x} = 1/\epsilon$. Subtraction gives $2\sqrt{x} = 1/\epsilon - \epsilon$.

In conclusion, if $\epsilon > 1$ then the inequality holds for all $x \geq 0$; if $\epsilon = 1$ the inequality holds for all $x > 0$; and if $\epsilon < 1$ then the inequality holds for all $x > \big(1/\epsilon - \epsilon\big)^2/4$.

Comment. If we don't recognize that the case $\epsilon > 1$ is special, and just blindly manipulate, we could wrongly decide that for any ϵ the inequality holds precisely when $x > \big(1/\epsilon - \epsilon\big)^2/4$. A look at the graph of $y = \sqrt{x+1} - \sqrt{x}$ helps avoid that mistake.

IV-49. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Find all integers n that satisfy the equation

$$\lfloor\sqrt{1}\rfloor + \lfloor\sqrt{2}\rfloor + \lfloor\sqrt{3}\rfloor + \cdots + \lfloor\sqrt{n}\rfloor = 3n.$$

Solution. Explore by calculating $\lfloor\sqrt{k}\rfloor$ for various k, starting at 1. We obtain 1, 1, 1, 2, 2, 2, 2, 2, 3, ... The pattern and its explanation are clear: $\lfloor\sqrt{k}\rfloor = j$ for $j^2 \le k < (j+1)^2$.

We reword the problem in terms of money. On day 1 we receive $\lfloor\sqrt{1}\rfloor$ dollars, on day 2 we receive $\lfloor\sqrt{2}\rfloor$ dollars, and so on. We would like to find the integer(s) n such that our average earnings per day over the first n days are 3 dollars.

After the first day, we are 2 dollars short of meeting our goal. We fall 2 dollars further behind on each of days 2 and 3. On each of days 4 through 8 we fall 1 dollar further behind. So after day 8 we are 11 dollars behind. On days 9 to 15 we are getting 3 dollars a day, so we fall no further behind.

On day 16 we begin to make progress. Day 26 brings us to an average of 3 dollars a day. After that, our average daily earnings are greater than 3, so $n = 26$ is the only solution.

Comment. If we try to solve the equation by symbol manipulation, the chances of success are small.

The mention of money in the solution is inessential but useful, since it makes the problem concrete. We can see immediately, for example, that if we replace square roots by cube roots, and $3n$ by $4n+3$, the reasoning proceeds in much the same way.

IV-50. Given that

$$\left(x+\frac{1}{x}\right)^2 = 50, \qquad \text{find} \qquad x^2+\frac{1}{x^2}.$$

Solution. Straightforward squaring gives

$$\left(x+\frac{1}{x}\right)^2 = x^2+2+\frac{1}{x^2}.$$

Since $(x+1/x)^2 = 50$, we conclude that $x^2+1/x^2 = 48$.

Another way: We could use a brute force attack. From the given equation, we get $x+1/x = \pm\sqrt{50}$. The solutions are the four roots of $x^2 \mp \sqrt{50}\,x+1 = 0$. We can write down the roots explicitly and for each of them compute x^2+1/x^2. It takes longer.

Comment. Suppose that $x+1/x = a$. We show how to express x^2+1/x^2, x^3+1/x^3, x^4+1/x^4, and so on in terms of a.

By the argument above, $x^2+1/x^2 = a^2-2$. For x^3+1/x^3, note that

$$\left(x+\frac{1}{x}\right)^3 = x^3+3x+\frac{3}{x}+\frac{1}{x^3}.$$

It follows that $x^3+1/x^3 = a^3-3a$. And since

$$\left(x+\frac{1}{x}\right)^4 = x^4+4x^2+6+\frac{4}{x^2}+\frac{1}{x^4},$$

$x^4+1/x^4 = a^4-4(a^2-2)-6 = a^4-4a^2+2$. One can continue, and find expressions for x^5+1/x^5, x^6+1/x^6, and so on in terms of a.

IV-51. Find the largest real number x that satisfies the equation

$$|x| + |x-2| = x^2.$$

Solution. Look first at real numbers x greater than 2. Then $|x|+|x-2| = 2x-2$, and we are studying the equation $x^2-2x+2 = 0$. Use the quadratic formula, or the fact that $x^2-2x+2 = (x-1)^2+1$, to conclude that the equation has no real solutions.

Suppose now that $0 \le x \le 2$. In that interval, $|x|+|x-2| = 2$. This is because $|x-a|$ is the distance from x to a, so $|x-0|+|x-2|$ is the distance of x from 0 plus the distance of x from 2. If $0 \le x \le 2$, this sum is clearly 2.

So we are looking at $x^2 = 2$ which has the solution $x = \sqrt{2}$ in our interval. There is no point in looking at negative x, since these can't be greater than $\sqrt{2}$.

Another way: A graphical solution is more informative. We can plot the curves $y = x^2$ and $y = |x|+|x-2|$. The first curve is familiar, and the second is easy to draw. It is symmetrical about $x = 1$, has constant value 2 between 0 and 2, and rises with slope 2 past $x = 2$. The picture shows that the parabola $y = x^2$ meets the curve at the positive solution of $x^2 = 2$, and also at one negative x.

IV-52. Find a polynomial $P(x)$, of as low a degree as possible, such that x^2 divides $P(x)$ and $(x-1)^2$ divides $P(x)-1$.

Solution. A quick glance shows that no polynomial of degree less than 3 can work, so we look for a suitable cubic $P(x)$. Since $P(x)$ must have the shape $P(x) = x^2(ax+b)$, we need to find $a \neq 0$ and b such that $(x-1)^2$ divides $x^2(ax+b)-1$.

Divide the polynomial $x^2(ax+b)-1$ by x^2-2x+1. The remainder turns out to be $(3a+2b)x-2a-b-1$. To make this identically 0, set $3a+2b = 0$ and $2a+b+1 = 0$. The only solution is $a = -2$, $b = 3$.

Or else we can argue that $x-1$ must divide $P(x)-1$, and therefore $P(x)-1 = 0$ when $x = 1$. From this we get $a+b = 1$. Now

$$ax^3+bx^2-1 = ax^3+x^2-ax^2-1 = a(x^3-x^2)+(x^2-1).$$

Divide by $x-1$. The quotient is $ax^2+(x+1)$, and must be divisible by $x-1$, so $a+2 = 0$. Now it is easy to find a and b.

IV-53. Solve the system

$$\begin{aligned} x-y &= 10 \\ x^3-y^3 &= 2170. \end{aligned}$$

Solution. Start with the identity $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$, divide by $x - y$, and conclude that $x^2 + xy + y^2 = 217$. Now what?

Note that $3xy = (x^2 + xy + y^2) - (x - y)^2$, so $xy = 39$, and therefore $x(x - 10) = 39$. By the quadratic formula, or otherwise, $x = 13$ or $x = -3$, and therefore $y = 3$ or $y = -13$.

More symmetrically, note that $(x + y)^2 = (x - y)^2 + 4xy$, so $(x + y)^2 = 256$, and therefore $x + y = \pm 16$. Given $x - y$ and $x + y$, it is easy to find x and y.

Or else note that $x = y + 10$, substitute for x in the equation $x^3 - y^3 = 2170$, and simplify. We quickly reach a quadratic equation in y.

Another way: Here is, in modern notation, the solution given by Diophantus, the probable originator of the problem. Note how beautifully he exploits symmetry.

Let $x = s + 5$. Then $y = s - 5$. The difference $(s + 5)^3 - (s - 5)^3$ is $30s^2 + 250$. Set this equal to 2170. We find easily that $s^2 = 64$. Neat!

Comment. Diophantus of Alexandria wrote a treatise called *Arithmetica*, maybe around the year 250. The original work consisted of 13 books. A book in the ancient Greek sense is something that can be rolled into a scroll of convenient size. The modern equivalent is probably a chapter.

In works that consist of several scrolls, the scrolls can become separated. That happened with *Arithmetica.* Until recently, it was thought that only six books remained. In the early 1970's, it became known that four more books had survived, in an Arabic translation by Qusta ibn Lūqā, in the Mashhad Shrine Library. This problem comes from one of the "lost" books.

Diophantus didn't have our modern algebraic notation—that was only developed in the seventeenth century. He had a shorthand for "the" unknown, but only for one unknown. The problem was stated without symbols (two numbers differ by 10 and their cubes differ by 2170). There were no negative numbers.

Arithmetica contains some virtuoso problem-solving that inspired the seventeenth century revival of Number Theory. In a margin of Bachet's translation of *Arithmetica* into Latin, Fermat wrote down the conjecture that became known as *Fermat's Last Theorem* and remained unsolved for more than 350 years.

IV-54. Find all c such that the polynomial $x^5 - 5cx + 1$ has (i) one real root; (ii) two real roots; (iii) three real roots.

Solution. Draw, or imagine drawing, the curves $y = x^5$ and $y = 5cx - 1$. If $c < 0$, then $y = 5cx - 1$ goes down as x increases. But $y = x^5$ goes up, so they cross at most once. They do cross, since $y = 5cx - 1$ is above $y = x^5$ when x is large negative and below $y = x^5$ when x is positive.

The curves also cross once if $c = 0$ or if c is positive but small, for $y = 5cx - 1$ is below $y = x^5$ at $x = 0$, and if the line has small slope it can never catch up.

As c grows, the line comes closer to catching $y = x^5$. For a certain positive c_0 the line just grazes $y = x^5$ and there are two roots, one negative and one positive. And if $c > c_0$, then $y = 5cx - 1$ passes $y = x^5$. But $x^5 > 5cx - 1$ when x is very large, so there are three roots.

We now compute c_0. If $y = 5c_0x - 1$ is tangent to $y = x^5$ at $x = r$, then r is a "double" root of $x^5 - 5c_0x + 1 = 0$, meaning that $(x - r)^2$ divides $x^5 - 5c_0x + 1$.

Do the division. The remainder turns out to be $5r^4x - 5c_0x + 1 - 4r^5$. This remainder identically 0 if $c_0 = r^4$ and $4r^5 = 1$. So $r = 4^{-1/5}$ and therefore $c_0 = 4^{-4/5}$.

IV-55. Given that

$$x_0 + x_1 = a_1, \quad x_0 + x_2 = a_2, \quad x_0 + x_3 = a_3, \ldots, x_0 + x_n = a_n,$$
$$x_0 + x_1 + \cdots + x_n = b,$$

find an explicit expression for x_0 in terms of b and $a_1, \ldots, a_n$.

Solution. "Add up" all of the equations except the last. There are n copies of x_0. Let one of them keep company with the rest of the x_i. We get

$$(n - 1)x_0 + (x_0 + x_1 + x_2 + \cdots + x_n) = a_1 + a_2 + \cdots + a_n.$$

But $x_0 + x_1 + \cdots + x_n = b$, so $x_0 = (a_1 + a_2 + \cdots + a_n - b)/(n - 1)$.

Comment. The rule for solving this system is attributed by the Neo-Pythagorean Iamblichus (fourth century?) to an earlier Pythagorean named Thymaridas (−350?). For obscure reasons, it is known as the *Bloom of Thymaridas.*

Make me a crown weighing sixty minae, mixing gold and brass, and with them tin and iron. Let the gold and bronze together form two-thirds, the gold and tin together three-fourths, and the gold and iron three-fifths. Tell me how much gold you must put in, ... (*M*etrodorus' Greek Anthology, maybe fifth century).

IV-56. Solve the equation

$$\frac{x^2 - 4x}{x - 2} = \frac{2x}{2 - x} + 1.$$

Solution. Perhaps we could multiply both sides by $x - 2$, and then simplify. We get $x^2 - 4x = -2x + (x - 2)$, and therefore $x^2 - 3x + 2 = 0$. This factors as $(x - 1)(x - 2) = 0$, giving $x = 1$ or $x = 2$.

It is easy to verify that 1 is indeed a solution of the original equation. But 2 is not—the expressions involved are not even *defined* when $x = 2$.

Comment. What happened? We showed that *if* the number x is a solution of the given equation, *then* the number x is a solution of the simplified equation $x^2 - 3x + 2 = 0$. That's perfectly correct: Any solution of the original equation is a solution of the transformed equation, just like any human being is a biped. So no number other than 1 or 2 can be a solution. But that doesn't mean that any solution of $x^2 - 3x + 2$ has to be a solution of the original equation: Some bipeds are chickens.

There is a similar logical flaw built into the symbol manipulations we use to solve equations. We transform the equation into another one, then another, and so on, until the solutions become obvious. Logically speaking, this only shows that any solution of the equation we started with must be a solution of the equation we end up with. That *does not* necessarily mean that a solution of the latter is a solution of the former.

IV-57. Let $c \geq 0$. Find the number of real solutions of $x^5+x^3-10x^2-10 = c$, explaining why your answer is certain to be right.

Solution. We could try to use a graphing calculator, but the parameter c causes difficulty. If we plot $y = x^5 + x^3 - 10x^2 - 10 - c$ for $c = 0.5$, $c = 1$, and so on, we may come to believe that for any $c \geq 0$ there is exactly one real solution. But it's hard to imagine examining every possible graph.

The problem can be solved without a calculator. Divide both sides of the equation by $x^2 + 1$. We obtain

$$x^3 = 10 + \frac{c}{x^2+1}.$$

Graph, or imagine graphing, the curves $y = x^3$ and $y = 10 + c/(x^2 + 1)$. The curves can't meet at a negative x. And the function x^3 is steadily increasing, while $10 + c/(x^2 + 1)$ is decreasing in the interval $[0, \infty)$. So the curves $y = x^3$ and $y = 10 + c/(x^2 + 1)$ can meet at most once.

A rough sketch shows that they do meet, but let's do the details. At $x = \sqrt[3]{10}$, the curve $y = x^3$ lies below $y = 10+c/(x^2+1)$. And $10+c/(x^2+1)$ is never larger than $10 + c$, so the curve $y = x^3$ is above $y = 10 + c/(x^2 + 1)$ when $x \geq \sqrt[3]{10+c}$. It follows that our equation has a solution somewhere between $\sqrt[3]{10}$ and $\sqrt[3]{10+c}$.

Another way: Rewrite the original equation as $x^2(x^3 + x - 10) = 10 + c$. We show that this equation has exactly one solution for any $c > -10$. This is a stronger result than was asked for.

If $c > -10$ then $10 + c$ is positive, so any solution of our equation must be positive. The function x^2 is an increasing function past $x = 0$. Since $x^3 + x - 10$ is an increasing function, $x^2(x^3 + x - 10)$ is an increasing function past $x = 0$. Thus $x^2(x^3 + x - 10)$ takes on the value $10 + c$ at most once. It does take on this value, for $x^2(x^3 + x - 10)$ is 0 when $x = 0$, and is very large when x is large, so somewhere it must be equal to $10 + c$.

Comment. Even though the solution is short, the general parameter c makes the problem difficult. It may be worthwhile to ask first about a specific example such as $c = 0.5$.

We asserted in the solution that it's hard to imagine examining the graph for every c. That's not really true. Look at the equation $z = x^5 + x^3 - 10x^2 - 10 - y$. This is the equation of a *surface* in three-dimensional space. There are quite a few computer programs that do a good job of visually representing surfaces. And computer-driven machines can accurately mold surfaces.

In the second solution, we could have noted that $x^3 + x - 10 = 0$ when $x = 2$, so $x^3 + x - 10$ can be factored. We chose the number 10 in order to make factoring approaches possible. The solutions given work if we replace $10x^2$ by something less round, such as $3.14x^2$.

IV-58. Find two rectangles with integer sides and equal perimeters such that the first rectangle has four times the area of the second.

Solution. If a, b, c, and d are positive integers such that

$$ab = 4cd, \qquad \text{and} \qquad a + b = c + d.$$

then the $a \times b$ and the $c \times d$ rectangles satisfy our conditions.

Let d be a multiple of b, say $d = kb$. If $a = 4kc$, then the first equation is satisfied. The second equation becomes $4kc+b = c+kb$, or equivalently $(4k-1)c = (k-1)b$. Thus if k is any integer greater than 1, and

$$d = kb, \quad a = 4kc, \quad \text{and} \quad (4k-1)c = (k-1)b,$$

then all our conditions are satisfied.

For a concrete example, take for instance $c = 1$ and $k = 2$. Then $b = 7$, $a = 8$, and $d = 14$.

IV-59. Solve the system $x + y = xy = x^2 - y^2$.

Solution. The system has the obvious solution $x = y = 0$. If $x + y = 0$, then since $x + y = xy$, one of the variables is 0, and therefore both are.

We may now assume that $x + y \neq 0$. Divide both sides of $x + y = x^2 - y^2$ by $x + y$. So $x - y = 1$. Substitute $y + 1$ for x in $x + y = xy$, simplify, and solve. The solutions are $y = (1 \pm \sqrt{5})/2$. So in addition to $x = 0$, $y = 0$, the system has the solutions $x = (3 + \sqrt{5})/2$, $y = (1 + \sqrt{5})/2$ and $x = (3 - \sqrt{5})/2$, $y = (1 - \sqrt{5})/2$.

Another way: Let $x = p + q$ and $y = p - q$. The given equations are equivalent to $2p = p^2 - q^2 = 4pq$. If $p = 0$, then $q = 0$, and we get the solution $x = 0$, $y = 0$.

If $p \neq 0$, then $q = 1/2$, and $p^2 - 2p - 1/4 = 0$. Thus $p = (2 \pm \sqrt{5})/2$. Now that we know p and q, we can easily find x and y.

IV-60. Show that if x and y are real numbers such that all of the square roots below are real, then

$$\sqrt{x \pm \sqrt{y}} = \sqrt{\left(x + \sqrt{x^2 - y}\right)/2} \pm \sqrt{\left(x - \sqrt{x^2 - y}\right)/2}.$$

Solution. There is a lot less to the problem than meets the eye. Call the right-hand side $A \pm B$. If we square, everything magically simplifies. It turns out that $A^2 + B^2 = x$ and $2AB = \sqrt{y}$.

Another way: Let u and v be non-negative numbers such that

$$u^2 = \left(x + \sqrt{x^2 - y}\right)/2 \qquad \text{and} \qquad v^2 = \left(x - \sqrt{x^2 - y}\right)/2.$$

Then $x = u^2 + v^2$ and $y = 4u^2v^2$. So we only need to show that $\sqrt{u^2 + v^2 \pm 2uv} = u \pm v$. This is obvious.

Comment. Numerical examples can be unsettling. For instance, take $x = 10$ and $y = 24$. We obtain

$$\sqrt{10 + 2\sqrt{6}} = \sqrt{5 + \sqrt{19}} + \sqrt{5 - \sqrt{19}},$$

a result which is hard to believe.

IV-61. Find all x that satisfy the inequality

$$\left(\frac{\sqrt{x+1} - 1}{x}\right)^2 > x + 1.$$

Solution. It is worthwhile to look for shortcuts before starting to calculate, for tricks can be useful, specially with problems manufactured by tricky people. In particular, when we see something like $\sqrt{x+1} - 1$, we should wonder whether its companion $\sqrt{x+1} + 1$ might be helpful—and here it is.

Multiply "top" and "bottom" of $(\sqrt{x+1} - 1)/x$ by $\sqrt{x+1} + 1$ and simplify. We get $1/(\sqrt{x+1} + 1)$. Write u for $\sqrt{x+1}$.

Our inequality becomes $1/(u+1)^2 > u^2$. Because u is non-negative, we can take square roots to obtain the equivalent inequality $1/(u+1) > u$, which in turn is equivalent to $u^2 + u - 1 < 0$.

The equation $u^2 + u - 1 = 0$ has roots $u = (-1 \pm \sqrt{5})/2$, and $u^2 + u - 1$ is negative only between these roots. But $u \geq 0$, so $0 \leq \sqrt{x+1} < (-1 + \sqrt{5})/2$, or equivalently $-1 \leq x < (1 - \sqrt{5})/2$.

IV-62. Find x and y such that $2^x \cdot 3^y = 6$ and $4^x \cdot 5^y = 12$.

Solution. Use logarithms. The base doesn't matter much—make it 10. Our equations are equivalent to the linear equations $x \log 2 + y \log 3 = \log 6$ and $x \log 4 + y \log 5 = \log 12$.

To find x, eliminate y in the usual way. We get

$$x = \frac{\log 6 \log 5 - \log 12 \log 3}{\log 2 \log 5 - \log 4 \log 3},$$

and now y is easy to find.

Because 4 is the square of 2, we could work without logarithms for a while. From the first equation we obtain, by squaring, $4^x \cdot 3^{2y} = 36$. But $4^x \cdot 5^y = 12$, and division gives $(9/5)^y = 3$. So $y = \log 3/(\log 9 - \log 5)$.

IV-63. Find all x such that $\log(x^4) < \left(\log(x^2)\right)^2$.

Solution. The curves $y = \log(x^4)$ and $y = \big(\log(x^2)\big)^2$ are symmetric about the y-axis, so first let x be positive and then reflect.

Using properties of logarithms, we then obtain the equivalent inequality

$$4 \log x < 4(\log x)^2, \quad \text{that is,} \quad (\log x)(1 - \log x) < 0.$$

This inequality holds if $\log x < 0$ (for then $1 - \log x > 0$), and also when $\log x > 1$. So the original inequality holds when $0 < |x| < 1$ and when $|x| > 10$.

IV-64. Find all x such that $5/(x-1) < 3/(x+1)$.

Solution. We could draw carefully the curves $y = 5/(x-1)$ and $y = 3/(x+1)$ or ask a graphing calculator to do it. The two hyperbolas meet where $x = -4$, and by looking at the picture it is easy to write down the answer.

The problem is not hard to solve even without a calculator, but inequalities can be treacherous, so we have to be careful. We use the strategy of breaking the problem into cases.

If $|x| > 1$, then $(x-1)(x+1) > 0$, while if $|x| < 1$, then $(x-1)(x+1) < 0$. Suppose first that $|x| > 1$. Since $(x-1)(x+1) > 0$, our inequality holds if and only if

$$\frac{5(x-1)(x+1)}{x-1} < \frac{3(x-1)(x+1)}{x+1},$$

that is, if and only if $5(x+1) < 3(x-1)$. We conclude that the inequality holds whenever $x < -4$.

Suppose next that $|x| < 1$. Since $(x-1)(x+1) < 0$, the same strategy yields the inequality $5(x+1) > 3(x-1)$, that is, $x > -4$. So the inequality also holds for all x in the interval $-1 < x < 1$.

Another way: Let $f(x) = 5/(x-1) - 3/(x+1)$. The only places where f can conceivably change sign are when $x = -1$ or $x = 1$, where f is not defined, and also where $f(x) = 0$, namely at $x = -4$. Our problem is to find where $f(x) < 0$.

In particular, $f(x)$ doesn't change sign in the infinite interval to the left of -4. Calculate $f(x)$ for some pleasant x in that interval, such as -9. The result is negative, so $f(x)$ is negative in the whole interval. Find other convenient *test values* of x, one between -4 and -1, another between -1 and 1, and a final one greater than 1, and calculate $f(x)$ at these test values. The results tell us where $f(x)$ is negative.

Comment. Here and elsewhere, we suggest using a graphing calculator to help us visualize. But it takes a surprising amount of practice to use the tool well: the graphs that calculators produce must be treated with some skepticism. The function in this problem "blows up" near -1 and near 1. Graphing calculators can be quite misleading near such *singularities*.

IV-65. Find a polynomial $P(x)$ with integer coefficients such that $P(\sqrt{6} - \sqrt{5}) = 0$.

Solution. Let $r = \sqrt{6} - \sqrt{5}$. Then $(r + \sqrt{5})^2 = 6$, so $r^2 + 2r\sqrt{5} + 5 = 6$. But then $2r\sqrt{5} = 1 - r^2$. Square both sides and simplify. We conclude that $r^4 - 22r^2 + 1 = 0$. Thus $\sqrt{6} - \sqrt{5}$ is a root of the polynomial $x^4 - 22x^2 + 1$. There are infinitely many other answers, but this one is the simplest.

Another way: Let $P(x)$ be the polynomial

$$(x - \sqrt{6} + \sqrt{5})(x + \sqrt{6} + \sqrt{5})(x - \sqrt{6} - \sqrt{5})(x + \sqrt{6} - \sqrt{5}).$$

Then $P(x)$ has our number as a root. We need to verify that $P(x)$ has integer coefficients.

The product of the first two terms is $x^2 + 2\sqrt{5}x - 1$, and the product of the last two is $x^2 - 2\sqrt{5}x - 1$. When we multiply these two quadratics, we get $x^4 - 22x^2 + 1$.

Comment. Recall that $\sqrt{2}$ and $-\sqrt{2}$ like to hang around together in formulas. In a similar way, the four numbers $\pm\sqrt{6} \pm \sqrt{5}$ and $\pm\sqrt{6} \mp \sqrt{5}$ are fellow travellers (the technical term is *conjugates*).

We sketch another proof of the fact that the polynomial $P(x)$ of the second solution does have integer coefficients. The idea exploits symmetries and has great conceptual importance in algebra.

Imagine multiplying the four factors of $P(x)$ together. The result is a polynomial that has first coefficient 1, and four other coefficients which we don't know because we haven't actually multiplied. Any coefficient must be of the shape

$$a + b\sqrt{6} + c\sqrt{5} + d\sqrt{6}\sqrt{5},$$

where a, b, c, and d are integers.

In the four roots, replace $\sqrt{5}$ everywhere by $-\sqrt{5}$. Then the coefficient $a + b\sqrt{6} + c\sqrt{5} + d\sqrt{6}\sqrt{5}$ becomes $a + b\sqrt{6} - c\sqrt{5} - d\sqrt{6}\sqrt{5}$. Since the replacement process only scrambles the roots, the coefficient is *unchanged.* We conclude that $c = 0$ and $d = 0$. Similarly, we can show that $b = 0$, so each coefficient of $P(x)$ is an integer!

IV-66. Find numbers a, b, and c such that

$$4x^2 - 5x + 1 = a + b(x-2) + c(x-2)^2 \qquad \text{for all } x.$$

Solution. We can hunt and peck our way to an answer. Put $x = 2$. Then the left-hand side is 7 while the right-hand side is a, so $a = 7$. The coefficient of x^2 on the left is 4, and on the right it is c, so $c = 4$. To find b, let $x = 0$. We get $1 = a - 2b + 4c$, so $b = 11$.

Another way: Let $f(x) = 4x^2 - 5x + 1$. Divide $f(x)$ by $x - 2$. It turns out that $f(x) = (4x + 3)(x - 2) + 7$. Divide $4x + 3$ by $x - 2$. We get $4x + 3 = 4(x - 2) + 11$. Thus

$$f(x) = (4(x-2) + 11)(x-2) + 7 = 4(x-2)^2 + 11(x-2) + 7.$$

The same idea works for any polynomial.

Another way: There is an efficient method that uses a small amount of calculus. More importantly, the method generalizes to functions other than polynomials, and enables us to express many of the functions that are useful in applications as *power series*.

Let $f(x) = 4x^2 - 5x + 1 = a + b(x-2) + c(x-2)^2$. Then $f(2) = 7 = a$, so $a = 7$. Also, $f'(x) = 8x - 5 = b + 2c(x-2)$. So $f'(2) = 11 = b$. Finally, $f''(x) = 8 = 2c$, so $c = 4$.

IV-67. Find a quadratic polynomial whose roots are the cubes of the roots of the polynomial $x^2 + bx + c$. Try to do it without computing the roots.

Solution. Let u and v be the roots of $x^2 + bx + c$. Then $u + v = -b$ and $uv = c$.

Let $x^2 + dx + e$ be a quadratic whose roots are u^3 and v^3. Then the product of u^3 and v^3 is e. Since $uv = c$, we have $u^3v^3 = c^3$, and therefore $e = c^3$.

To find d, note that $d = -(u^3+v^3) = -(u+v)(u^2-uv+v^2)$. But $u^2-uv+v^2 = (u+v)^2 - 3uv$, so $d = b(b^2 - 3c)$.

IV-68. Suppose that $abc \neq 0$. Solve the equation

$$\frac{a}{x-b-c} + \frac{b}{x-a-c} + \frac{c}{x-a-b} = 3.$$

Solution. We could "cross-multiply" but it seems a shame to uglify so soon. To make things look nicer, let $x = y + a + b + c$. The equation becomes

$$\frac{a}{y+a} + \frac{b}{y+b} + \frac{c}{y+c} = 3.$$

Rewrite this as

$$\frac{a}{y+a} - 1 + \frac{b}{y+b} - 1 + \frac{c}{y+c} - 1 = 0,$$

or equivalently

$$\frac{y}{y+a} + \frac{y}{y+b} + \frac{y}{y+c} = 0.$$

Since a, b, and c are non-zero, the equation has the obvious solution $y = 0$.

To find the other roots we need to solve

$$\frac{1}{y+a} + \frac{1}{y+b} + \frac{1}{y+c} = 0.$$

If we multiply through by $(y+a)(y+b)(y+c)$, we get the quadratic equation

$$3y^2 + 2(a+b+c)y + bc + ac + ab = 0,$$

whose roots are

$$y = \frac{-(a+b+c) \pm \sqrt{a^2+b^2+c^2-bc-ac-ab}}{3}.$$

Finally, add $a + b + c$ to get x.

IV-69. It is easy to verify that

$$(a+b)^3 = a^3 + b^3 + 3ab(a+b).$$

(a) Find numbers a and b such that $a^3 + b^3 = 1$ and $3ab = -3$.

(b) Use the identity given above to find an expression for the real solution of $x^3 + 3x - 1 = 0$.

Solution. (a) When we substitute $-1/a$ for b in $a^3 + b^3 = 1$ and simplify, we get $a^6 - a^3 - 1 = 0$. This is a quadratic equation in a^3. One solution is $a^3 = (1+\sqrt{5})/2$. For brevity, call this number τ. Then one solution of our system of equations is $a = \tau^{1/3}$, $b = -\tau^{-1/3}$.

(b) From the identity given above, we see that $x = a+b$ is a solution of $x^3 - 3abx - a^3 - b^3 = 0$. So one solution of the given equation is $\tau^{1/3} - \tau^{-1/3}$, and it is real. To see that this is the only real solution, note that the functions x^3 and x are both steadily increasing, and therefore so is $x^3 + 3x - 1$, so the graph of $y = x^3 + 3x - 1$ can only cross the x-axis once.

Comments. 1. The following identity, like most identities, is easy to check:

$$x^3 - 3abx - a^3 - b^3 = (x - a - b)(x^2 + ax + bx + a^2 + b^2 - ab).$$

Once we have found the real root $a + b$, the other roots can be found by solving a quadratic equation.

2. In the sixteenth century, through the work of a series of interesting Italian Renaissance figures, among them del Ferro, Tartaglia, and Cardano, a formula was found for the roots of the general cubic. By a suitable change of variable, the general cubic equation can be reduced to an equation of type $y^3 + py + q = 0$, which can then be solved with the method we used for $x^3 + 3x - 1 = 0$. See also V-37.

IV-70. Solve the equation $\sqrt{x+1} + \sqrt{x-1} = 3$.

Solution. Rewrite the equation as $\sqrt{x-1} = 3 - \sqrt{x+1}$, square both sides, and simplify. We get $6\sqrt{x+1} = 11$. Square both sides and extract x. We get $x = 85/36$. Maybe we should check that 85/36 satisfies the equation. But we don't need to. The function $\sqrt{x+1} + \sqrt{x-1}$ is 1 at 1 and is large when x is large, so it takes on all values greater than or equal to 1, in particular 3. Since there is a solution, and our manipulation has only produced one candidate, it must work.

Another way: Multiply both sides of the equation by $\sqrt{x+1} - \sqrt{x-1}$ and simplify. We get $\sqrt{x+1} - \sqrt{x-1} = 2/3$. "Add" the original equation. We get $2\sqrt{x+1} = 11/3$, so $x = 85/36$.

IV-71. Find all x such that $\dfrac{x^2 - 5x + a}{a - 4} > x$. Consider all possible values of a.

Solution. The problem divides naturally into two cases: (i) $a > 4$ and (ii) $a < 4$. We deal first with case (i). Then $a-4$ is positive, and so our inequality is equivalent to $x^2 - 5x + a > x(a - 4)$, that is, to $x^2 - (1 + a)x + a > 0$. This inequality can be rewritten as $(x - 1)(x - a) > 0$. Since $a > 1$, we conclude that if $a > 4$, then the inequality holds when $x > a$ and also when $x < 1$.

Now we deal with case (ii). Since $a - 4$ is negative, the inequality is equivalent to $x^2 - 5x + a < x(a - 4)$, that is, to $(x - 1)(x - a) < 0$. The inequality cannot hold if $a = 1$. If $a > 1$, the inequality holds when $1 < x < a$, while if $a < 1$, the inequality holds if $a < x < 1$.

IV-72. Suppose that the polynomial $ax^4 + bx^3 - x^2 + bx + a$ is divisible by $x^2 - x - 2$. Find the possible values of a and b.

Solution. Imagine dividing the polynomial by x^2-x-2. Let $Q(x)$ be the quotient, and $dx + e$ the remainder. Then

$$ax^4 + bx^3 - x^2 + bx + a = (x^2 - x - 2)Q(x) + dx + e.$$

We want to find conditions on a and b which are equivalent to $d = e = 0$.

Note that $x^2 - x - 2 = 0$ when $x = -1$ and when $x = 2$. Substituting these values of x into the equation above, we get $a - b - 1 - b + a = -d + e$ and $16a + 8b - 4 + 2b + a = 2d + e$. Thus $d = e = 0$ if and only if $2a - 2b = 1$ and $17a + 10b = 4$. Now solve for a and b. We get $a = 1/3$ and $b = -1/6$.

Another way: Use ordinary polynomial division. The calculation is straightforward. After a while, we find that the quotient is $ax^2+(a+b)x+3a+b-1$ and the remainder is $(5a+4b-1)x+7a+2b-2$. This remainder is identically 0 precisely if $5a+4b-1 = 0$ and $7a + 2b - 2 = 0$. Now solve for a and b.

IV-73. Let $M = (1/2, 1/2)$. Find points A and B on the curve $y = 1/x^2$ such that M is the midpoint of the segment AB.

Solution. Make an informal sketch of the curve. A little play shows that there are such points A and B, and that they must lie on opposite sides of the y-axis.

Let the x-coordinates of A and B be u and v. The y-coordinates are then $1/u^2$ and $1/v^2$. From the usual expression for the point midway between two given points, we get $(u + v)/2 = 1/2$ and $(1/u^2 + 1/v^2)/2 = 1/2$. So we want

$$u + v = 1 \qquad \text{and} \qquad \frac{1}{u^2} + \frac{1}{v^2} = 1.$$

The system takes some effort to solve. One way is to rewrite the second equation as $u^2+v^2 = (uv)^2$. By squaring both sides of $u+v = 1$, we find that $u^2+v^2 = 1-2uv$. Thus $(uv)^2 = 1 - 2uv$, and by the quadratic formula $uv = -1 \pm \sqrt{2}$.

Things are now routine. From the identity $(u - v)^2 = (u + v)^2 - 4uv$, we conclude that $(u - v)^2 = 5 + 4\sqrt{2}$ (one answer was rejected because $(u - v)^2$ can't be negative).

So $u - v = \pm\sqrt{5 + 4\sqrt{2}}$. Now that we know $u + v$ and $u - v$, u and v are easy to find. Since there is a solution, by symmetry there are 2 possible values of u, so both our solutions must be valid.

Another way: We can preserve symmetry yet get down to one variable by letting $u = 1/2 - t$ and $v = 1/2 + t$. The equation $1/u^2 + 1/v^2 = 1$ becomes

$$\frac{1}{(1/2 - t)^2} + \frac{1}{(1/2 + t)^2} = \frac{1/2 + 2t^2}{(1/4 - t^2)^2} = 1.$$

Simplify. We obtain a quadratic equation in t^2, and now we can find t in the usual way.

Comment. The same type of problem, for other functions, produces other challenging systems of equations.

IV-74. For what values of c does the equation $|u - 3| + |u - 333| = c$ have (i) no solutions; (ii) two solutions; (iii) more than two solutions?

Solution. The first step we take is unnecessary, but it feels like a good idea. Note that 168 is midway between 3 and 333. Let $v = u - 168$. We are interested in $|v + 165| + |v - 165|$. Since 165 isn't an attractive number, maybe we should let $v = 165x$ and $c = 165d$. We want to study the number of solutions of $|x+1|+|x-1| = d$.

When $x < -1$, both $x + 1$ and $x - 1$ are negative, so the sum of their absolute values is $-2x$. When $-1 \le x \le 1$, the absolute value of $x + 1$ is $x + 1$, while the absolute value of $x - 1$ is $-(x - 1)$, so the sum of the absolute values is 2. And when $x > 1$, the sum of the absolute values is $2x$. The graph of the curve $y = |x + 1| + |x - 1|$ is now easy to draw. The original equation has no solutions if $c < 330$, exactly two solutions if $c > 330$, and infinitely many solutions when $c = 330$.

Another way: Two people have jobs on the x-axis, one at $x = 3$ and the other at $x = 333$. They build a house at $(u, 0)$. Let $d(u)$ be the sum of their distances from work. Then $d(u) = |u - 3| + |u - 333|$.

If the house is built between 3 and 333, then the sum of their commuting distances is 330. If it is built distance t to the right of 333 or distance t to the left of 3, then the sum of their commuting distances is $330 + 2t$.

So the sum can't be below 330. For any number $c > 330$, the sum can be equal to c in two ways, and it can be exactly 330 in infinitely many ways.

Comment. A graphing calculator can be used to graph $|u - 3| + |u - 333|$, but one may have to experiment in order to find the appropriate viewing window. The reduction to $|x + 1| + |x - 1|$ solves the window problem.

IV-75. Find numbers A, B, and C such that

$$\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + 1}$$

for all x where the functions are defined.

Solution. Bring the sum on the right to the common denominator $(x-1)(x^2+1)$. The numerator is then $A(x^2+1)+(Bx+C)(x-1)$. Compare with the expression on the left. We need to have

$$x = A(x^2+1)+(Bx+C)(x-1),$$

at least for all $x \neq 1$. But if two polynomials are equal at all but one point (or even at infinitely many points) they are equal everywhere, so the equation above has to hold identically. Thus the polynomial x is the same polynomial as

$$(A+B)x^2+(-B+C)x+A-C.$$

We conclude that $A+B=0$, $-B+C=1$, and $A-C=0$. Solve. We get $A=1/2$, $B=-1/2$, and $C=1/2$.

Comment. Problems that look somewhat like this one, but are far more complicated, come up naturally in applications, for example in the design of electrical circuits. Calculations can be speeded up in various ways. For example, start with $x = A(x^2+1)+(Bx+C)(x-1)$, and put $x=1$. We find immediately that $A=1/2$.

IV-76. Solve the equation

$$\frac{x+1}{x+3}+\frac{x+3}{x+5}=\frac{x+5}{x+7}+\frac{x+7}{x+9}.$$

Solution. We can cross-multiply and simplify, hoping that things won't get too ugly. In fact they don't, but it's worthwhile to play a bit first. Note that

$$\frac{x+1}{x+3}=1-\frac{2}{x+3} \qquad \text{and} \qquad \frac{x+3}{x+5}=1-\frac{2}{x+5},$$

and so on like that for the others. So the original equation reduces to

$$\frac{1}{x+3}+\frac{1}{x+5}=\frac{1}{x+7}+\frac{1}{x+9}.$$

Now we can compute, but maybe to bring out the symmetry let $x+6=y$. After some manipulation we obtain

$$\frac{1}{y-3}-\frac{1}{y+3}=\frac{1}{y+1}-\frac{1}{y-1}.$$

Bring each side to a common denominator and simplify. We get $y^2=3$, so $x = -6\pm\sqrt{3}$.

IV-77. Let p, q, and r be the roots of $x^3-3x+1=0$. Find a cubic polynomial whose roots are p^2, q^2, and r^2.

Solution. We could find explicit expressions for p, q, and r by looking at the solution of IV-69. In sixteenth-century Italy, explicit though complicated formulas were found for the roots of cubic equations. But we don't need these formulas to solve the problem.

Note that $p + q + r = 0$, $pq + qr + rp = -3$, and $pqr = -1$. We are looking for a cubic $x^3 - ax^2 + bx - c$, where $a = p^2 + q^2 + r^2$, $b = p^2q^2 + q^2r^2 + r^2p^2$, and $c = p^2q^2r^2$. Immediately we find that $c = 1$.

Since $p^2 + q^2 + r^2 = (p+q+r)^2 - 2(pq+qr+rp) = 6$, we conclude that $a = 6$. It remains to find b. Square $pq + qr + rp$. We get

$$9 = (pq + qr + rp)^2 = p^2q^2 + q^2r^2 + r^2p^2 + 2pqr(p + q + r).$$

Thus $p^2q^2 + q^2r^2 + r^2p^2 = 9$, and $x^3 - 6x^2 + 9x - 1$ is a cubic polynomial with roots p^2, q^2, and r^2.

IV-78. Solve the equation $\sqrt[3]{x} + \sqrt[3]{95 - x} = 5$.

Solution. Usually we try to eliminate variables. Here we make the equation more symmetric by using two variables. Let $u^3 = x$ and $v^3 = 95 - x$. Our problem reduces to solving the system $u + v = 5$, $u^3 + v^3 = 95$. Much more attractive!

Using the identity $(u^3 + v^3)/(u + v) = u^2 - uv + v^2$, we obtain the equivalent system $u+v = 5$, $u^2 - uv + v^2 = 19$. But $u^2 + 2uv + v^2 = 25$, and therefore $3uv = 6$. But $(u - v)^2 = (u + v)^2 - 4uv = 17$, so $u - v = \pm\sqrt{17}$. We now know $u + v$ and $u - v$, and conclude that $u = (5 \pm \sqrt{17})/2$ and $v = (5 \mp \sqrt{17})/2$.

Should we check that these are indeed solutions? Maybe it's a good idea, for at one point we squared, and $u^2 + 2uv + v^2 = 25$ is *not* equivalent to $u + v = 5$. But in fact we don't need to check.

By the symmetry between u and v we know that if $u = a$, $v = b$ is a solution then so is $u = b$, $v = a$. And it is easy to see that the original equation has solutions, for when $x = 0$ the left side is $\sqrt[3]{95}$, which is less than 5, while at $x = 47.5$ the left side is greater than 5. So we know there are two solutions, we have found two solutions, and therefore the solutions we found are not "extraneous."

IV-79. Solve the system $xy + z = xz + y = yz + x = 3$.

Solution. Rewrite the first equation as

$$xy - xz = y - z, \qquad \text{or equivalently} \qquad (x - 1)(y - z) = 0.$$

By symmetry, we also need $(z - 1)(x - y) = 0$ and $(y - 1)(x - z) = 0$.

Not all of x, y, and z can be 1. So assume that $x \neq 1$. Then $y = z$. There are then two possibilities: (i) $x = y$, in which case $x^2 + x = 3$, that is, $x = y = z = (-1 \pm \sqrt{13})/2$, or (ii) $x \neq y$, in which case $y = z = 1$ and $x = 2$. By symmetry we also have the solutions $x = 1$, $y = 1$, $z = 2$ and $x = 1$, $y = 2$, $z = 1$.

Another way: Let $x = u + 1$, $y = v + 1$, and $z = w + 1$. Substitute and simplify. We get $uv = vw = wu = 1 - u - v - w$. Better!

If none of the variables is 0, then by cancellation $u = v = w$ and therefore $u^2 = 1 - 3u$, so $u = (-3 \pm \sqrt{1})/2$. If one of the variables is 0 let it be say u. Then $vw = 1 - v - w = 0$, so one of v or w is 0, say v. But then $w = 1$.

IV-80. Find all k such that $x^2 + kx + 1$ and $x^2 + x + k^2$ have at least one root in common.

Solution. The number u is a root of both polynomials if and only if u is a root of the first one, and is also a root of the difference of the two polynomials, that is, iff $u^2 + ku + 1 = 0$ and $(k-1)u = k^2 - 1$.

The case $k = 1$ is uninteresting, for then the equations are the same. So suppose $k \neq 1$. Then we can cancel $k - 1$ from both sides of $(k-1)u = k^2 - 1$, and conclude that $u = k + 1$.

So $k + 1$ should be a root of $x^2 + kx + 1$. Substitute $k + 1$ for x, simplify, and set the result equal to 0. We get $2k^2 + 3k + 2 = 0$, which has solutions $k = -2$ and $k = -1/2$. The k that work are therefore -2, $-1/2$, and 1.

IV-81. Let $i = \sqrt{-1}$. Find all real numbers x and y such that

$$(x + iy)^2 = 2 + 6i.$$

Solution. We are trying to find all (complex) square roots of $2 + 6i$. For $x + iy$ to be such a square root, we need

$$(x + iy)^2 = x^2 + 2xyi + y^2 i^2 = x^2 - y^2 + 2xyi = 2 + 6i.$$

So we are looking for the intersection points of the hyperbolas

$$x^2 - y^2 = 2 \qquad \text{and} \qquad 2xy = 6.$$

Substitute $3/x$ for y in the first equation. We arrive at $x^4 - 2x^2 - 9 = 0$. By the quadratic formula, $x^2 = 1 + \sqrt{10}$. Thus $x = \pm\sqrt{1 + \sqrt{10}}$ and $y = \pm 3/\sqrt{1 + \sqrt{10}}$.

Comment. The above solution is the natural one, unless we know better. One should probably suggest looking for a solution of the shape

$$x + iy = r(\cos\theta + i\sin\theta).$$

Square both sides and use double angle identities. We obtain

$$r^2 \cos 2\theta = 2 \qquad \text{and} \qquad r^2 \sin 2\theta = 6.$$

Using these equations, we get

$$(r^2 \cos 2\theta)^2 + (r^2 \sin 2\theta)^2 = r^4 = 40,$$

so $r^2 = \sqrt{40}$, $\cos 2\theta = 2/\sqrt{40}$, and $\sin 2\theta = 6/\sqrt{40}$. Now with a calculator we can find θ, and then $\cos\theta$ and $\sin\theta$. But we don't need a calculator. Once we know

$\cos 2\theta$ and $\sin 2\theta$ we can find $\cos\theta$ by using $\cos 2\theta = 1 - 2\cos^2\theta$, and then $\sin\theta$ from $\sin 2\theta = 2\sin\theta\cos\theta$.

The approach through trigonometric functions is a special case of the important *de Moivre Formula*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Chapter 5

Trigonometric Functions

Introduction

Most of the problems use familiar identities such as $\cos^2 x + \sin^2 x = 1$ (or equivalently $1 + \tan^2 \theta = \sec^2 \theta$) and the addition formulas

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad \text{and}$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y,$$

with special cases $\sin 2x = 2\sin x \cos x$ and $\cos 2x = \cos^2 x - \sin^2 x$ (and variants). The less familiar addition formula for the tangent function, namely $\tan(x+y) = (\tan x + \tan y)/(1 - \tan x \tan y)$, is occasionally useful.

Problem V-37 deals with Viète's method for solving certain cubic equations by expressing $\cos 3\theta$ as a polynomial in $\cos \theta$. (The cubic is also solved in IV-69.) Problem V-38 briefly introduces the hyperbolic functions and could serve as the beginning of an exploration.

Problems and Solutions

V-1. In Figure 5.1, $BD = 2$, $DC = 3$, D lies on line BC, $AD = 1$, and AD bisects $\angle BAC$. Find $\angle BAC$ to the nearest tenth of a degree.

Solution. View $\triangle ABD$ and $\triangle ADC$ as triangles with the bases AD and DB. Since the bases are in the ratio $2:3$, so are the areas. Let $\theta = \angle BAD = \angle DAB$. Then the areas of $\triangle ABD$ and $\triangle ADC$ are $(1/2)(AB)(AD)\sin\theta$ and $(1/2)(AC)(AD)\sin\theta$, so AB and AD are in the ratio $2:3$.

Let $AB = 2t$ and $AD = 3t$, and let $\angle ADB = \theta$. Note that $\angle ADC = 180° - \theta$, and $\cos(180° - \theta = -\cos\theta$. By the Cosine Law

$$4t^2 = 5 - 4\cos\theta \qquad \text{and} \qquad 9t^2 = 13 + 12\cos\theta.$$

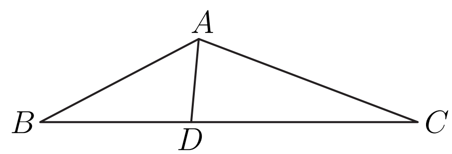

Figure 5.1: Finding $\angle BAC$

We conclude that $t^2 = 4/3$. Let $\phi = \angle BAC$. Again by the Cosine Law

$$25 = \frac{16}{3} + 12 - 16\cos\phi,$$

so $\cos\phi = -23/48$, and therefore θ is about 118.6 degrees.

V-2. Solve the equation $\sin^{-1} x + \sin^{-1} 2x = \pi/2$.

Solution. Recall that $\sin^{-1} t$ is the angle between $-\pi/2$ and $\pi/2$ whose sine is t. Let $y = \sin^{-1} x$ and $z = \sin^{-1} 2x$. Then

$$\sin(y+z) = 1, \qquad \text{so} \qquad \sin y \cos z + \cos y \sin z = 1.$$

We have $\sin y = x$ and $\sin z = 2x$. But $\cos y = \sqrt{1-x^2}$ and $\cos z = \sqrt{1-4x^2}$, since both angles are between $-\pi/2$ and $\pi/2$, and therefore

$$x\sqrt{1-4x^2} + 2x\sqrt{1-x^2} = 1.$$

Rewrite this as $2x\sqrt{1-x^2} = 1 - x\sqrt{1-4x^2}$, square both sides, and simplify. We get $3x^2 - 1 = -2x\sqrt{1-4x^2}$. Square both sides again. We get $25x^4 - 10x^2 + 1 = 0$, which has the roots $1/\sqrt{5}$ and $-1/\sqrt{5}$. The second root obviously doesn't work. To see that the first does, we only need to show that the original equation *has* a solution. This is clear, for the function $\sin^{-1} x + \sin^{-1} 2x$ grows from 0 to beyond $\pi/2$ as x grows from 0 to 1/2.

V-3. Suppose that $\cos x + 4\sin x$ is as large as possible. Find $\cos x$. Hint: Divide by $\sqrt{1^2+4^2}$ and use a trigonometric identity.

Solution. Let $f(x)$ be the function we wish to maximize. Then

$$f(x) = \sqrt{17}\left(\frac{1}{\sqrt{17}}\cos x + \frac{4}{\sqrt{17}}\sin x\right).$$

Let α be the angle between 0 and $\pi/2$ whose sine is $1/\sqrt{17}$. By the addition law for the sine function,

$$\sqrt{17}f(x) = \sin(\alpha + x).$$

Thus $f(x)$ reaches a maximum whenever $\alpha + x = \pi/2 + 2n\pi$ for some integer n. But then $\cos x = \cos(\pi/2 - \alpha) = \sin\alpha = 1/\sqrt{17}$.

V-4. Suppose that $\cos\theta = (1-t^2)/(1+t^2)$, where $0 \le \theta < \pi$. Express $\tan(\theta/2)$ in terms of t.

Solution. Since $\cos\theta = 2\cos^2(\theta/2) - 1$,

$$2\cos^2(\theta/2) = \frac{1-t^2}{1+t^2} + 1 = \frac{2}{1+t^2},$$

and therefore $\sec^2(\theta/2) = 1 + t^2$. Now use the identity $1 + \tan^2 u = \sec^2 u$ to conclude that $\tan^2(\theta/2) = t^2$. Since $\tan(\theta/2) \ge 0$, it follows that $\tan(\theta/2) = |t|$.
Another way: The identity $1 + \tan^2\theta = \sec^2\theta$ can be bypassed. Note that

$$\sin^2\theta = 1 - \frac{(1-t^2)^2}{(1+t^2)^2}.$$

The right-hand side simplifies to $4t^2/(1+t^2)^2$, so $\sin\theta = 2|t|/(1+t^2)$. Now use the fact that $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$ and $\cos\theta = 2\cos^2(\theta/2) - 1$ to conclude that $\tan(\theta/2) = |t|$.

Comment. Note that

$$\frac{1-t^2}{1+t^2} = \frac{2}{1+t^2} - 1.$$

As t travels upward from 0, $2/(1+t^2)$ travels from 2 toward 0. So the function $(1-t^2)/(1+t^2)$ takes on each value in $(-1, 1]$ exactly once, and therefore the cosine of any angle in $[0, \pi)$ can be expressed as $(1-t^2)/(1+t^2)$. The angle π doesn't quite make it, unless we allow $t = \infty$. See 3 for a problem on the same theme.

V-5. The usual identities for $\cos(x+y)$ and $\cos(x-y)$ yield

$$\cos x \cos y = \frac{\cos(x+y) + \cos(x-y)}{2}.$$

Your calculator is broken: it can only add, subtract, divide by 2, and find $\cos u$ and $\cos^{-1} v$ for any u and v. Let $a = 0.87654321$ and $b = 0.31415926$. Use the broken calculator and the identity above to calculate ab with good accuracy.

Solution. First find x and y such that $\cos x = a$ and $\cos y = b$. It doesn't matter whether we work in radians or degrees; for the same of tradition use degrees. Using the $\cos^{-1}$ button, we find that we can take $x = 28.77185651$ and $y = 71.68993344$, and therefore $x+y = 100.46179$ and $x-y = -42.91807693$. Now calculate $\cos(x+y)$, $\cos(x-y)$, add, and divide the result by 2. We conclude that $ab = 0.275374165$.

Comment. In the sixteenth century, astronomers sometimes used a variant of this method to multiply! The procedure was called *prosthaphaeresis.* In astronomical calculations, it was often necessary to find (the equivalent of) $\sin x \sin y$. High accuracy tables of the sine function were a standard tool of the trade, so the procedure described above was considered practical.

In the early seventeenth century, Napier and Bürgi discovered logarithms, and good tables were soon available. These "log tables" remained a standard tool for multiplying until the 1960's.

There is in fact a deep structural connection between the sine and cosine functions and the exponential function—see for example V-38. And the identity that was used in the computation has the same feel as $e^x e^y = e^{x+y}$, if you are willing to overlook small differences of detail.

V-6. The triangle PQR of Figure 5.2 has $PQ = PR = a$ and $QR = 2b$. The bisectors of $\angle PQR$ and $\angle PRQ$ meet at O. Express the area of $\triangle OQR$

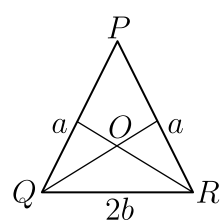

Figure 5.2: The Area of $\triangle OQR$

in terms of a and b.

Solution. There are a number of geometric approaches, but in keeping with the theme of this chapter the solution uses trigonometric identities. Let M be the midpoint of QR, and let $\angle PQR = 2\theta$. Then $OM = b\tan\theta$, and therefore the area of $\triangle OQR$ is $b^2 \tan\theta$.

It remains to express $\tan\theta$ in terms of a and b. Since

$$\frac{b}{a} = \cos 2\theta = 2\cos^2\theta - 1,$$

it follows that $\sec^2\theta = 2a/(a+b)$, and therefore $\tan^2\theta = (a-b)/(a+b)$. So the required area is $b^2\sqrt{(a-b)/(a+b)}$.

V-7. Given that $\sin x \neq 0$, find a simple expression for

$$(\cos x)(\cos 2x)(\cos 4x)\cdots(\cos 256x)(\cos 512x).$$

Hint: Multiply by $\sin x$.

Solution. Let our product be $P(x)$, and calculate $(\sin x)P(x)$. Note that $\sin x \cos x = (1/2)\sin 2x$. But $\sin 2x \cos 2x = (1/2)\sin 4x$, and therefore $\sin x \cos x \cos 2x = (1/4)\sin 4x$. Continue in this way. After a few steps we find that

$$(\sin x)P(x) = \frac{\sin 1024x}{1024} \quad \text{and therefore} \quad P(x) = \frac{\sin 1024x}{1024 \sin x}.$$

Comment. Multiplication by $\sin x$ acted somewhat like a catalyst does in a chemical reaction, for it helped the other terms to combine with each other.

V-8. The sides of a triangle are in arithmetic progression and one of the angles is 120°. Find the cosine of the smallest angle of the triangle.

Solution. The sides can be taken to be $a - d$, a, and $a + d$. By scaling if necessary we can assume that $a = 1$. The largest side is opposite the 120° angle. By the Cosine Law

$$(1 + d)^2 = (1 - d)^2 + 1^2 + (1)(1 - d).$$

When we simplify, we find to our pleasure that the d^2 terms cancel and $d = 2/5$. So the sides can be taken to be 3/5, 1, and 7/5, or more simply 3, 5, and 7.

The smallest angle θ is opposite the smallest side. By the Cosine Law $9 = 25 + 49 - 70 \cos\theta$ and therefore $\cos\theta = 13/14$.

V-9. Find the smallest positive x such that $\cos 3x + \sin 2x = 0$, preferably in several ways.

Solution. We can let microprocessors do the work, either by reading the answer off a graph or by using the *Solve* button on a calculator.

Another way: We can bring out the machinery of trigonometric identities. The identity $\cos 3x = \cos 2x \cos x - \sin 2x \sin x$, together with the usual double angle identities, yields after a while $\cos 3x = 4\cos^3 x - 3\cos x$.

The original equation can thus be rewritten as $4\cos^3 x - 3\cos x + 2\cos x \sin x = 0$. The common factor $\cos x$ produces the obvious solution $x = \pi/2$.

Now look for solutions of $4\cos^2 x - 3 + 2\sin x = 0$. This can be rewritten as $4\sin^2 x - 2\sin x - 1 = 0$. The solutions are $\sin x = (1 \pm \sqrt{5})/4$. So we want the least positive solution of $\sin x = (1 + \sqrt{5})/4$.

Perhaps it is time to go to the calculator. We find that x is approximately 0.942477796 radians. Out of curiosity, we might go to degrees. To the limit of calculator accuracy, the result seems to be 54°. Interesting! Maybe we should try

Another way: We need to find where the two curves $y = \cos 3x$ and $y = -\sin 2x$ meet. Since radian measure is less familiar, we will work with degrees. A casual sketch shows that the two curves first meet at a point a little short of 60°.

Recall that $\cos(90° + u) = -\sin u$. So we want the first positive solution of $\cos 3x = \cos(90° + 2x)$. If we bear in mind that the answer is a little short of 60°, we can see that the angles $3x$ and $90° + 2x$ should be symmetrical about $180°$, and therefore

$$90° + 2x - 180° = 180° - 3x, \qquad \text{so} \qquad x = 54°.$$

Comment. The identity $\cos 3x = 4\cos^3 x - 3\cos x$ can be used to solve cubic equations, and is crucial in proving that the general angle cannot be trisected with straightedge and compass. See V-37.

The third approach works for $\cos ax + \sin bx = 0$, where a and b are any real numbers, and for other closely related equations.

V-10. Let $\theta = \pi/12$. Find a simple expression for

$$\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}.$$

Solution. Multiply top and bottom by $\cos\theta + \sin\theta$. The denominator becomes $\cos^2\theta - \sin^2\theta$, that is, $\cos 2\theta$, which in this case is $\sqrt{3}/2$. The numerator becomes $\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta$, which simplifies to $1 + \sin 2\theta$, that is, $3/2$. Now divide: the result is $\sqrt{3}$.

V-11. Find all solutions of the system

$$\sin x + \sin y = \sin(x + y)$$
$$\cos x + \cos y = \cos(x + y)$$

in the interval $-\pi < x, y \leq \pi$.

Solution. This is an exercise in trigonometric identities. Square both sides of each equation and add. By using the identity $\sin^2 u + \cos^2 u = 1$, and some manipulation we obtain

$$\cos x\cos y + \sin x\sin y = -\frac{1}{2},$$

that is, $\cos(x - y) = -1/2$.

Multiply both sides of the first given equation by $\sin x$, both sides of the second by $\cos x$, and add. Note that

$$\cos(x + y)\cos x + \sin(x + y)\sin x = \cos(x + y - x) = \cos y.$$

So we obtain $1 + \cos(x - y) = \cos y$, and conclude that $\cos y = 1/2$. By symmetry, $\cos x = 1/2$.

The only possibilities for x and y are therefore $\pm\pi/3$—nothing else can work, but that doesn't mean all four combinations will work. The solutions turn out to be $x = \pi/3$, $y = -\pi/3$ and $x = -\pi/3$, $y = \pi/3$.

V-12. Is there a number a such that

$$a(\sin^4 x + \cos^4 x) - (\sin^6 x + \cos^6 x)$$

is independent of x?

Solution. It seems unlikely that there is such a number. But if there is, then there is a constant b such that

$$a(\sin^4 x + \cos^4 x) = (\sin^6 x + \cos^6 x) + b$$

for all x. Put $x = 0$ in the above (hypothetical) equation. Then $a = 1 + b$. Put $x = \pi/4$. Then $a/2 = 1/4 + b$. We conclude that the only possible a is $3/2$. We must verify that this a actually works for *all* x. To get rid of fractions, multiply by 2. So we examine

$$3(\sin^4 x + \cos^4 x) - 2(\sin^6 x + \cos^6 x).$$

Rewrite the above expression as

$$(\sin^4 x)(3 - 2\sin^2 x) + (\cos^4 x)(3 - 2\cos^2 x),$$

and use the fact that $3-2\sin^2 x = 1+2(1-\sin^2 x) = 1+2\cos^2 x$, and $3-2\cos^2 x = 1 + 2\sin^2 x$ to obtain

$$\sin^4 x + (2\sin^2 x\cos^2 x)(\sin^2 x + \cos^2 x) + \cos^4 x.$$

But this is simply the square of $\sin^2 x + \cos^2 x$, so it is identically equal to 1.
Another way: Recall that in general $u^3 + v^3 = (u + v)(u^2 - uv + v^2)$. If we set $u = \sin^2 x$, $v = \cos^2 x$, we find that

$$\begin{aligned}\sin^6 x + \cos^6 x &= (\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x)\\ &= \sin^4 x - \sin^2 \cos^2 x + \cos^4 x.\end{aligned}$$

It follows that

$$3(\sin^4 x + \cos^4 x) - 2(\sin^6 x + \cos^6 x) = \sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x,$$

and this last expression is identically equal to 1.

V-13. Given that $\sec x + \tan x = 2$, find $\csc x + \cot x$. Generalize.

Solution. Suppose that $\sec x + \tan x = a$. Multiply both sides by $\sec x - \tan x$. Since $\sec^2 x - \tan^2 x = 1$, there is no risk that we are multiplying by 0. We have

$$\begin{aligned}1 &= a(\sec x - \tan x) \qquad \text{and}\\ a^2 &= a(\sec x + \tan x).\end{aligned}$$

Solve for $\sec x$ and $\tan x$:

$$\sec x = \frac{a^2+1}{2a} \qquad \text{and} \qquad \tan x = \frac{a^2-1}{2a}.$$

Since $\cot x = 1/\tan x$ and $\csc x = \sec x/\tan x$, we conclude that

$$\csc x + \cot x = \frac{a^2+1}{a^2-1} + \frac{2a}{a^2-1} = \frac{a+1}{a-1}.$$

V-14. Find $(\sin 15^\circ + \sin 75^\circ)^6$ without using a calculator.

Solution. Note that $\sin 75^\circ = \cos 15^\circ$. But

$$(\sin 15^\circ + \cos 15^\circ)^2 = \sin^2 15^\circ + \cos^2 15^\circ + 2\sin 15^\circ \cos 15^\circ$$
$$= 1 + \sin 30^\circ = \frac{3}{2}.$$

and therefore $(\sin 15^\circ + \sin 75^\circ)^6 = 27/8$. (The instruction not to use a calculator shouldn't be taken too seriously—it's worth peeking, maybe to get insight into what the answer might be, or to check at the end that there is no mistake.)

Another way: Note that $15^\circ = 45^\circ - 30^\circ$ and $75^\circ = 45^\circ + 30^\circ$. By using the formulas for the sine of the difference and of the sum of two angles, we find that $\sin 15^\circ + \sin 75^\circ$ is equal to

$$(\sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ) + (\sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ).$$

This simplifies to $\sqrt{3}/\sqrt{2}$. Now calculate the sixth power.

V-15. Let A be the point with coordinates $(4, 3)$, and B the point obtained by rotating A counterclockwise about the origin through $\pi/6$ radians. Use trigonometric identities to find the coordinates of B.

Solution. Let α be the angle through which we must rotate the positive x-axis counterclockwise so that it will pass through A, and let $\beta = \alpha + \pi/6$. If we rotate the positive x-axis counterclockwise through an angle β, then the resulting half-line passes through B.

The line joining the origin to A has length $\sqrt{4^2+3^2}$, and therefore $\cos\alpha = 4/5$ and $\sin\alpha = 3/5$. Thus

$$\cos\beta = \cos(\alpha + \pi/6) = \cos\alpha\cos(\pi/6) - \sin\alpha\sin(\pi/6),$$

and therefore $\cos\beta = (4/5)(\sqrt{3}/2) - (3/5)(1/2)$. A similar calculation shows that $\sin\beta = (3/5)(\sqrt{3}/2) + (4/5)(1/2)$. Since the line segment joining the origin to B has length 5, it follows that B has x-coordinate $(4\sqrt{3} - 3)/2$ and y-coordinate $(3\sqrt{3} + 4)/2$.

Comment. It should be stressed that the sum formulas for cosine and sine are really all about combining rotations. The above problem is artificial, in that numbers work out "exactly," but calculations of the same type are the most common practical application of the trigonometric identities. For example, combining rotations efficiently is key to making computer animation look realistic. Computer hardware for games must be able to calculate sines and cosines very quickly.

V-16. Find an explicit formula for the solutions of the equation $\tan(\pi x) = \tan(\pi/x)$.

Solution. In general, $\tan u = \tan v$ precisely if u and v differ by an integer multiple of π. So our equation holds if and only if $\pi x - \pi/x = n\pi$ for some integer n. The equation simplifies to $x^2 - nx - 1 = 0$, which has the solutions $(n \pm \sqrt{1+4n^2})/2$.

V-17. Find the solutions of the equation $(\cos^2\theta \sin^2\theta)\, x^2 - x + 1 = 0$, where θ is a given number. Simplify as much as possible.

Solution. If $\sin\theta = 0$ or $\cos\theta = 0$, the only solution is $x = 1$. In all other cases, the quadratic formula gives

$$x = \frac{1 \pm \sqrt{1 - 4\cos^2\theta\sin^2\theta}}{2\cos^2\theta\sin^2\theta}.$$

Now we simplify as much as possible, whatever that means. Maybe we can use the fact that $2\sin\theta\cos\theta = \sin 2\theta$. If we do that, we obtain

$$\frac{1 \pm \sqrt{1 - \sin^2 2\theta}}{2\cos^2\theta\sin^2\theta},$$

which simplifies to

$$\frac{1 \pm |\cos 2\theta|}{2\cos^2\theta\sin^2\theta}.$$

In the above formula, $\pm|\cos 2\theta|$ can be replaced by $1 + \cos 2\theta$. Now use the relation $\cos 2\theta = 2\cos^2\theta - 1$. The numerators are $1 + (2\cos^2\theta - 1)$ and $1 - (1 - 2\sin^2\theta)$, that is, $2\cos^2\theta$ and $2\sin^2\theta$. There is cancellation, and the roots are $\csc^2\theta$ and $\sec^2\theta$.

Another way: When we simplified the discriminant, we took an unnecessary round trip through double angle identities, for

$$1 - 4\cos^2\theta\sin^2\theta = 1 - 4(1-\sin^2\theta)\sin^2\theta = (2\sin^2\theta - 1)^2.$$

Another way: The relation

$$(\cos^2\theta\sin^2\theta)\, x^2 - x + 1 = ((\cos^2\theta)\, x - 1)((\sin^2\theta)\, x - 1).$$

holds identically. To check this, just multiply out. The coefficient of x in the product is $-(\cos^2\theta + \sin^2\theta)$, and that is -1 as desired. The quadratic is now factored, so we can write down the roots immediately, in pretty simple form.

V-18. Given that $\cos 2\theta = 2\cos\theta$ and $0 < \theta < \pi$, find $\sin\theta$.

Solution. We really should sketch the curves $y = \cos 2x$ and $y = 2\cos x$ to find out what's going on. Even a rough picture shows that there is exactly one solution θ in the specified interval, and tells us approximately where it is.

But sometimes (often?) there is no penalty for pushing things through the formula sausage machine. Use the identity $\cos 2x = 2\cos^2 x - 1$. Our original equation becomes $2\cos^2\theta - 2\cos\theta - 1 = 0$. Solve this quadratic equation in $\cos\theta$. We get $\cos\theta = (1-\sqrt{3})/2$ (the other root is bigger than 1, so can't be the cosine of anything).

At this stage we could use a calculator to find θ, and then $\sin\theta$. But it is not difficult, and aesthetically more pleasing, to find an exact expression.

Note that $\sin^2\theta = 1-\cos^2\theta = 1-(1-\sqrt{3})^2/4$. This simplifies to $\sin^2\theta = \sqrt{3}/2$. But $\sin\theta$ is positive, since $0 < \theta < \pi$, and therefore $\sin\theta = \sqrt{\sqrt{3}/2}$.

Another way: We can compute $\sin\theta$ directly. Note that $\cos 2\theta = 1 - 2\sin^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$. If θ satisfies the original equation, then $\cos^2 2\theta = 4\cos^2\theta$, and therefore

$$(1 - 2\sin^2\theta)^2 = 4(1 - \sin^2\theta).$$

This simplifies nicely: after a short while we find that $\sin^4\theta = 3/4$, and since $\sin\theta$ is positive, conclude that $\sin\theta = \sqrt{\sqrt{3}/2}$.

Any anxiety we might feel about having introduced a spurious root through squaring—which can happen—is allayed by a sketch. The sketch shows there *is* a solution, and the calculations produced only one candidate, so it must work.

Comment. The first argument is probably the natural one, since we can *see* that a familiar identity yields a quadratic equation, so $\cos\theta$ is "known," and therefore $\sin\theta$ is. After we see this, it's all over except for minor details.

V-19. The familiar identity $\cos 2x = 2\cos^2 x - 1$ says that $\cos 2x = P(\cos x)$ where $P(t)$ is the polynomial $2t^2 - 1$. (a) Find a polynomial $Q(t)$ of degree 8 such that $\cos 8x = Q(\cos x)$. (b) How many *real* numbers t are there such that $Q(t) = 1/2$? List them all.

Solution. (a) We have

$$\cos 4x = 2\cos^2 2x - 1 = 2(2\cos^2 x - 1)^2 - 1.$$

But $\cos 8x = 2\cos^2 4x - 1$ and therefore

$$\cos 8x = (2(2\cos^2 x - 1)^2 - 1)^2 - 1.$$

We conclude that $Q(t) = (2(2t^2-1)^2 - 1)^2 - 1$. "Simplification" seems pointless. It may be closer to uglification.

(b) The equation $Q(t) = 1/2$ has at most 8 real roots, since $Q(t)$ has degree 8. We show it has exactly 8 roots, and identify them. Look for solutions of the form

$t = \cos x$. Then $Q(t) = \cos 8x$, so $\cos x$ is a solution if and only if $\cos 8x = 1/2$. But $\cos 8x = 1/2$ iff $8x$ is, in radians, of the form $2\pi k \pm \pi/3$.

We find 8 values of $\cos x$ that work. Let $x = (2\pi k + \pi/3)/8$ for $k = 0, 1, 2, \dots, 7$. That gives $x = \pi/24$, $7\pi/24$, $13\pi/24$, $19\pi/24$, $25\pi/24$, $31\pi/24$, $37\pi/24$, $43\pi/24$—degree notation is more pleasant!

By looking at the graph of $y = \cos x$, we can see that the cosines of these 8 angles are distinct. We have found 8 roots, but there are no more than 8, so we have found them all. We needn't even look at a graph, since we can use a calculator to compute $\cos x$ for the 8 values of x. But what comes out of the calculator is a mess of digits. The *expressions* for the roots are far more informative.

Comments. 1. The formula $Q(t) = (2(2t^2 - 1)^2 - 1)^2 - 1$ is more *practical* than the multiplied out form would be. To compute $Q(t)$ for some real t, takes, if we use the above expression, three rounds of squaring, multiplying by 2, and subtracting 1. If we needed to work out something similar for say $\cos 64x$, the number of rounds would climb to six, but that is *far* easier than multiplying out the polynomial and evaluating in the "normal" way. What is appropriate simplified form for one purpose may not be appropriate for another.

2. For any non-negative integer n there is a polynomial $T_n(t)$, of degree n such that $\cos nx = T_n(\cos x)$. The first five of these polynomials are 1, t, $2t^2 - 1$, $4t^3 - 3t$, and $8t^4 - 8t^2 + 1$. The $Q(t)$ that we calculated is just $T_8(t)$. These polynomials are called the *Chebyshev polynomials*, after the nineteenth-century Russian mathematician. So why the letter T? His name used to be transcribed from the Cyrillic to the Latin alphabet as, among other things, *Tchebychef.* The Chebyshev polynomials have important engineering applications, for example in loudspeaker design, the design of electrical filters, and the analysis of heat flow.

V-20. A certain angle θ, where $0 < \theta < \pi/2$, satisfies the equation $6^{\cos^2\theta} + 6^{\sin^2\theta} = 5$. Find θ, in radians, to 7 decimal places.

Solution. It is tempting to substitute $1 - \sin^2\theta$ for $\cos^2\theta$ to see whether anything interesting happens. It does. But it is more efficient to multiply both sides of the given equation by $6^{\sin^2\theta}$. Since $\sin^2\theta + \cos^2\theta = 1$, the given equation is equivalent to

$$6 + 6^{2\sin^2\theta} = 5 \cdot 6^{\sin^2\theta}.$$

Let $x = 6^{\sin^2\theta}$. Then $6 + x^2 = 5x$, and therefore $x = 2$ or $x = 3$.

The solution $x = 2$ gives $6^{\sin^2\theta} = 2$. Take logarithms, say to the base 10. We get $\sin\theta = \sqrt{\log 2/\log 6}$, since $\sin\theta$ is positive. The calculator says that θ is about 0.6712623 radians.

We could work with $x = 3$ in the same way. But note that $\sin\theta$ and $\cos\theta$ occur symmetrically in the problem, and therefore if θ is a solution, so is $\pi/2 - \theta$.

V-21. On an exam, a student needed to find the cosine of x degrees. So the student keyed in x and pressed the cos key. By mistake, the calculator had been left in radian mode, but as luck would have it, the calculator's answer turned out to be exactly right. Find the smallest possible x, given that $x > 50°$.

Solution. Since the calculator was in radian mode, it thought it was taking the cosine of x radians, that is, $180x/\pi$ degrees, and the cosine of this turned out to be exactly the same as the cosine of $x°$. It follows that $180x/\pi = 360n \pm x$ for some integer n.

Solve for x. We obtain $x = 360\pi n/(180 \mp \pi)$. So, approximately, $x = 6.175n$ or $x = 6.395n$. To make $x > 50$ but as small as possible, use the second expression for x and put $n = 8$. We conclude that x is about 51.16 degrees.

V-22. Find all possible values of $\sin x/\sin 3x$.

Solution. A graphing calculator is useful here. We can confine attention to the interval from $-\pi/2$ to $\pi/2$, and in fact, since $\sin(-u) = -\sin u$, we can restrict attention to the interval from 0 to $\pi/2$. In graphing, we need to stay somewhat away 0 and $\pi/3$, since the denominator is not defined at these points. In particular, our function blows up near $\pi/3$, so the calculator has to be used carefully. But if we do, we can see that the range of values seems to consist of the interval $(-\infty, -1]$, together with $[a, \infty)$, where a looks as if it is not far from 1/3.

Another way: Note that

$$\sin 3x = \sin x \cos 2x + \cos x \sin 2x = (\sin x)(1 - 2\sin^2 x) + (2\sin x)(\cos^2 x).$$

If we replace $\cos^2 x$ by $1 - \sin^2 x$ and simplify, we obtain $\sin 3x = 3\sin x - 4\sin^3 x$. Thus, except where the denominator is 0, $\sin x/\sin 3x = 1/(3 - 4\sin^2 x$.

The denominator takes on all values between -1 and 3. Thus, the range of $\sin x/\sin 3x$ consists of all numbers in the interval $(-\infty, -1]$ together with $[1/3, \infty)$.

V-23. Assume that the Earth is a perfect sphere of radius r. We travel around the Earth at latitude 50° North. How far do we travel?

Solution. Let P be a point at latitude 50° North, let N be the North Pole, and C the center of the Earth. Drop a perpendicular from P to the line CN, meeting CN say at Q. In going around the Earth at latitude 50°, we travel around a circle of radius PQ. The plane through N, P, and C meets the Equator at two points, a "near" one E and a diametrically opposite "far" one (see Figure 5.3). Latitude 50° North means that $\angle ECP = 50°$ and P is in the northern hemisphere. It follows that $\angle CPQ = 50°$. But $CP = r$ and $\angle CQP$ is a right angle, so $PQ = r\cos 50°$, and therefore the distance travelled is $2\pi r \cos 50°$.

V-24. Suppose that $\cos\theta + \sin\theta = 5/4$, and $0 < \theta < 90°$. Find the possible values of $\tan\theta$, preferably without using a calculator.

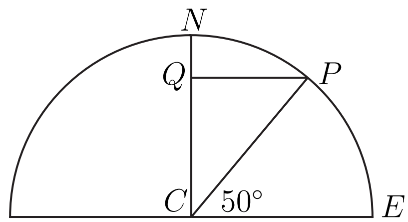

Figure 5.3: Travelling Around the Earth at 50° North

Solution. The expression $\cos\theta + \sin\theta$ becomes more attractive when squared. So square both sides. We obtain

$$\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta = 25/16.$$

Now use trigonometric identities to conclude that $1 + \sin 2\theta = 25/16$, and hence $\sin 2\theta = 9/16$. We can now use the calculator to find 2θ with good accuracy. There are two values, about 34.229° and $180^\circ - 34.229^\circ$. Using the calculator, compute θ, and then $\tan\theta$. The answers are x, where x is about 0.3079159, and $1/x$.

A calculator-free way is to note as above that $\sin 2\theta = 9/16$, and conclude that $\cos^2 2\theta = 1 - 81/256$. Thus we know $\cos 2\theta$. But $2\cos^2\theta = \cos 2\theta + 1$, and $\tan\theta$ is then the ratio of the known quantities $2\sin\theta\cos\theta$ and $2\cos^2\theta$.

Another way: It seems wasteful to go to trigonometric functions of 2θ, and then back to θ. By squaring both sides, we found that $1 + 2\cos\theta\sin\theta = 25/16$, and therefore $\sin\theta\cos\theta = 9/32$. Thus

$$\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{9/32}.$$

But $(\sin^2\theta + \cos^2\theta)/\sin\theta\cos\theta = \tan\theta + 1/\tan\theta$. We arrive at the equation $\tan^2\theta - (32/9)\tan\theta + 1 = 0$. Solve: $\tan\theta = (16 \pm 5\sqrt{7})/9$.

Comment. The original equation is symmetrical in $\sin\theta$ and $\cos\theta$. Equivalently, since $\sin(\pi/2 - \theta) = \cos\theta$ and $\cos(\pi/2 - \theta) = \sin\theta$, the equation has symmetry about $\theta = \pi/4$. Since $\tan(\pi/2 - \theta) = 1/\tan\theta$, if $\tan\theta = a$ is a solution then so is $\tan\theta = 1/a$.

V-25. Show that

$$\sin(x + y)\sin(x - y) = (\sin x + \sin y)(\sin x - \sin y).$$

Solution. We play with the standard identities

$$\sin(x + y) = \sin x\cos y + \cos x\sin y \quad \text{and} \quad \sin(x - y) = \sin x\cos y - \cos x\sin y.$$

Multiply. The middle terms cancel, and we get $\sin^2 x\cos^2 y - \cos^2 x\sin^2 y$. Finally, replace $\cos^2 u$ by $1 - \sin^2 u$ in the preceding expression, and simplify. There is some cancellation, and we get $\sin^2 x - \sin^2 y$.

V-26. Let ABC be a right-angled isosceles triangle, with the right angle at C, and let M be the midpoint of BC. Find the cotangents of $\angle MAC$ and $\angle MAB$.

Solution. Let the equal legs of $\triangle ABC$ have length 2. The cotangent of $\angle MAC$ can be read off from Figure 5.4: it is 2. Now compute the cotangent of $\angle MAB$.

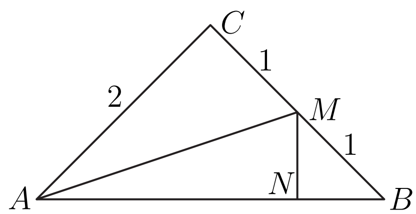

Figure 5.4: Cotangents of Two Angles

One way is to obtain a general formula for $\cot(x-y)$. Recall that $\cos(x-y) = \cos x \cos y + \sin x \sin y$ and $\sin(x-y) = \sin x \cos y - \cos x \sin y$. Divide, and then divide "top" and "bottom" by $\sin x \sin y$. We obtain

$$\cot(x-y) = \frac{\cos x \cos y + \sin x \sin y}{\sin x \cos y - \cos x \sin y} = \frac{\cot x \cot y + 1}{\cot y - \cot x}.$$

Use the above formula with $x = \angle CAB$ and $y = \angle MAC$. The cotangent of $\angle MAB$ is turns out to be 3. (We could also work with the more familiar tangent function.)

Another way: Drop a perpendicular from M to the hypotenuse, meeting AB at N. By the Pythagorean Theorem, $NM = NB = 1/\sqrt{2}$. But again by the Pythagorean Theorem, $AB = 2\sqrt{2}$ and therefore $AN = (3/2)\sqrt{2}$. It follows that the cotangent of $\angle MAB$ is 3.

V-27. Let P be a point where the curves $y = \cos x$ and $y = \tan x$ meet. The line through P parallel to the y-axis meets the curves $y = \csc x$ and $y = \sin x$ at points Q and R. Find the length of QR.

Solution. Rewrite the equation $\cos x = \tan x$ as $\cos x = \sin x / \cos x$. Simplify, using the identity $\cos^2 x = 1 - \sin^2 x$. We obtain $\sin^2 x + \sin x - 1 = 0$. Thus $\sin x = (\sqrt{5} - 1)/2$.

The length of QR is $|\csc x - \sin x|$. Calculate, using the computed value of $\sin x$. Fairly quickly, we get the answer 1. Nice answer!

Another way: Start not by trying to find the intersection points, but by looking at what is asked for. We want $\csc x - \sin x$. From $\cos x = \tan x$, we obtain $\cos^2 x = \sin x$, that is, $1 - \sin^2 x = \sin x$. Divide both sides by $\sin x$. We get $\csc x - \sin x = 1$.

Comment. It turns out that we didn't need to know that $\sin x = (\sqrt{5} - 1)/2$ and therefore $\csc x = (\sqrt{5} + 1)/2$. But if we solve the problem in the second way, we will miss seeing the Golden Ratio $(\sqrt{5} + 1)/2$ strike again.

V-28. Find exact expressions for $\sin(\pi/8)$ and $\sin(\pi/16)$.

Solution. Since $\sin(\pi/4) = 1/\sqrt{2}$, it is reasonable to use the double angle formula $\sin 2\theta = 2\sin\theta\cos\theta$. We rewrite the formula in a more convenient form. Let $b = \sin 2\theta$. Then

$$b^2 = 4\sin^2\theta\cos^2\theta = 4\sin^2\theta(1-\sin^2\theta),$$

or equivalently $4\sin^4\theta - 4\sin^2\theta + b^2 = 0$. This is a quadratic equation in $\sin^2\theta$. Solve, obtaining $\sin^2\theta = (1 \pm \sqrt{1-b^2})/2$.

We will be working with angles θ between 0 and $\pi/8$, and therefore

$$\sin\theta = \frac{\sqrt{2-\sqrt{4-4\sin^2 2\theta}}}{2}.$$

In the formula above, let $2\theta = \pi/4$. We conclude that

$$\sin(\pi/8) = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

Next let $2\theta = \pi/8$. We conclude that

$$\sin(\pi/16) = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}.$$

Comment. In exactly the same way, we can continue by finding $\sin(\pi/32)$, then $\sin(\pi/64)$, and so on forever. We show how these computations can be used to calculate π.

Inscribe a regular 2^n-sided polygon in a circle of radius 1. Any side subtends angle $2\pi/2^n$ at the center of the circle. A sketch shows that the polygon has side length $2\sin(\pi/2^n)$, and so has half-circumference $2^n\sin(\pi/2^n)$. But the half-circumference of the polygon, should, for large n, be quite close to the half-circumference of the circle, namely π.

We have seen that $\sin(\pi/2^n)$ can be computed just using basic arithmetic operations and square root. In particular, take $2^n = 16$. The half-perimeter of the regular 16-gon is roughly 3.121445. Not a bad estimate for π!

Archimedes, around -250, used this idea to show that π lies between $3\frac{10}{71}$ and $3\frac{1}{7}$. His method was more sophisticated, in that he used both inscribed and circumscribed polygons. His polygons had 96 sides.

The same basic approach to the computation of π was used up to the end of the seventeenth century. Ludolph van Ceulen, in 1610, in a heroic computation of limited utility, found π to 35 decimal places using a polygon with 2^{62} sides. This impressed people so much that in the Netherlands π is still sometimes called the *Ludolphine* number.

V-29. Suppose that $\cos x - \sin x = 3/5$. Find $\cos^3 x - \sin^3 x$.

Solution. We could calculate $\cos x$ and $\sin x$. Rewrite the given equation as $\cos x = \sin x + 3/5$, square both sides, and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain the equation $2\sin^2 x + \sin x - 16/25 = 0$. This can be solved for $\sin x$, and after some mildly tedious computation we find the answer. This would be the reasonable approach if we were dealing for example with the equation $2\cos x - 3\sin x = 3/4$. But in our case it is tempting, because of the symmetry, to try

Another way: We have

$$\cos^3 x - \sin^3 x = (\cos x - \sin x)(\cos^2 x + \cos x \sin x + \sin^2 x).$$

Since $\cos^2 x + \sin^2 x = 1$, we only need to find $\cos x \sin x$. But

$$\frac{9}{25} = (\cos x - \sin x)^2 = \cos^2 x - 2\cos x \sin x + \sin^2 x.$$

Thus $\cos x \sin x = 8/25$, and therefore $\cos^3 x - \sin^3 x = 99/125$.

V-30. A mountain rises from a large plain. Show how to find the height of the mountain above the plain, using old-fashioned surveying equipment.

Solution. To find the height of a *flagpole,* choose a point P at an accurately measured distance d from the base of the flagpole, and find the angle of elevation of the top of the pole from P. If this angle is θ, then the height of the pole is $d\tan\theta$. (That's not quite right. Let the distance of the eyepiece of the measuring instrument from the ground be k. Then the height of the pole is $k + d\tan\theta$.)

With a mountain, the flagpole procedure can't be used unless we are prepared to do many miles of digging. Instead, take a point P on the plain, and another point Q on the plain such that a vertical pole through P is on the line of sight between Q and T, as in Figure 5.5. Measure the distance PQ, and call it d. Measure θ, the

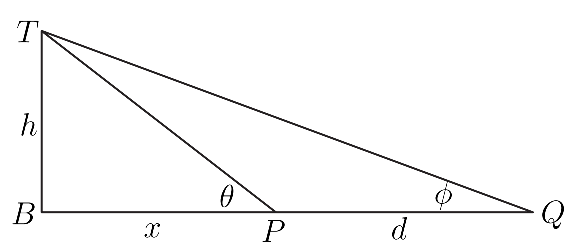

Figure 5.5: Measuring the Height of a Mountain

angle of elevation of T from P, and ϕ, the angle of elevation of T from Q. Assume that the Earth is flat. This will not lead to a large error.

Let h be the height of the mountain, and let x be the distance from P to the point B at the level of the plain which is directly beneath T. Neither h nor x is directly measurable. But $h/x = \tan\theta$, and $h/(x+d) = \tan\phi$, and therefore $h = d\tan\theta\tan\phi/(\tan\theta - \tan\phi)$.

Comment. Surveying is an ancient application of mathematics, and the inaccessible mountain problem was surely often solved over the millenia. The oldest surviving solution comes from Liu Hui (around 250). His *Haidao suanjing* (Sea Island Mathematical Manual) contains nine surveying problems. The first and simplest has to do with finding, from the mainland, the height of a mountain on a sea island, and the distance to the island.

V-31. A triangle has two $22.5°$ angles and its longest side is 2. Find the area of the triangle without using a calculator.

Solution. We need to find the height of the triangle. The base angles are each $22.5°$; call this angle θ. The problem will be solved once we know $\tan\theta$.

Note that $\cos 2\theta = 1/\sqrt{2}$. Thus

$$\tan^2\theta = \frac{2\sin^2\theta}{2\cos^2\theta} = \frac{1-\cos 2\theta}{1+\cos 2\theta} = \frac{1-1/\sqrt{2}}{1+1/\sqrt{2}}.$$

After rationalizing the denominator, we find that $\tan^2\theta = (\sqrt{2}-1)^2$. So the area is $\sqrt{2}-1$.

Another way: There are other ways to find $\tan\theta$ by using trigonometric identities. For example, we can use the double angle formula

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}.$$

Since $\tan(45°) = 1$, we find that $\tan^2\theta + 2\tan\theta - 1$, and solve the quadratic equation.

Another way: Use the Cosine Law. If a is the length of one of the short sides, then

$$4 = a^2 + a^2 - 2a^2\cos 135°$$

and therefore $a^2 = 4/(2+\sqrt{2}) = 4-2\sqrt{2}$. By the Pythagorean Theorem, the square of the height is $3-2\sqrt{2}$. It so happens that $3-2\sqrt{2} = (\sqrt{2}-1)^2$, so the height is $\sqrt{2}-1$.

V-32. A cylindrical glass with inner radius 4 cm and inner height 12 cm is full of water. (a) To the nearest tenth of a degree, through what angle should we tilt the glass so that 70 cubic cm of water flow out? (b) What about if we start with the water level 1 cm below the top?

Solution. (a) Let the angle of tilt be θ degrees. The amount that flows out is much less than half the volume, so the tilt is small. In particular, the bottom of the glass remains covered with water

At the point on the side of the glass where the water level goes down the most, it goes down by an amount h, where $h = 8\tan\theta$—see Figure 5.6. The volume of a column of water of radius 4 and height h is $16\pi h$. But only half this much has

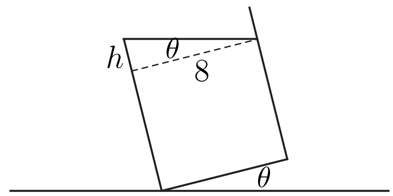

Figure 5.6: The Correct Angle of Tilt

spilled out, so the amount of water that flowed out is $(16\pi)(8\tan\theta)/2$. Set this equal to 70, and solve for $\tan\theta$. We get that $\tan\theta$ is about 0.34815. The calculator says that θ is about 19.9°.

(b) Fill the glass with water. That requires adding 16π cubic centimeters. Then pour out $16\pi + 70$. Using the same idea as in part (a), we find that if the angle of tilt is θ then $\tan\theta = (16\pi + 70)/64\pi$. Thus $\tan\theta$ is about 0.59815, and θ is about 30.9°.

Comment. If we want to pour out more than half of the water, the geometry becomes more complicated.

V-33. Two circular steel wheels are joined by a tightly stretched leather drive belt. The wheels have radius 2 and 3, and the distance between their centers is 12. How long is the belt?

Solution. Look at Figure 5.7. The centers of the wheels are labelled A and B. Let P, P' be the points where the belt ceases to be in contact with the left wheel, and let Q and Q' be the corresponding points for the right wheel. By symmetry, we can concentrate on the top half of the belt. The line PQ is tangent to both circles, so

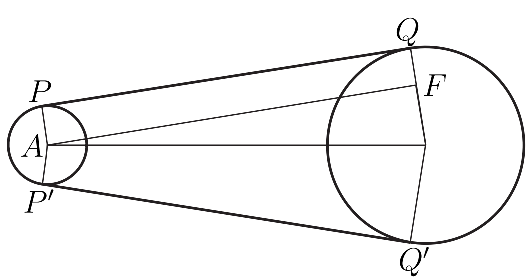

Figure 5.7: The Length of the Drive Belt

$\angle APQ$ and $\angle BQP$ are right angles. Drop a perpendicular from A to a point F on BQ. Then $PQ = AF$. But $BF = 1$. By the Pythagorean Theorem, $PQ = \sqrt{143}$, so the straight parts of the belt add up to $2\sqrt{143}$.

It remains to calculate the length of the curvy bits of the belt. Let θ be the size (in radians) of $\angle FAB$. Then $\sin\theta = 1/12$.

So $\angle PAP' = \pi + 2\theta$, and therefore the length of belt wrapped around the small circle is $2(\pi - 2\theta)$, while $3(\pi + 2\theta)$ is wrapped around the large circle, for a total of $5\pi + 2\theta$.

The length of the belt is therefore $2\sqrt{143} + 5\pi + 2\theta$. It is finally time to use the calculator. We have $\sin\theta = 1/12$. So (using the $\sin^{-1}$ key on the calculator, and making sure we are in radian mode) we find that θ is roughly 0.083430086. The total length is therefore about 39.7.

Comments. 1. We measured angles in radians for convenience. Take an arc that subtends an angle θ at the center of a circle of radius r. If θ is measured in radians, the length of the arc is $r\theta$. If θ is measured in degrees, the expression for the length is less attractive.

2. There is an interesting bonus to using radian notation. In our case we had $\sin\theta = 1/12$, so $\sin\theta$ is about 0.08333. In radians, θ turned out to be about 0.08343. So θ and $\sin\theta$ are almost equal! Indeed $(\sin\theta)/\theta$ is about 0.999. In general, if we use radian notation, and θ is not far from 0, then $(\sin\theta)/\theta$ is close to 1. We could have done the whole calculation, to high accuracy, without pressing the $\sin^{-1}$ key!

3. We explore briefly a problem that may appear difficult. Let the wheels have radius 2 and 3, and suppose that the belt has length 50. How far apart, approximately, are the centers of the wheels?

Let this distance be d. The argument used above can be adapted to show that

$$2\sqrt{d^2 - 1} + 5\pi + 2\theta = 50,$$

where $\sin\theta = 1/d$. We can't hope to find an exact expression for the solution. What are we to do?

There are *numerical* techniques for finding solutions of quite general equations. Sophisticated calculators have a *Solve* key that will do the job. But we can get by without such a tool. For note that θ is small. Throw away that term and solve for d. We find that d is about 17.17. Since $\sin\theta$ and θ are nearly equal, θ is roughly 0.05824. Substitute this into the main equation, and solve again for d. We get $d = 17.117$, an approximation good enough for all practical purposes.

V-34. By combining the identities for $\sin(x+y)$ and $\cos(x+y)$ we can show that

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

Let α, β, and γ be the acute angles whose tangents are 1/2, 1/5, and 1/8. Without using a calculator, find $\alpha + \beta + \gamma$.

Solution. Using the formula for the tangent of a sum, we get

$$\tan(\alpha+\beta) = \frac{\frac{1}{2}\cdot\frac{1}{5}}{1-\frac{1}{10}} = \frac{7}{9}.$$

Use the formula again, thinking of $\alpha+\beta+\gamma$ as $(\alpha+\beta)+\gamma$. We get $\tan(\alpha+\beta+\gamma) = (7/9+1/8)/(1-7/72)$. This simplifies to 1! But $\alpha+\beta+\gamma$ is acute, so $\alpha+\beta+\gamma$ has radian measure $\pi/4$.

Comment. This is one of a large family of formulas discovered by Euler. Until quite recently, formulas related to this one, most notably Machin's formula $\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$, were standard tools for accurate calculation of π. What made it all work is that when $|x|$ is substantially below 1, the MacLaurin series $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ converges rapidly to $\arctan x$.

V-35. Cut off a strip of width w from a 1×1 square, where w is chosen so that the strip can be placed diagonally on what is left of the square, as in Figure 5.8. Find w.

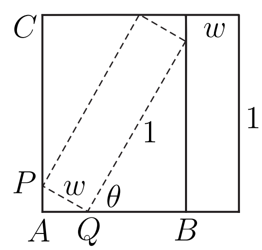

Figure 5.8: Cutting a Strip off a Square

Solution. Most approaches lead to messy expressions with several square roots, while trigonometric functions lead to a clean solution. Let labels be as in Figure 5.8. Then $AP = w\cos\theta$ and $PC = \sin\theta$. Also, $AQ = w\sin\theta$ and $QB = \cos\theta$.

Working up from A, we find that $w\cos\theta + \sin\theta = 1$. Working to the right from A, we find that $w\sin\theta + \cos\theta + w = 1$. We want w, but it is easier to find θ first.

Eliminate w using our two equations:

$$w = \frac{1-\sin\theta}{\cos\theta} = \frac{1-\cos\theta}{1+\sin\theta}.$$

"Cross-multiply" and simplify, using the fact that $1-\sin^2\theta = \cos^2\theta$. We obtain $2\cos^2\theta - \cos\theta = 0$, so $\cos\theta = 1/2$. Now we can notice that $\theta = 60°$ or just compute $\sin\theta$ and then w. It turns out that $w = 2-\sqrt{3}$.

V-36. Sightings were taken of the top of an apartment building from 100 meters from the base and from 500 meters from the base. The difference between the angles of elevation was 30°. Find the height of the building. Hint: $\tan(\alpha - \beta) = (\tan\alpha - \tan\beta)/(1 + \tan\alpha \tan\beta)$.

Solution. Let the height be h. For convenience, we measure in units of 100 meters. Let α be the angle of elevation from 100 meters away, and β the angle of elevation from 500 meters. Then $\tan\alpha = h$ and $\tan\beta = h/5$. But $\alpha - \beta = 30°$. By using the formula of the hint, we find that $(h - h/5)/(1 + h^2/5) = 1/\sqrt{3}$, which simplifies to $h^2 - 4\sqrt{3}\,h + 5 = 0$. This has the solutions $h = 2\sqrt{3} \pm \sqrt{7}$. The solution $2\sqrt{3} + \sqrt{7}$ gives a height of about 611 meters, implausibly large, so the height is about 81.8 meters.

Comment. We tacitly assumed that the two observation points are at the same height. And unless sightings were taken from ground level, which is unlikely, we need to add the height of the sighting instrument to the computed height.

V-37. By using the addition formula

$$\cos(x + y) = \cos x \cos y - \sin x \sin y,$$

we can express $\cos 3\theta$ in terms of $\cos 2\theta$, $\cos\theta$, $\sin 2\theta$, and $\sin\theta$, and further manipulation yields

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

Use this formula and trigonometric function keys on the calculator to solve the equation $4x^3 - 3x = 1/2$.

Solution. Let $\cos 3\theta = 1/2$. From the given formula, we find that $\cos\theta$ is a root of the equation $4x^3 - 3x = 1/2$. Since $\cos 3\theta = 1/2$, one possibility is $3\theta = 60°$, so $\theta = 20°$. It follows that $x = \cos 20°$ is a solution. The calculator says that this is about 0.9396926, but that's an ugly and therefore in the long run less useful answer.

There are a couple of approaches to the other solutions. One is to recall that if a is a solution of the polynomial equation $P(x) = 0$, then $x - a$ divides $P(x)$. For neatness, write a instead of $\cos 20°$. *Do not use* the calculator approximation found above. We carry out the division, and find

$$\frac{4x^3 - 3x - 1/2}{x - a} = 4x^2 + 4ax + 4a^2 - 3.$$

So to find the remaining roots, we find the roots of the quadratic equation $4x^2 + 4ax + 4a^2 - 3 = 0$ in the usual way. The process of dividing by $x - a$ once we have found a root a is called *depressing the degree.* It is built into most computer programs for approximating roots of polynomials.

For our polynomial there is a nicer idea than depressing the degree. We were a little hasty in concluding $3\theta = 60°$ from $\cos 3\theta = 1/2$. For we could have $3\theta = 300°$, giving $x = \cos 100°$ as a solution. And we can go beyond 360°, with $3\theta = 420$ and $x = \cos 140°$. The three numbers $\cos 20°$, $\cos 100°$, and $\cos 140°$ are distinct, they are all solutions of our equation. Since the equation is a cubic they must be all the solutions.

Comments. Analysis of the equation $4x^3 - 3x = 1/2$ is a key step in proving that the general angle cannot be trisected by straightedge and compass. What is actually proved is that even the 60° angle can't be so trisected.

The idea we used with $4x^3 - 3x = 1/2$ can be adapted to solve some other cubic equations. We sketch the procedure. The equation $x^3 + ax^2 + bx + c = 0$ can be changed to one with no x^2 term by the change of variable $x = y - a/3$. And we can multiply through by a non-zero constant, so we may assume that we are solving the equation $4y^3 - py = q$.

Make the change of variable $y = z/m$, where m will be chosen later. The equation becomes $4z^3 - pm^2 z = qm^3$. We want $pm^2 = 3$, that is, $m = \sqrt{3/p}$. (If $p = 0$ then solving the cubic is trivial.) We end up with the equation $4z^3 - 3z = q(3/p)^{3/2}$. Call the right-hand side k. If $|k| \le 1$, then $k = \cos 3\theta$ for some θ, and we continue like we did in the case $k = 1/2$. If $|k| > 1$ then the method fails. This method of solving some cubic equations was found by Viète in the late sixteenth century.

The author was once asked a question that reduced to solving cubics of Viète's type. A *formula* was needed, for use with a spreadsheet. There was even some money in it—not much, but at the time it bought a few bottles of cheap wine. And there was pleasure in seeing an antique formula, probably not used for practical purposes for many years, turn out to be useful. See also IV-69.

V-38. Let $a > 1$. Write $\cosh_a(t)$ for the function $(a^t + a^{-t})/2$, and $\sinh_a(t)$ for $(a^t - a^{-t})/2$. (a) Let $x = \cosh_a(t)$ and $y = \sinh_a(t)$. Find $x^2 - y^2$, simplifying as much as possible. (b) Express $\sinh_a(2t)$ in terms of $\sinh_a(t)$ and $\cosh_a(t)$. (c) Let $a = 10$. Solve the equation $\cosh_a(t) = 20$.

Solution. (a) Funny names for functions! We have $x^2 = (a^t + a^{-t})^2/4$. Expand, noting that $a^t \cdot a^{-t} = 1$. So $x^2 = (a^{2t} + 2 + a^{-2t})/4$. In a similar way, we find that $y^2 = (a^{2t} - 2 + a^{-2t})/4$. Subtract. Almost everything cancels, and $x^2 - y^2 = 1$.
(b) We have

$$\sinh_a(2t) = \frac{a^{2t} - a^{-2t}}{2} = 2 \cdot \frac{a^t + a^{-t}}{2} \cdot \frac{a^t - a^{-t}}{2},$$

and therefore $\sinh_a(2t) = 2\cosh_a(t)\sinh_a(t)$.
(c) The equation can be rewritten as $10^t + 10^{-t} = 20$. Let $x = 10^t$. Then $x + 1/x = 20$. This simplifies to $x^2 - 20x + 1 = 0$, which has the solutions $x = 10 \pm \sqrt{99}$. So $t = \log x = \log(10 \pm \sqrt{99})$.

Comment. The functions $\cosh_a$ and $\sinh_a$ have properties strikingly reminiscent of the cosine and sine functions. Basically, they are to the hyperbola $x^2 - y^2 = 1$ as cosine and sine are to the circle $x^2 + y^2 = 1$. That explains the funny name: cosh stands for *hyperbolic cosine.* Almost always, a is taken to be e, the base for natural logarithms, about 2.71828, and the functions are simply called cosh and sinh. All of the familiar trigonometric identities have analogues in the "hyperbolic" world.

The hyperbolic functions have many applications. For example, a cable hanging between two poles of equal height takes on a shape that can be described using the cosh function. Some calculators permit evaluation of cosh and sinh with one keypress.

It is tempting to mention now the beautiful and mysterious formulas $\cos x = (e^{ix} + e^{-ix})/2$ and $\sin x = (e^{ix} - e^{-ix})/2i$, but we won't.

Chapter 6

Sequences

Introduction

These problems are mainly about arithmetic and geometric sequences, that is, sequences (a_n) that satisfy the recurrences $a_{n+1} = a_n + d$ and $a_{n+1} = ra_n$ respectively. Many of the remaining questions deal with sequences given by less familiar recurrences.

The solutions take it for granted that students are familiar with sums of arithmetic and geometric progressions. That is regrettably not always true. The main difficulty is that the basic facts are viewed as "formulas" rather than ideas.

Look at $1 + 3 + \cdots + 997 + 999$, or any other arithmetic series with positive terms. The way to understand the sum is to associate with it a number of clear concrete images. There might be a number of men, having respectively 1, 3, 5, …, 499 dollars to their name, and an equal number of women, having 501, 503, …, 999 dollars. The two people who have 1 and 999 dollars pool their fortunes, as do the two people who have 3 and 997 dollars, and so on. There are now 250 joint accounts, each with \$1000, and therefore

$$1 + 3 + \cdots + 997 + 999 = 250000.$$

There are other concrete ways of *seeing* what the sum of particular arithmetic progressions ought to be, including various versions of stacking. At a more algebraic level, note that

$$a + (a + d) + \cdots + (a + (n - 1)d) = an + d\,(1 + 2 + \cdots + (n - 1))\,,$$

so by the magic of linearity we need only calculate $1 + 2 + \cdots + (n - 1)$.

One could explain why $1 + 2 + \cdots + n$ is equal to $\binom{n+1}{2}$. (How many ways are there of choosing two numbers from the $n + 1$ numbers 0 to n? If the bigger of the numbers is 1, there is 1 way of choosing the other number. If the bigger is 2, there are 2 ways of picking the other number, and so on. Finally, if the bigger is n, there are n ways of picking the other.) Everything is connected to everything else.

If several sums are reasoned through in various ways, then one can know, *forever*, what is going on in this corner of mathematics.

The chapter stresses *telescoping series* more than usual. The idea is often presented as an isolated trick for finding

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)n},$$

but in fact it is a method with many uses. Problem VI-7, and several others in this chapter, make it clear that telescoping series can be used to prove the correctness of many of the standard formulas.

Problems and Solutions

VI-1. (a) Show that every multiple of 20 can be expressed as a sum of 10 consecutive odd integers. What about if the consecutive odd integers must be positive? Generalize. (b) What about if 10 is replaced by 11?

Solution. (a) The usual idea for summing arithmetic progressions shows that

$$(2k+1)+(2k+3)+\cdots+(2k+19)=10(2k+10)=20(k+5).$$

By selecting the integer k appropriately, we can make $20(k+5)$ equal to any multiple of 20. If k must be non-negative, then only the multiples of 20 greater than or equal to 100 are representable.

Replace 10 by the positive even integer n. We have

$$(2k+1)+(2k+3)+\cdots+(2k+2n-1)=2n\left(k+\frac{n}{2}\right).$$

If the integer k is unrestricted, then any multiple of $2n$ is representable. If we must have $k \geq 0$, then only the multiples of $2n$ greater than or equal to n^2 are representable.

(b) Let n be odd. Note that

$$(2k+1)+(2k+3)+\cdots+(2k+2n-1)=n(2k+n).$$

If the integer k is unrestricted, then $2k + n$ can be any odd integer, so any odd multiple of n is representable. If we must have $k \geq 0$, then the representable numbers are the odd multiples of n greater than or equal to n^2.

Another way: The 10 odd integers from -9 to 9 have sum 0. To get a sum of $20q$, just add $2q$ to each number. If integers must be positive, then $q \geq 5$ so $20q \geq 100$. The same argument works if 10 is replaced by any even integer.

A mild variant works for 11, or indeed any odd integer. The 11 *even* integers from -10 to 10 have sum 0. To get a sum of $11q$, just add q to each of them. But we want consecutive *odd* integers, so q must be odd. If the integers must be positive, then $q \geq 11$.

Comment. The second solution is quite geometric. It uses symmetry, which is at its roots a visual notion. Note that for 11 we even (temporarily) lifted the requirement that the numbers be odd in order to hang on to symmetry. And adding the same thing to each number is geometric—we are just shifting a structure left or right.

It is a nice exercise to find purely pictorial arguments. Hint: Let O be the origin and $P = (18, 9)$. Look at the points with integer coordinates (dots) that are in the first quadrant and on or to the left of line segment OP. Slide OP to the right and count dots. Or slide the square of dots in Figure 2.5 in a northeastely direction.

VI-2. Find an infinite geometric sequence all of whose terms come from 1, 1/2, 1/4, 1/8, and so on and whose sum is 1/60.

Solution. Let the first term a of our sequence be $1/2^s$ and let the common ratio r be $1/2^t$. We want

$$\frac{4}{15} = \frac{a}{1-r} = \frac{1/2^s}{1-1/2^t} = \frac{2^{t-s}}{2^t-1}.$$

Since $60 = 4 \cdot 15$, putting $t = 4$ and $s = 6$ yields the desired sum.

VI-3. From $\sin(x+y) = \sin x \cos y + \cos x \sin y$ and the related identity for $\sin(x-y)$ we conclude that

$$2\cos x \sin y = \sin(x+y) - \sin(x-y).$$

Use the above identity to show that if $\sin\theta \neq 0$ then

$$\cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}.$$

Hint: Multiply by $2\sin\theta$.

Solution. The given identity specializes to

$$(2\sin\theta)\cos(2k-1)\theta = \sin 2k\theta - \sin(2k-2)\theta.$$

Let k range from 1 to n and add up. Note the cancellation on the right. We obtain

$$(2\sin\theta)\big(\cos\theta + \cos 3\theta + \cdots \cos(2n-1)\theta\big) = \sin 2n\theta.$$

Finally, divide both sides by $2\sin\theta$.

Comment. The same technique works for $\cos a_1 + \cos a_2 + \cdots \cos a_n$ where a_1, a_2, $\ldots a_n$ is any arithmetic progression, and also if cos is replaced by sin. There is a deep structural connection between the exponential function and the sine and cosine functions. The sum in this problem is closely connected to the sum of a geometric series.

VI-4. Let a, ar, ar^2, $\ldots$, ar^n be a finite geometric sequence with $a > 0$ and $0 < r \le 1/2$. Show that any term is greater than the sum of all the terms that follow it.

Solution. We can use formulas. Let ar^k be a term of the sequence with $k < n$. Then the sum of the terms after ar^k is $ar^{k+1} + ar^{k+2} + \cdots + ar^n$. By a standard formula this sum is

$$\frac{ar^{k+1}(1-r^{n-k})}{1-r}.$$

We need to show that the above quantity is less than ar^k, or equivalently that

$$\frac{r(1-r^{n-k})}{1-r} < 1.$$

Since $1 - r^{n-k} < 1$, it is enough to show that $r/(1-r) \le 1$. This is clear, for $r \le 1/2$, and therefore $1 - r \ge 1/2$.
Another way: We want to show that

$$ar^{k+1} + ar^{k+2} + \cdots + ar^n < ar^k,$$

or equivalently that $r + \cdots + r^{n-k} < 1$. Since $0 < r \le 1/2$, it is enough to show that

$$\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-k}} < 1.$$

This is geometrically clear. A grasshopper starts at one end of a stick of length 1, hops a distance 1/2 toward the other end, then a distance 1/4, then a distance 1/8, and so on. With each hop the grasshopper's distance from the other end is divided by 2, but the grasshopper can't reach the other end in a finite number of hops.

Comment. The second solution doesn't use the formula for the sum of a finite geometric progression. In many cases, we do not know a closed form for the sum of a sequence, so the analogue of the first approach is simply not available. Often we need instead to *compare,* term by term, the sequence we are studying with a simpler sequence whose behaviour is known.

VI-5. The sum $a_1 + a_3 + a_5 + \cdots + a_{99}$ of the odd-numbered terms of the finite arithmetic progression a_1, a_2, $\ldots$, a_{99} is 200. Find the sum of the even-numbered terms.

Solution. The sum of the 50 odd-numbered terms and the sum of all the terms are respectively

$$50\,\frac{a_1+a_{99}}{2} \qquad \text{and} \qquad 99\,\frac{a_1+a_{99}}{2}.$$

It follows that the sum of the even-numbered terms is $49(a_1+a_{99})/2$. We have been told that $50(a_1+a_{99})/2 = 200$. So the sum of the even-numbered terms is $200(49/50)$, that is, 196.

Comment. The wording of the problem suggests that the sum of the even-numbered terms is completely determined by the sum of the odd-numbered terms. If that's true—a big if!—then we can assume that the terms are all 200/50, so the sum of the 49 even-numbered terms is 196. That kind of reasoning is useful in multiple-choice tests but not elsewhere.

Note for example that the sum of the even-numbered terms in the arithmetic sequence a_1, a_2 is not determined by the sum of the odd-numbered terms: knowing a_1 tells us nothing about a_2. In general, if we know the sum of the even-numbered terms in the arithmetic sequence $a_1, a_2, \dots, a_{2n}$, then we know nothing about the sum of the odd-numbered terms.

VI-6. Show that for any positive integer n

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}<2.8.$$

Solution. Use the calculator to add the first two terms, the first three, and so on for a while. The sums, truncated to three decimal places, are 2.500, 2.666, 2.708, 2.716, 2.718, and 2.718. Things seem to be settling down—it looks as if we won't even get close to 2.8. But we need to *prove* that we won't.

The key observation is that the terms are going down very fast, far faster than the terms of a geometric sequence. More precisely,

$$\frac{1}{3!}=\left(\frac{1}{2!}\right)\left(\frac{1}{3}\right), \qquad \frac{1}{4!}<\left(\frac{1}{2!}\right)\left(\frac{1}{3}\right)^2, \qquad \frac{1}{5!}<\left(\frac{1}{2!}\right)\left(\frac{1}{3}\right)^3,$$

and so on. In particular, our sum is less than

$$2+\left(\frac{1}{2!}\right)\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots+\frac{1}{3^{n-2}}\right).$$

The infinite geometric series $1+1/3+1/9+\cdots$ has sum $3/2$. We conclude that our sum is less than $2+(1/2)(3/2)$, that is, 2.75.

Comment. The full sum

$$1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots$$

is the base e for natural logarithms, one of the most important numbers in all of mathematics. It turns out that e is about 2.718281828 (the repetition of digits is an "accident," e is not a rational number). By using ideas like the ones used in solving the problem, we can easily compute e to guaranteed high accuracy.

VI-7. The sum s_n of the first n terms of a sequence is equal to $n^2(n+1)^2/4$. Find a simple expression for the n-th term of the sequence.

Solution. Let a_n be the n-th term of the sequence. Then

$$a_n = s_n - s_{n-1} = \frac{n^2(n+1)^2}{4} - \frac{(n-1)^2n^2}{4} = n^3.$$

Comment. It is much more interesting to read the result "backwards." Note that if s_n denotes the sum of the first n terms of *any* sequence, then because of wholesale cancellation, usually called *telescoping,* we have

$$(s_1 - s_0) + (s_2 - s_1) + (s_3 - s_2) + \cdots + (s_N - s_{N-1}) = s_N - s_0.$$

In the case $s_n = n^2(n+1)^2/4$, we have $s_n - s_{n-1} = n^3$, so

$$1^3 + 2^3 + 3^3 + \cdots + N^3 = \frac{N^2(N+1)^2}{4}.$$

Similarly, we can prove the correctness of the usual formula for the sum of the first N perfect squares from the identity

$$\frac{(n^2)(n+1) + n(n+1)^2}{6} - \frac{(n-1)^2(n) + (n-1)(n^2)}{6} = n^2.$$

And we can prove the correctness of the usual formula for the sum of the first N terms of the geometric sequence with first term 1 and common ratio $r \neq 1$ by using the identity

$$\frac{r^n}{r-1} - \frac{r^{n-1}}{r-1} = r^{n-1}.$$

VI-8. By a suitable choice of + signs and − signs, can we make

$$\pm 1 \pm 2 \pm 3 \pm \cdots \pm 99 \pm 100$$

equal to 3999? What about 4000?

Solution. However we put in + or − signs, we have a "sum" of 50 odd numbers and 50 even numbers. A sum of 50 odd numbers is even, as is any sum of even numbers, so 3999 is not representable in the desired form.

To get a representation of 4000, play with the numbers. First try to get close to 4000 by summing the integers from somewhere to 100. For example, if we sum from 46 to 100, the result is 4015. Subtract $1 + 2 + 3 + 4 + 5$ from this, and we have reached 4000. But we need to use also the 40 numbers from 6 to 45 without disturbing the sum. This is easy: just add $(6 - 7 - 8 + 9)$, and $(10 - 11 - 12 + 13)$, and so on up to $(42 - 43 - 44 + 45)$. There are *many* other representations of 4000.

Comment. More generally, what numbers N are representable by expressions of the form $\pm 1 \pm 2 \pm 3 \pm \cdots \pm n$, where n is given?

There are size restrictions on N. If we use only plus signs, we can only get to $n(n+1)/2$, so $|N| \le n(n+1)/2$. There is also a parity restriction. Whatever choice of signs we make, the "sum" is even if $1 + 2 + \cdots + n$ is even, and odd otherwise. But the sum of the first n positive integers is $n(n+1)/2$, and this is even precisely if one of n or $n+1$ is a multiple of 4. So *if* N is representable, then N must be even if n leaves a remainder of 0 or 3 on division by 4, and N must be odd otherwise.

It can be shown that the restrictions of the preceding paragraph are the *only* restrictions. So for example since 100 is divisible by 4 and 4000 is smaller than 5050 and even, 4000 *had* to be representable. It is usually not hard to find an explicit representation of a representable number N but sometimes experimentation and ingenuity are required. Problems of this type, for various values of n and N, make good concrete computational exercises.

VI-9. Let $a_1 = 1$, and for $n \ge 1$ let $a_{n+1} = \sqrt{90 + a_n}$. Show that for all n

$$0 < 10 - a_{n+1} < \frac{10 - a_n}{10}.$$

Solution. We have

$$10 - a_{n+1} = 10 - \sqrt{90 + a_n} = \frac{10 - a_n}{10 + \sqrt{90 + a_n}}$$

(we multiplied "top" and "bottom" by $10 + \sqrt{90 + a_n}$ to get from the second expression to the third).

Since $10 - a_1$ is positive, the result above shows that $10 - a_2$ is positive, but then $10 - a_3$ is positive, and so on forever. And since $10 + \sqrt{90 + a_n} > 10$, we conclude that $10 - a_{n+1} < (10 - a_n)/10$.

Comment. So a_n stays under 10, and its distance from 10 is divided by more than 10 each time, indeed by more than 19.5, since $10 + \sqrt{90 + a_n} > 19.5$. We conclude that the sequence a_1, a_2, a_3, ... approaches 10 rapidly. If we start with $a_1 > 10$, there is again a rapid approach to 10, this time from above.

The fact that a_n is always under 10 can be proved without rationalizing the numerator. We have $a_1 < 10$. But because $a_1 < 10$, we have

$$a_2 = \sqrt{90 + a_1} < \sqrt{90 + 10} = 10.$$

In the same way, from the fact that $a_2 < 10$ we conclude that $a_3 < 10$, which leads to $a_4 < 10$, and so on forever.

VI-10. Let a_n be the integer nearest to $\sqrt{n}$. Calculate

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_{999}}.$$

Solution. Using a calculator if necessary, find a_1, a_2, and so on. The first few a_i are 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 5. There are two 1's, four 2's, six 3's, eight 4's. We show that the pattern continues: there are $2k$ values of n such that $a_n = k$.

The square root of an integer can't be halfway between two integers, so $a_n = k$ if and only if $k - 1/2 < \sqrt{n} < k + 1/2$. If we square everything, we obtain equivalently $k^2 - k + 1/4 < n < k^2 + k + 1/4$. So n ranges from $k^2 - k + 1$ to $k^2 + k$; there are $2k$ integers in this interval.

We can now evaluate the sum efficiently. Note that $1/a_1 + 1/a_2 = 1/1 + 1/1 = 2$, and $1/a_3 + 1/a_4 + 1/a_5 + 1/a_6 = 4/2 = 2$, $1/a_7 + \cdots + 1/a_{12} = 6(1/3) = 2$ and so on.

The largest k such that $k^2 + k \leq 999$ is 31. So the sum can be broken up into 31 sums each equal to 2, plus the terms $1/a_{993}$ to $1/a_{999}$, that is, 7 terms each equal to 1/32. The full sum is therefore $62 + 7/32$.

VI-11. Note that $3^2 + 4^2 = 5^2$ and $10^2 + 11^2 + 12^2 = 13^2 + 14^2$. Find seven consecutive positive integers such that the sum of the squares of the first four is equal to the sum of the squares of the last three. Generalize.

Solution. It is convenient to let the seven numbers be $n-3$, $n-2$, $n-1$, n, $n+1$, $n+2$, and $n+3$. Then n must satisfy the equation

$$(n-3)^2 + (n-2)^2 + (n-1)^2 + n^2 = (n+1)^2 + (n+2)^2 + (n+3)^2.$$

Expand and simplify. We obtain $n^2 - 4n(1+2+3) = 0$. Since $n \neq 0$, it follows that $n = 4(1+2+3) = 24$. So there is exactly one example, namely $21^2+22^2+23^2+24^2 = 25^2 + 26^2 + 27^2$.

In general, we look for $2k+1$ consecutive positive integers such that the sum of the squares of the first $k+1$ is equal to the sum of the squares of the last k. Let the middle number be n. Just as in the case $k = 3$,

$$(n-k)^2 + \cdots + (n-1)^2 + n^2 = (n+1)^2 + \cdots + (n+k)^2.$$

Expand and simplify. We obtain the equation

$$n^2 - 4n(1 + 2 + \cdots + k) = 0.$$

Since $n \neq 0$, we have $n = 4(1 + 2 + \cdots + k) = 2k(k+1)$.

Comment. If instead we let the numbers be n, $n+1$, $\ldots$, $n+6$, then after some work we arrive at $n^2 - 18n - 63 = 0$, whose only positive root is 21. But the calculation takes longer and the general case is far less obvious. A symmetrical choice of names often pays off.

VI-12. Let $a_0 = 0$, $a_1 = 1$, and for $n \geq 1$ let

$$a_{n+1} = 2a_n - a_{n-1} + 2.$$

Find a simple formula for a_n and show that it is correct.

Solution. Compute for a while:

$$a_2 = 2 - 0 + 2 = 4, \quad a_3 = 8 - 1 + 2 = 9, \quad a_4 = 16, \quad \text{and} \quad a_5 = 25.$$

It is reasonable to conjecture that $a_n = n^2$ for all n. To prove this, let $b_n = n^2$. We need to show that the sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are the same.

It is easy to check that $b_0 = a_0$ and $b_1 = a_1$. Also,

$$2b_n - b_{n-1} + 2 = 2n^2 - (n-1)^2 + 2 = n^2 + 2n + 1,$$

and therefore $b_{n+1} = 2b_n - b_{n-1} + 2$. Since the two sequences begin in the same way and satisfy the same recurrence formula, they are the same everywhere.

VI-13. Suppose that $x \neq -1$, and let S be the infinite sum

$$\frac{x}{(1)(1+x)} + \frac{x^2}{(1+x)(1+x+x^2)} + \frac{x^3}{(1+x+x^2)(1+x+x^2+x^3)} + \cdots .$$

Find a simple expression for S. The answer will look different when $|x| < 1$ than when $|x| > 1$.

Solution. Note that

$$\frac{1}{1+x+\cdots+x^{k-1}} - \frac{1}{1+x+\cdots+x^k} = \frac{x^k}{(1+x+\cdots+x^{k-1})(1+x+\cdots+x^k)}.$$

Add up from $k = 1$ to $k = n$. There is wholesale cancellation, and the sum S_n of the first n terms is given by

$$S_n = 1 - \frac{1}{1+x+\cdots+x^n}.$$

Now we study the behaviour of $1/(1+x+\cdots+x^n)$ as n grows without bound (as $n \to \infty$.)

Suppose first that $|x| < 1$. Then $1+x+\cdots+x^n$ approaches $1/(1-x)$ as $n \to \infty$. So S_n approaches $1-(1-x)$, and therefore $S = x$.

Let $x = 1$. Then $S_n = 1 - 1/(n+1)$. As $n \to \infty$, $S_n \to 1$, and therefore $S = 1$. If $x > 1$, then $1 + x + \cdots + x^n > n + 1$, so again $S = 1$.

Suppose finally that $x < -1$. By the usual formula for the sum of a finite geometric progression,

$$1 + x + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

As $n \to \infty$, $|x^{n+1}|$ grows large without bound, and therefore so does $|1+x+\cdots+x^n|$, so again $S = 1$.

VI-14. The sum of an infinite geometric progression is 2. Find all possible values of the first term.

Solution. Let the first term be a and let the common ratio be r. Then the sum of all the terms is $a/(1-r)$. Thus $a/(1-r) = 2$, and therefore $r = 1 - a/2$.

But (apart from the dull case $a = 0$) the sum of an infinite geometric progression makes sense only when $-1 < r < 1$. So $-1 < 1 - a/2 < 1$, and therefore $0 < a < 4$.

VI-15. Let k be positive. Show that

$$\frac{1}{2\sqrt{k+1}} < \sqrt{k+1} - \sqrt{k} < \frac{1}{2\sqrt{k}}$$

and us e these inequalities to give a rough estimate of S, where

$$S = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots + \frac{1}{\sqrt{5000}}.$$

Solution. We rationalize the numerator and obtain

$$\sqrt{k+1} - \sqrt{k} = \frac{(\sqrt{k+1} - \sqrt{k})(\sqrt{k+1} + \sqrt{k})}{\sqrt{k+1} + \sqrt{k}} = \frac{1}{\sqrt{k+1} + \sqrt{k}}.$$

The desired inequalities now follow from the fact that

$$2\sqrt{k} < \sqrt{k+1} + \sqrt{k} < 2\sqrt{k+1}.$$

To estimate S, use first the inequality $\sqrt{k+1} - \sqrt{k} < 1/2\sqrt{k}$. Let $k = 1$, 2, 3, and so on up to 5000, and add up. Note the wholesale cancellation. We obtain

$$\sqrt{5001} - 1 < \frac{1}{2\sqrt{1}} + \frac{1}{2\sqrt{2}}\frac{1}{2\sqrt{3}} + \cdots + \frac{1}{2\sqrt{5000}},$$

and conclude that $S > 139.4$.

Now use the inequality $1/2\sqrt{k+1} < \sqrt{k+1} - \sqrt{k})$. A calculation similar to the one above shows that $S < 2\sqrt{5000} - 1$, so $S < 140.5$. We conclude that S is roughly 140.

Comment. The inequalities are least "sharp" when k is small. So to find more accurate estimates of S, add up on the calculator something like the first dozen terms, and use the inequalities to estimate the rest of the sum.

VI-16. Let a_1, a_2, $\ldots$, a_{100} be integers. Show that

$$|a_1 - a_2| + |a_2 - a_3| + |a_3 - a_4| + \cdots + |a_{99} - a_{100}| + |a_{100} - a_1|$$

is even.

Solution. Let beads labelled 1, 2, 3, ..., 100 be arranged around a circular necklace, where bead i is white if a_i is even and black if a_i is odd. Let 101 be another name for bead 1. Then $|a_i - a_{i+1}|$ is odd if the neighbouring beads i and $i+1$ are of different colours, and $|a_i - a_{i+1}|$ is even if beads i and $i+1$ have the same colour.

Travel around the necklace, from bead 1 all the way back to bead 1. The number of colour changes is even—if it weren't, bead 1 would be of a different colour from itself! So evenly many among the numbers $|a_1 - a_2|$, $|a_2 - a_3|$, ..., $|a_{100} - a_1|$ are odd. Since the sum of an even number of odd numbers is even, as is the sum of any number of even numbers, the desired result follows.

Another way: We prove that if $a_1, a_2, \ldots, a_n$ is a sequence of integers, and $b_1, b_2, \ldots, b_n$ is *any* rearrangement of that sequence, then

$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|$$

is even. In any sum, if you replace a number by one of the same *parity*—that is, replace an even number with an even number, or an odd with an odd—then the parity of the sum doesn't change.

In general, $|z|$ is even if and only if z is even. Therefore our sum has the same parity as

$$(a_1 - b_1) + (a_2 - b_2) + \cdots + (a_n - b_n).$$

But since the b_i are just the a_i in some order, the above sum is 0, which is even. So the original sum is even.

VI-17. Find a simple expression for

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{98}{99!} + \frac{99}{100!}.$$

Solution. Add the first two terms and simplify. The result is 5/6. Similarly, the sum of the first three terms is 23/24, and the sum of the first four terms is 119/120. And lest we forget, the sum of the first "one terms" is 1/2. It is natural to conjecture that the sum of the first n terms is $1 - 1/(n+1)!$.

To *prove* this, let $a_n = 1 - 1/(n+1)!$. Calculate $a_n - a_{n-1}$. We get

$$a_n - a_{n-1} = \left(1 - \frac{1}{(n+1)!}\right) - \left(1 - \frac{1}{n!}\right) = \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!} = \frac{n}{(n+1)!}.$$

Thus the original sum is equal to

$$(a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_n - a_{n-1}).$$

Note that $a_0 = 0$ and that almost everything cancels. The sum is a_n, exactly as conjectured.

Comment. Equivalently, we can pull the identity

$$\frac{k}{(k+1)!} = \frac{k+1}{(k+1)!} - \frac{1}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!}.$$

out of a hat, and sum from $k = 1$ to $k = 99$, noting the telescoping. The result is $1 - \frac{1}{100!}$. But it is more natural to make a conjecture about the sum and then use an essentially mechanical procedure to prove the conjecture.

VI-18. Show that no geometric sequence can have in it all three of the numbers 3, 4, and 5.

Solution. Suppose to the contrary that there is a geometric sequence which contains our three numbers. Without loss of generality we may assume that 3 is the first term of the sequence. Let r be the common ratio. Then $4 = 3r^m$ and $5 = 3r^n$ for some positive integers m and n.

Raise both sides of the first equation to the n-th power and both sides of the second equation to the m-th power. We obtain

$$4^n = 3^n r^{mn} \qquad \text{and} \qquad 5^m = 3^m r^{mn}.$$

Use the above equations to eliminate the term r^{mn}. We obtain

$$3^m 4^n = 3^n 5^m.$$

But this equation doesn't have positive integer solutions, since 2 divides the left-hand side but not the right-hand side.

VI-19. Find a simple expression for

$$1 + 11 + 111 + 1111 + \cdots + 111\ldots111,$$

where the last number in the sum has 100 ones.

Solution. The number with n 1's in its decimal expansion is equal to $(10^n - 1)/9$. We first calculate

$$(10^1 - 1) + (10^2 - 1) + \cdots + (10^{100} - 1).$$

By rearranging the above sum, we get

$$(10^1 + 10^2 + \cdots + 10^{100}) - 100.$$

The first part is the sum of a geometric progression with first term 10 and common ratio 10. By a standard result, this sum is $(10^{101} - 10)/9$. Simplify and divide by 9. The required sum is $(10^{101} - 910)/81$.

Comment. An integer whose decimal representation contains only 1's is called a *repunit.* There are many puzzles and problems about repunits. Puzzles about the decimal representation of numbers are often mathematically uninteresting, since 10 is a boring number. But repunits have "structure," since they are the numbers of the form $(10^n - 1)/9$, and we can generalize by replacing 10 by any integer $b > 1$.

VI-20. The product of the first five terms of a geometric progression is 2. The product of the first ten terms is 128. Find the product of the first fifteen terms.

Solution. Let a be the first term and r the common ratio. The product of the first five terms is $a^5r^{1+2+3+4}$, that is, a^5r^{10}. Similarly, the product of the first ten terms is $a^{10}r^{45}$, and the product of the first fifteen is $a^{15}r^{105}$.

So $a^5r^{10} = 2$ and $a^{10}r^{45} = 128$. We could solve for r and then for a, but that's not necessary. For

$$a^{15}r^{105} = \frac{\left(a^{10}r^{45}\right)^3}{\left(a^5r^{10}\right)^3},$$

and therefore the product of the first fifteen terms is $(128/2)^3$.

Another way: If a were negative, then the product of the first five terms would be negative, but it isn't. If r were negative, then the product of the first ten terms would be negative, but it isn't. So all terms are positive, and we can take logarithms. Any base will do, but base 2 is the most natural.

We have $\log_2(ar^n) = \log_2 a + n\log_2 r$. Taking logarithms transforms the geometric progression into an arithmetic progression, and products into sums—that is, after all, what logarithms are about. Let b, $b + d$, ... be the sequence of the logarithms of our geometric progression. The first five terms add up to $\log_2 2$, that is, 1, so $5b + 10d = 1$. Similarly, $10b + 45d = 7$. We want $15b + 105d$.

We could solve the system of two linear equations and find b and d. More directly, note that $15b+105d = 3(10b+45d)-3(5b+10d)$. It follows that $15b+105d = 18$. So the logarithm of the product of the first fifteen terms is 18, and therefore that product is 2^{18}.

VI-21. Divide the positive integers into groups as follows: 1; 2, 3; 4, 5, 6; 7, 8, 9, 10; 11, 12, ... Find the sum of the numbers in the fiftieth group

Solution. First find the smallest member of group 50. Altogether, groups 1 through 49 have $1 + 2 + \cdots + 49$ members, so group 50 begins with 1226, and ends with 1275. There are 50 integers in the group, with average value $(1226 + 1275)/2$. For the sum, multiply by 50. We get 62525.

Comment. The first number in the k-th group is $(k-1)k/2+1$, and the last number in the k-th group is $k(k + 1)/2$, for an average of $(k^2 + 1)/2$. Thus the sum of the numbers in the k-th group is $k(k^2 + 1)/2$.

The same type of question can be asked for any arithmetic progression. An interesting example is 1; 3, 5; 7, 9, 11; 13, 15, ... In that case, the sums of the numbers in each group are 1, 8, 27, ..., the perfect cubes. It is not hard to deduce that the sum of the first n positive cubes is the square of the sum of the first n positive integers.

VI-22. Find an increasing sequence of nine positive odd integers which add up to 1815 and such that the middle number is as big as possible.

Solution. The four smallest must add up to at least $1+3+5+7$, that is, 16. If $2n+1$ is the middle number, then the sum of the four that are larger than $2n+1$ is at least $(2n+3)+(2n+5)+(2n+7)+(2n+9)$. Add up all the terms. The sum is at least $10n+41$, and therefore $10n+41 \le 1815$. Thus $10n \le 1774$.

But n is an integer, so $n \le 177$, and therefore the middle number is no bigger than 355. Finally, we look for 9 odd numbers, the middle one of which is 355, that add up to 1815. The sequence 1, 3, 5, 7, 355, 357, 359, 361, 367 works. There are others.

VI-23. Compute $1^2 - 2^2 + 3^2 - 4^2 + \cdots + 997^2 - 998^2 + 999^2$. Generalize.

Solution. Rewrite the sum as

$$(1^2 - 0^2) + (3^2 - 2^2) + \cdots + (997^2 - 996^2) + (999^2 - 998^2).$$

Note that $(a+1)^2 - a^2 = 2a+1$. So we want to compute

$$(1 + 2\cdot 0) + (1 + 2\cdot 2) + \cdots + (1 + 2\cdot 996) + (1 + 2\cdot 998).$$

This is the sum of an arithmetic sequence with 500 terms. The terms average out to 999, so the sum is 499500.

The sum $1^1 - 2^2 + 3^2 - 4^2 + \cdots - (2n)^2 + (2n+1)^2$ is computed in exactly the same way. The result is $(2n+1)(n+1)$.

Another way: If we happen to know that in general

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6},$$

(a fact that is proved in VI-7, in VI-37 and also in VI-51) there is an alternate approach. Note that

$$2^2 + 4^2 + \cdots + 998^2 = 4(1^2 + 2^2 + \cdots + 499^2).$$

Our sum is therefore equal to

$$(1^2 + 2^2 + \cdots + 999^2) - (2)(4)(1^2 + 2^2 + \cdots + 499^2).$$

A short calculation yields 499500.

Comment. The second solution is, for a number of reasons, less attractive than the first. For one thing, it relies on information about sums of squares which, though "standard," is more complicated than the fact we are trying to establish!

The actual numerical calculation at the end is more unpleasant. Also, the answer, a number of modest size, is expressed as the difference of two much larger numbers. Because of calculator limitations, it is best to avoid such differences.

VI-24. Find four different numbers in arithmetic progression whose sum is equal to the sum of their squares.

Solution. To use an old-fashioned word, the problem is *indeterminate,* that is, it has more than one answer. Start with 1, 2, 3, 4. Their sum is 10 and the sum of their squares is 30. Multiply each of 1, 2, 3, 4 by a scale factor s. The sum is now $10s$ and the sum of the squares is $30s^2$. To make these equal, set $s = 1/3$. Instead of starting with 1, 2, 3, 4, we can start with *any* four distinct real numbers in arithmetic progression whose sum is not 0.

VI-25. Let $a_1, a_2, a_3, \ldots, a_{99}$ and $b_1, b_2, b_3, \ldots, b_{99}$ each be rearrangements of the sequence $1, 2, 3, \ldots, 99$. Show that

$$(a_1 - b_1)(a_2 - b_2)(a_3 - b_3)\cdots(a_{99} - b_{99})$$

is even.

Solution. Before trying to write out an argument, we should experiment with numbers much smaller than 99, such as 3 and 4 and 5. With 4, can we arrange for the analogous product to be odd? Certainly. Let one of the rearrangements be 1, 2, 3, 4 and the other 2, 1, 4, 3. Each of the differences is odd, so their product is odd.

What about with 3? No matter what we try, an odd number ends up paired with an odd number, and thus we get an even product. What about with 5? We make the problem more concrete as follows.

On one side of a room, there are 5 people from class A, three boys wearing numbers 1, 3, 5 and two girls wearing numbers 2 and 4. On the other side of the room are 5 people from class B, again three boys and two girls, with the same numbering scheme. People from class A are supposed to pair up with people from class B to dance. No matter how the pairing is done, some boy will be paired with a boy—this is obvious, there are just too many boys.

We now write out the idea more formally, for 99 people on each side of the room. Note that 50 of the a_i are odd, and only 49 of the b_i are even. So there is at least one i for which both a_i and b_i are odd. But then $a_i - b_i$ is even, and therefore so is our product.

Another way: Let $c_i = a_i - b_i$. The sums of the a_i and of the b_i are equal, so the sum of the c_i is 0. In particular, the sum of the c_i is even. But there are 99 (an odd number) of c_i, and therefore not all the c_i can be odd. Thus c_i is even for at least one i, and hence our product is even.

Comment. The second solution is short and slick—maybe too slick. The first solution arises more naturally out of consideration of the situation.

VI-26. Find the sum of the series

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{8} + \frac{1}{10} + \frac{1}{16} + \frac{1}{20} + \cdots.$$

The denominators are all positive integers which have no prime divisor other than 2 or 5.

Solution. First add up the fractions whose denominator is a power of 2. So we want the sum of the geometric series $1 + 1/2 + 1/4 + \cdots$. This sum is 2.

Next add up the fractions whose denominator is divisible by 5 but by no higher power of 5. That sum is $(1/5)(1 + 1/2 + 1/4 + \cdots)$, that is, $2/5$. Then add up the fractions whose denominator is divisible by 25 but by no higher power of 5. This sum is $2/25$. Continue in this way forever.

Finally, calculate the sum of the infinite geometric series $2 + 2/5 + 2/25 + \cdots$. This sum is $5/2$.

Another way: Imagine expanding out the product

$$\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right)\left(1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots\right).$$

We obtain precisely the sum of the fractions $1/n$ where n has no prime divisor other than 2 or 5. The two infinite geometric series have sum 2 and $5/4$ respectively. Now multiply.

Comments. 1. The second solution is a bit shorter, and generalizes nicely. Let $p_1, p_2, \ldots, p_k$ be distinct primes, and let S be the sum of all numbers $1/n$ where n ranges over the positive integers which have no prime factor other than the p_i. Then

$$S = \left(1 + \frac{1}{p_1} + \frac{1}{p_1^2} + \cdots\right)\left(1 + \frac{1}{p_2} + \frac{1}{p_2^2} + \cdots\right)\cdots\left(1 + \frac{1}{p_k} + \frac{1}{p_k^2} + \cdots\right).$$

Now sum the k infinite geometric series above. We get

$$\frac{1}{S} = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right).$$

The idea of this solution is due to Euler, the greatest mathematician of the eighteenth century. Elaborate variants have been used to study the distribution of primes ever since.

2. We operated with infinite sums as if they were long finite sums, and in particular rearranged the order of summation freely. That can be done without risk for series that only have positive terms. But this kind of rearrangement may be illegitimate in some cases. For example, it turns out that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4},$$

but the series can be rearranged so as to have sum 1024, or any other number we like!

VI-27. Male bees have only one parent—a mother, of course—while female bees have two, one of each sex. Bea, a female bee, is working on her family tree. Find the largest number of female ancestors that Bea could have who *exactly* ten generations back from her.

Solution. Draw a diagram, in this case a family tree. We are only tracing back for ten generations, so numbers stay small. But we need to stay organized and look carefully at what happens one, two, three, and four generations back. After a while we get structural insight, and everything becomes clear.

Let F_n be the *largest* number of female ancestors that Bea could have who are exactly n generations back from her, and let M_n be the corresponding number of male ancestors. Bea has one mother and one father, so $F_1 = M_1 = 1$. It follows that $F_2 = 2$ and $M_2 = 1$, and therefore $F_3 = 3$ and $M_3 = 2$, and therefore $F_4 = 5$ and $M_4 = 3$.

In general, the reproductive habits of bees are summarized by the recurrences

$$F_{n+1} = F_n + M_n, \qquad \text{and} \qquad M_{n+1} = F_n.$$

These two recurrences can be turned into a computer program, and F_n can be calculated for reasonable-sized n (after a while, the numbers are enormous). It is not hard to compute up to $n = 10$ by hand, but first we eliminate the males from the equation.

Replace n by $n + 1$ in the first equation; so $F_{n+2} = F_{n+1} + M_{n+1}$. Since $M_{n+1} = F_n$, we obtain

$$F_{n+2} = F_{n+1} + F_n,$$

the famous recurrence that produces the *Fibonacci sequence* and related sequences. Note that $F_1 = 1$ and $F_2 = 2$. So the sequence goes 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, and so on, and $F_{10} = 89$.

Comments. 1. In most cases, because of inbreeding, the number of *different* female ancestors at exactly 10 generations back is less than 89. And F_{100} is exceedingly large; the ancestors 100 generations back can't possibly be all different.

2. For those familiar with matrices, the recurrences have an interesting matrix interpretation. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad V_n = \begin{pmatrix} F_n \\ M_n \end{pmatrix}.$$

The recurrences can be rewritten as $V_{n+1} = AV_n$. In particular, $V_2 = AV_1$, $V_3 = A(AV_1) = A^2V_1$, and in general $V_n = A^{n-1}V_1$. From the matrix point of view, the sequence of vectors V_1, V_2, V_3 and so on feels like a geometric sequence. Everything is connected to everything else!

VI-28. Draw a 1×1 square. Next to it, place a second square with side three-fifths the side of the first square. Next to that, place a third square with side three-fifths the side of the second square, and go on like that forever. Find the total area covered by the squares. (Please see Figure 6.1.)

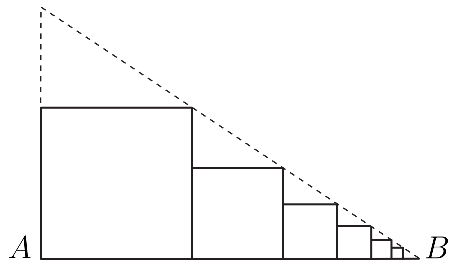

Figure 6.1: Summing Infinitely Many Squares

Solution. We calculate the areas of the successive squares. The first square has area 1. The second has side t, where $t = 3/5$, and therefore it has area t^2. Similarly, the third square has area t^4, the fourth has area t^6, and so on.

We need to compute the infinite sum $1 + t^2 + t^4 + t^6 + \cdots$. By the standard formula for the sum of an infinite geometric progression, this sum is $1/(1-t^2)$. For $t = 3/5$, the sum is $25/16$.

Another way: Let the region covered by the squares be $\mathcal{C}$, and let c be its area. Then $\mathcal{C}$ can be viewed as a 1×1 square, with a three-fifths scale version of $\mathcal{C}$ appended on the right. But a linear scale factor of $3/5$ scales areas by the factor $(3/5)^2$, and therefore

$$c = 1 + \frac{9c}{25}.$$

Solve for c. It turns out that $c = 25/16$.

As a variant, we can enlarge $\mathcal{C}$ by a scale factor of $5/3$. So the new area is $(5/3)^2 c$. The scaled region consists of a $(5/3) \times (5/3)$ square, together with a copy of $\mathcal{C}$. We obtain the equation $25c/9 = 25/9 + c$ and then solve for c.

Comment. In the second solution we used a geometric trick and the nowadays fashionable notion of self-similarity to sum the series. The idea is the same as the usual method in which we look at $a + ar + ar^2 + \cdots$, let the sum be S, note that $Sr = ar + ar^2 + ar^3 + \cdots$, and conclude that $S - Sr = a$. But the squares version feels more concrete, less the result of symbol manipulation. The same argument can be used to find the sum $a + ar + ar^2 + \cdots$ for any r with $0 < r < 1$. Instead of using $3/5$ as the linear scale factor, we use $\sqrt{r}$.

We can get additional information out of Figure 6.1. Let $r = 3/5$. The length of AB is the sum $1 + r + r^2 + \cdots$. A similar triangles argument, or a slope calculation using the upper right corners of the two largest squares, shows that this length is $1/(1-r)$.

VI-29. Look at the sequence 1, 2, 3, 6, 7, 14, 15, ... (To get the "next" term, alternately double and add 1.) Find the 100-th term. Find the sum of the first 100 terms.

Solution. Let the n-th term of the sequence be a_n. Suppose first that n is odd, say $n = 2k - 1$. Note the pattern 1, 3, 7, 15, ..., which suggests that a_{2k-1} may be $2^k - 1$.

We give an informal argument that the pattern continues. Suppose that we hired a consultant who did an expensive series of calculations and reported that $a_1 = 2^1 - 1$, $a_3 = 2^2 - 1$, $a_5 = 2^3 - 1$, and so on up to $a_{75} = 2^{38} - 1$. Then the money ran out.

Can we check cheaply whether the pattern still holds when $k = 39$? To get a_{76} we double a_{75}. So $a_{76} = 2^{39} - 2$. Then we add 1 to find a_{77}, so $a_{77} = 2^{39} - 1$. It follows that the pattern continues to hold when $k = 39$. Go quickly to the next k, by doubling and adding 1. The result is $2^{40} - 1$, so yes, things are fine here too. And so on.

Since $a_{2k-1} = 2^k - 1$, it follows that $a_{2k} = 2^{k+1} - 2$, and in particular $a_{100} = 2^{101} - 2$.

To add up the first 100 terms, note that $a_{2k} = 2a_{2k-1}$, and therefore $a_{2k-1} + a_{2k} = 3a_{2k-1}$. So the required sum is

$$3(1 + 3 + \cdots + (2^{50} - 1)), \quad \text{that is,} \quad 3(2 + 4 + 8 + \cdots + 2^{50}) - 150.$$

Finally, use standard facts about $1 + 2 + 4 + \cdots + 2^{50}$ to conclude that our sum is $3 \cdot 2^{51} - 156$.

Comment. The argument that uses a hypothetical consultant is an attempt to capture the idea of mathematical induction without introducing "theoretical" material about properties $S(n)$, and $S(k)$ implying $S(k + 1)$. That part can be done *after* we really know what induction is about.

VI-30. Let a_n be the n-digit number whose first $n-1$ digits are 3 and whose last digit is 5. What does the decimal representation of a_n^2 look like?

Solution. Calculate 5^2, 35^2, 335^2, 3335^2. From the results, it is natural to conjecture that the decimal expansion of a_n^2 consists of $n - 1$ consecutive 1's followed by n 2's, and finally a 5.

We could prove this conjecture by looking carefully at the usual multiplication process. But it is easier to note that the n-digit number all of whose digits are 3 is $(10^n - 1)/3$, and therefore $a_n = (10^n - 1)/3 + 2 = (10^n + 5)/3$. It follows that $a_n^2 = (10^n + 5)^2/9$. But

$$(10^n + 5)^2 = 10^{2n} + 10^{n+1} + 25 = (10^{2n} - 1) + (10^{n+1} - 1) + 27.$$

Divide by 9. The result is $x + y + 3$ where x has decimal expansion that consists of $2n$ consecutive 1's, while y has $n + 1$ consecutive 1's. Add x and y in the usual way. We get $n - 1$ consecutive 1's followed by $n + 1$ 2's. Now add the 3.

VI-31. The sequence (q_n) is defined as follows: $q_1 = 1$, and for any $n \geq 1$, $q_{n+1} = q_n + 3n$. Find an explicit formula for q_n in terms of n.

Solution. We have $q_2 = q_1 + 3 \cdot 1$, $q_3 = q_2 + 3 \cdot 2$, $q_4 = q_3 + 3 \cdot 3$, and so on, with finally $q_n = q_{n-1} + 3 \cdot (n-1)$. Add up the left-hand sides, add up the right-hand sides. We get

$$q_2 + \cdots + q_n = q_1 + \cdots + q_{n-1} + 3\left(1 + \cdots + (n-1)\right).$$

Now cancel $q_2 + \cdots + q_{n-1}$ from both sides:

$$q_n = 1 + 3\left(1 + 2 + \cdots + (n-1)\right) = (3n^2 - 3n + 2)/2.$$

Another way: Guess, somehow, that the answer looks like $an^2 + bn + c$ for some constants a, b, and c. We know that $q_1 = 1$, $q_2 = 4$, and $q_3 = 10$. When we substitute in turn 1, 2, and 3 for n in the formula $q_n = an^2 + bn + c$, we get

$$a + b + c = 1, \qquad 4a + 2b + c = 4, \qquad \text{and} \qquad 9a + 3b + c = 10.$$

Solve this system to find a, b, and c. It turns out that $a = 3/2$, $b = -3/2$, and $c = 1$.

So far we have proved roughly speaking nothing. All we know is that with these values of a, b, and c, $q_n = an^2 + bn + c$ for $n = 1$, 2, and 3. Since a, b, and c were chosen to make this happen, the result isn't impressive!

Let $r_n = (3n^2 - 3n + 2)/2$. We know that $q_n = r_n$ when $n = 1$. We now show that $q_n = r_n$ for *every* n.

Calculate $r_{n+1} - r_n$. It turns out to be $3n$. So the sequence (r_n) agrees with (q_n) when $n = 1$, and satisfies the same recurrence relation. The two sequences are therefore the same everywhere. The idea that we used has many applications. It is called the *Method of Undetermined Coefficients.*

Another way: Here is a more streamlined version of the preceding solution. Let $q_n = an^2 + bn + c$. Then

$$q_{n+1} - q_n = \left(a(n+1)^2 + b(n+1) + c\right) - \left(an^2 + bn + c\right) = 2an + (a + b).$$

The choice $a = 3/2$, $b = -3/2$ ensures that the recurrence $q_{n+1} = q_n + 3n$ holds. The constant c is still at our disposal: the choice $c = 1$ makes $q_1 = 1$.

VI-32. Let P be the infinite product $2^{1/2} \cdot 4^{1/4} \cdot 8^{1/8} \cdots$. Find a simple expression for P.

Solution. Note that $4^{1/4} = 2^{2/4}$, $8^{1/8} = 2^{3/8}$, and so on. Thus if infinite products behave like finite products, then

$$P = 2^S \qquad \text{where} \qquad S = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots.$$

We calculate S by imitating the usual argument for the sum of a geometric series. Note that

$$\begin{aligned} S = 2S - S = 1 &+ \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \cdots \\ &- \frac{1}{2} - \frac{2}{4} - \frac{3}{8} - \frac{4}{16} - \cdots. \end{aligned}$$

Thus $S = 1 + 1/2 + 1/4 + 1/8 + \cdots = 2$, and therefore $P = 2^2 = 4$.

VI-33. (a) Observe that $\frac{1}{1-u} + \frac{1}{1+u} = \frac{2}{1-u^2}$. If $x \neq \pm 1$, find a simple expression for $S_n(x)$, where

$$S_n(x) = \frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \cdots + \frac{2^n}{1+x^{2^n}}.$$

(b) Find $S_9(2)$ correct to 200 decimal places.

Solution. (a) Calculate $1/(1-x) + S_n(x)$. Start adding: $1/(1-x) + 1/(1+x) = 2/(1-x^2)$. Now add $2/(1+x^2)$ and simplify: we get $4/(1-x^4)$. Add $4/(1-x^4)$: we get $8/(1-x^8)$. Keep on adding. After a while we find that

$$\frac{1}{1-x} + S_n(x) = \frac{2^{n+1}}{1-x^{2^{n+1}}} \qquad \text{so} \qquad S_n(x) = \frac{2^{n+1}}{1-x^{2^{n+1}}} - \frac{1}{1-x}.$$

Note that $1/(1-x)$ acted like a catalyst: it helped combine things, and then got taken away.
(b) Set $x = 2$ in the result of part (a):

$$S_9(2) = 1 - \frac{1024}{2^{1024}-1}.$$

Since $2^{1024} - 1 > 2^{1023}$, $S_9(2)$ differs from 1 by less than 2^{-1013}. To estimate 2^{-1013}, we can use logarithms to the base 10. More simply, note that since $2^{10} > 10^3$,

$$2^{1013} > 2^{1010} > 10^{303},$$

so $S_9(2) = 1.000\ldots$ to more than 303 decimal places!

VI-34. Jo got a job with a starting wage of \$1 a day. Each day, Jo's wage went up by a dollar, so Jo made \$2 on day 2, \$3 on day 3, and so on. Jo worked every day of a 365 day year. How much did Jo earn that year? Hint: Jo's friend Mo started at \$365 a day, and every day Mo's wage went *down* by a dollar. How much did Jo and Mo earn together on the first day? On the second?

Solution. Since Jo's wage goes up by a dollar a day, while Mo's goes down by the same amount, their combined daily income stays constant at 366. So in a year they earn a total of $365 \cdot 366$. By symmetry, Jo earns precisely as much during the year as Mo does, so Jo earns $(365 \cdot 366)/2$.

Comment. The wording is meant to make more concrete the idea behind the usual proof that $1+2+3+\cdots+n = n(n+1)/2$. Most people have a tactile grasp of money by an early age, so the story is more persuasive than the "symbol manipulation" reversal of the sum.

VI-35. In the eighteenth century, Euler showed that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots=\frac{\pi^2}{6}.$$

Use Euler's result to calculate A, where

$$A=\frac{1}{1^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\cdots$$

(the denominators are the perfect squares divisible by neither 2 nor 3).

Solution. It is easier to add up the terms in Euler's sum that are *missing* from the above sum. We calculate B, where

$$B=\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\cdots$$

(the denominators are the perfect squares divisible by 2 or 3 or both).

First add up the terms whose denominator is the square of an even number:

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots=\left(\frac{1}{4}\right)\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)=\frac{\pi^2}{24}.$$

Next add up the terms whose denominator is the square of a multiple of 3. A calculation like the one above shows that this sum is $\pi^2/54$.

The terms whose denominator is the square of a multiple of 6 occurred in both sums, so they were "counted twice." These terms add up to $\pi^2/216$. It follows that

$$B=\frac{\pi^2}{24}+\frac{\pi^2}{54}-\frac{\pi^2}{216}=\frac{\pi^2}{18}$$

and therefore $A=\pi^2/6-\pi^2/18=\pi^2/9$.

Another way: We can let the algebra do more of the thinking. Let E be Euler's sum. Then

$$E=A\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\cdots\right)\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\cdots\right).$$

(Imagine multiplying out the three series: every perfect square appears once and only once as a denominator.) The two geometric series on the right have sum $1/(1-1/2^2)$ and $1/(1-1/3^2)$, and therefore

$$A=E\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right).$$

The same idea can be used to compute for example the sum of the terms whose denominators are, for example, divisible by none of 2, 5, or 7, or indeed by none of a specified finite collection of primes.

VI-36. Let $f(x) = 1 + 3x + 5x^2 + 7x^3 + \cdots + 9999x^{4999}$. Find a simple closed form expression for $f(x)$, valid when $x \neq 1$.

Solution. We might as well sum to $(2n+1)x^n$—for one thing, n takes less effort to type than 4999. Bump exponents up by 1 as follows:

$$\begin{aligned} f(x) &= 1 + 3x + 5x^2 + 7x^3 + \cdots + (2n+1)x^n \\ xf(x) &= \qquad x + 3x^3 + 5x^3 + \cdots + (2n-1)x^n + (2n+1)x^{n+1}. \end{aligned}$$

Subtract. We get $(1-x)f(x) = 1 + 2x(1 + x + x^2 + \cdots + x^{n-1}) - (2n+1)x^{n+1}$.

For $x \neq 1$, the series $1 + x + x^2 + \cdots + x^{n-1}$ has sum $(1-x^n)/(1-x)$. Substitute and simplify. After a while we get

$$f(x) = \frac{1 + x - (2n+3)x^{n+1} + (2n+1)x^{n+2}}{(1-x)^2}.$$

Comments. 1. Using similar ideas, we can tackle sums such as

$$1^2 + 2^2x + 3^2x^2 + \cdots + n^2x^{n-1}.$$

Let $g(x)$ be the above sum. Then $(1-x)g(x)$ is a close relative of $f(x)$.

2. Things become prettier if instead of summing to n, we sum "to infinity" (this only makes sense if $|x| < 1$). There is a natural calculus-based approach which we illustrate with the sum $1 + 2x + 3x^2 + \cdots$.

Let $h(x) = 1 + x + x^2 + \cdots$. Then, *maybe,* $h'(x) = 1 + 2x + 3x^2 + \cdots$. The "maybe" is there because we are assuming that the differentiation of "infinite sums" behaves like differentiation of finite sums. It doesn't always, but does in many important cases including this one.

We know that $h(x) = 1/(1-x)$ if $|x| < 1$. Thus $h'(x) = 1/(1-x)^2$, and therefore $1 + 2x + 3x^2 + \cdots = 1/(1-x)^2$.

VI-37. It so happens that there are constants a, b, c, and d such that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = an^3 + bn^2 + cn + d.$$

for all n. Starting from first principles, find the constants and prove that the formula holds.

Solution. If the formula holds for all n, it holds in particular when $n = 1$. That gives the equation $1 = a + b + c + d$. From the fact that the equation holds when $n = 2$, we get $5 = 8a + 4b + 2c + d$. Similarly, from the fact that the formula holds at $n = 3$ and 4, we get two more linear equations. Finally, solve the four linear equations in four unknowns. This is not very pleasant, and after all that work we still have to check that the formula so obtained really does work for every n, not just at 1, 2, 3, and 4.

Another way: Let $s_n = 1^2 + 2^2 + \cdots + n^2$ and $q_n = an^3 + bn^2 + cn + d$. We want to choose the constants so that $q_n = s_n$ for all n. Note that $s_n = s_{n-1} + n^2$ and $s_1 = 1$. So we need to have $q_n - q_{n-1} = n^2$. But

$$q_n - q_{n-1} = 3an^2 - 3an + 2bn + a - b + c,$$

so we want the right-hand side to be identically equal to n^2. That forces $a = 1/3$, $b = 1/2$, and $c = 1/6$.

With this choice of a, b, and c, the sequences (s_n) and (q_n) satisfy the same recurrence relation. To make the sequences identical, we need to choose d so that $q_1 = 1$. But $q_1 = a + b + c + d = 1 + d$, so $q_1 = 1$ if $d = 0$.

Comment. The same strategy will produce a formula for the sum of the first n cubes—we look for an expression of shape $an^4 + bn^3 + cn^2 + dn + e$—the first n fourth powers, and so on. Things do get messier, but not very fast.

VI-38. Let $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Show that

$$F_n = \left(\frac{1}{\sqrt{5}}\right)\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \qquad \text{for all } n.$$

Hint: Show that $\left(\frac{1+\sqrt{5}}{2}\right)^n$ and $\left(\frac{1-\sqrt{5}}{2}\right)^n$ satisfy the recurrence.

Solution. Let G_n be the messy expression to the right of the equals sign. We want to show that $F_n = G_n$ for all n. It is easy to see that $G_0 = 0$ and $G_1 = 1$, so the sequence of G's begins in the same way as the sequence of F's. If we can show that $G_{n+2} = G_{n+1} + G_n$ for all n, it will follow that the two sequences are the same everywhere.

To make things less messy, let $U_n = \left(\frac{1+\sqrt{5}}{2}\right)^n$, and let $V_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$. If we can prove that $U_{n+2} = U_{n+1} + U_n$, and the analogous formula for the V's, the desired recurrence formula for the G's will follow.

Use the abbreviation τ for $(1+\sqrt{5})/2$. Note that τ is a root of the equation $x^2 - x - 1 = 0$, and therefore $\tau + 1 = \tau^2$. We have

$$U_{n+1} + U_n = \tau^{n+1} + \tau^n = \tau^n(\tau + 1) = \tau^n\tau^2 = \tau^{n+2} = U_{n+2}.$$

The calculation for the V's is almost identical.

Comments. 1. The sequence (F_n) is the famous *Fibonacci sequence*. The above "closed form" formula for F_n was mentioned by de Moivre in 1718 but is usually called *Binet's Formula.* The closed form formula, though interesting, is less useful than the recurrence.

The number τ, often called the *golden ratio,* turns up in many areas of mathematics. For example, the length of a diagonal of a regular pentagon is τ times the side. As a consequence, $\cos 36° = \tau/2$.

There is much *non-mathematical* interest in τ—see the Internet for some strange speculations. It is often written that the ancient Greeks thought that the rectangle with long side τ times the short has the most beautiful shape, and that they used the golden ratio in their architecture and sculpture. But there is no evidence that any ancient Greek ever expressed an opinion on the relative beauty of rectangles.

2. Let (W_n) be a sequence that satisfies the recurrence $W_{n+2} = pW_{n+1} + qW_n$. If the roots of $x^2 - px - q = 0$ are distinct and real, say λ and μ, then $W_n = a\lambda^n + b\mu^n$ for some numbers a and b that can be calculated once we know W_0 and W_1.

VI-39. How many triangles with integer sides have one side equal to 12 and no side greater than 12? What about if we replace 12 by 13? Generalize.

Solution. Let the other two sides be a and b. The only constraint on a and b is that they be positive integers no greater than 12 such that $a + b > 12$. (The sum of two sides of a triangle is greater than the third side.)

Without loss of generality we may take $a \leq b$. Start counting. If $a = 1$, there is one possibility for b, namely 12. If $a = 2$, then b can be 11 or 12 . If $a = 3$, then b can be 10 or 11 or 12. Go on until $a = 6$, when b can be any of 7, 8, ..., 11, or 12. If $a = 7$, the possible b are again 7 to 12, if $a = 8$ they are 8 to 12, and so on, until at $a = 12$ we can only have $b = 12$. So there are $2(1 + 2 + \cdots + 5 + 6)$, namely 42, triangles.

For 13 the calculation is essentially the same. The only difference is that we get the pattern 1, 2, 3, 4, 5, 6, 7, 6, 5, 4, 3, 2, 1. The sum is 49.

The general case is done in exactly the same way. When n is even we get the pattern $1, 2, 3, \ldots, k-1, k, k, k-1, \ldots, 1$, where $k = n/2$. The sum is $(n^2+2n)/4$. If n is odd, let $n = 2k+1$. We get the pattern $1, 2, \ldots, k, k+1, k, \ldots, 1$, and the sum is $(n^2 + 2n + 1)/4$. Formulas as nice as these urge us to look for

Another way: Look at Figure 6.2, which illustrates the cases $n = 8$ and $n = 9$, but the square on the left works for any even n, and the one on the right for any odd n.

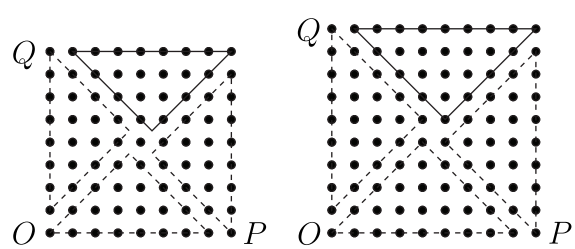

Figure 6.2: Counting Triangles with Integer Sides

Let O be the origin and let OP be along the x-axis, with $OP = n$ (in the picture, $OP = 8$). The triangle $\mathcal{T}$ surrounded by the solid line contains all pairs (x, y) of integers (dots) such that $x \leq y \leq n$ and $x + y > n$. The triangles surrounded by the dashed lines all have the same number of dots as $\mathcal{T}$. Together, the four triangles contain all the points of the $(n+1)\times(n+1)$ dot array except for the central dot, so $\mathcal{T}$ contains $((n+1)^2-1)/4$, that is, $(n^2+2n)/4$ dots. The picture on the right yields almost identical information, except that there is no central dot, so $\mathcal{T}$ contains $(n+1)^2/4$ dots.

Comment. We sketch a recurrence relation approach. Let $\mathcal{T}_n$ be the set of triangles with integers sides, with one side n and no side greater than n. Let t_n be the number of triangles in $\mathcal{T}_n$.

Take a triangle in $\mathcal{T}_n$ with sides a, b, and n. By adding 1 to each of a and b, and 2 to n, we get a triangle in $\mathcal{T}_{n+2}$. The only triangles we can't get this way are the ones that have two or more sides equal to $n+2$; there are $n+2$ of these.

We conclude that the sequence (t_n) satisfies the recurrence $t_{n+2}=t_n+n+2$. It is clear that $t_1=1$ and $t_2=2$. We find a formula for t_n when n is even. (The odd case is handled similarly.)

Let $n=2k$. We have $t_4=t_2+2+2$, $t_6=t_4+4+2$, $t_8=t_6+6+2$ and so on up to $t_{2k}=t_{2k-2}+2k-2+2$. Add up the left-hand sides, add up the right-hand sides, then note that almost all the t_i cancel. We get

$$t_{2k}=t_2+(2+4+6+\cdots+(2k-2))+2(k-1).$$

But $t_2=2$ and $2+4+6+\cdots+(2k-2)=(k-1)k$, so $t_{2k}=k^2+k$. Thus $t_n=(n^2+2n)/2$.

VI-40. An arithmetic progression has four terms. The product of the end terms is 20, and the product of the middle terms is 70. Find the sum of the terms. And how many such arithmetic progressions are there?

Solution. We should take as much advantage of symmetry as possible. So let the terms be $b-3k$, $b-k$, $b+k$, and $b+3k$. Then $b^2-9k^2=20$ and $b^2-k^2=70$. Thus $k^2=25/4$ and $b^2=305/4$.

The sum of the terms, namely $4b$, is therefore $\pm 2\sqrt{305}$. Since b and k can, independently, each take on two values, there are four different arithmetic progressions with the desired properties. They are not very different: if a_1, a_2, a_3, a_4 is one of them, we get the others by writing the sequence backwards and/or multiplying by -1.

VI-41. In a four-term geometric progression, the end terms add up to 61 and the middle terms add up to 36. Find the terms.

Solution. Let the terms be a, ar, ar^2, and ar^3. Then

$$a+ar^3=61 \qquad \text{and} \qquad ar+ar^2=36.$$

Thus $36a(1+r^3)=61a(r+r^2)$. But $1+r^3=(1+r)(1-r+r^2)$. Since neither a nor $1+r$ can be 0, we cancel $a(1+r)$ from both sides and obtain $36r^2-97r+36=0$.

The solutions are $r = 9/4$ and $r = 4/9$. If $r = 9/4$, then $a = 64/13$, so the terms are 64/13, 144/13, 324/13, and 729/13. If $r = 4/9$ we get the same terms in reverse order.

Comment. This problem, with the sum of the middle terms equal to 12 and the sum of the end terms equal to 20, occurs in Isaac Newton's *Lectures on Algebra.*

VI-42. Find a simple expression for

$$\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdot\frac{4^3-1}{4^3+1}\cdots\frac{100^3-1}{100^3+1}\cdot\frac{101^3-1}{101^3+1}.$$

Solution. Use the factorization

$$\frac{k^3-1}{k^3+1}=\frac{k-1}{k+1}\cdot\frac{k^2+k+1}{k^2-k+1}.$$

First multiply together the terms of the form $(k-1)/(k+1)$ as k goes from 2 to 101. We get

$$\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdot\frac{4}{6}\cdots\frac{98}{100}\cdot\frac{99}{101}\cdot\frac{100}{102}.$$

There is a lot of cancellation: the product is $2/(101\cdot 102)$. Now evaluate

$$\frac{2^2+2+1}{2^2-2+1}\cdot\frac{3^2+3+1}{3^2-3+1}\cdot\frac{4^2+4+1}{4^2-4+1}\cdots\frac{100^2+100+1}{100^2-100+1}\cdot\frac{101^2+101+1}{101^2-101+1}.$$

Note that in general $k^2+k+1=(k+1)^2-(k+1)+1$. So any numerator is equal to the "next" denominator, there is a lot of cancellation, and we get $(101^2+101+1)/(2^2-2+1)$.

The full product turns out to be 10303/15453.

Comment. If we multiply instead up to the term $(n^3-1)/(n^3+1)$, the same argument shows that the product is $(2/3)(n^2+n+1)/(n^2+n)$. As $n\to\infty$, the ratio $(n^2+n+1)/(n^2+n)$ approaches 1, and therefore the "infinite" product is 2/3.

VI-43. Let $a_1=1$, $a_2=-1$, $a_3=1$. For $k\geq 1$ let $a_{k+3}=a_k a_{k+2}$. Calculate a_N where $N=10000$.

Solution. We can calculate easily as far as we want, but going all the way to 10000 is not an attractive prospect. Compute for a while. The first few terms are 1, −1, 1, 1, −1, −1, −1, 1, −1, 1, 1, −1, and so on.

Note that the term a_8, a_9, and a_{10} are respectively equal to a_1, a_2, and a_3. Thus the initial segment $a_1, a_2, \ldots, a_7$ repeats endlessly. If k is a multiple of 7, then a_k is at the end of a cycle. Since $10000=7\cdot 1428+4$, we conclude that $a_N=a_4=1$.

Comment. The cycling was pleasant, but it shouldn't come as a surprise. Imagine looking at a_1, a_2, a_3, then at a_2, a_3, a_4, then at a_3, a_4, a_5, and so on. Each is a string of length 3 made up of the letters 1 and/or -1. There are only 8 such strings, so by the time we have looked at a_9, a_{10}, a_{11} we must have seen some string twice.

In fact, the string $1, 1, 1$ can't occur: if it ever does, then by calculating backwards we find that all the a_i must be 1. So by the time we have looked at a_8, a_9, a_{10} we must have seen some string twice. Thus it turns out that we were unlucky: the cycle length is the longest it could conceivably be. And it isn't just by chance that a_1, a_2, a_3 is the *first* string that we see again. (Hint: Extend the calculation backwards, to a_0, a_{-1}, and so on, then go forward again.)

We can thus freely invent similar problems with confidence that there will be cycling. The same problem can also be given in disguise. Let $b_1 = 0$, $b_2 = 1$, and $b_3 = 0$. For any $k > 3$, let b_k be the remainder when $b_{k-3} + b_{k-1}$ is divided by 2.

VI-44. Find 50 consecutive odd numbers whose sum is 10000.

Solution. Let the first number in the sequence be $2a + 1$. Then the numbers are $2a + 1$, $2a + 3$, $\ldots$, $2a + 99$. We have an arithmetic progression with 50 terms. By the usual method for summing an arithmetic progression, the sum of the 50 numbers is $100a + 2500$. But this sum is 10000 and therefore $a = 75$. The numbers are 151, 153, $\ldots$, 249.

Another way: The sum of the first and the fiftieth is the same as the sum of the second and the forty-ninth, and so on. There are 25 such pairs, so each sum is 400. Thus the sum of the two middle terms is also 400, and therefore these terms are 199 and 201. Now we can write down the whole sequence.

Another way: Let's try to guess the answer. Guess for example that the numbers are -49, -47, $\ldots$, -1, 1, $\ldots$, 47, 49. Not a very good guess—the sum is 0! To get a sum of 10000, add 10000/50 to each number.

VI-45. A path is 2 feet wide and 18 feet long. We want to pave the path with 1×2 concrete paving blocks. How many ways are there to accomplish the task? Hint: Start at one end and put down a paving block.

Solution. Let $F(n)$ be the number of ways of paving a $2 \times n$ path. To get some insight, we should start by calculating $F(1)$, $F(2)$, and so on for a while. We find that $F(1) = 1$, $F(2) = 2$, and $F(3) = 3$. Is it that simple? No, $F(4) = 5$, and with a little effort $F(5) = 8$. Interesting!

Suppose that the path goes from East to West, as in Figure 6.3. The first block can be put at the eastern end of the path with its long side (i) going North–South or (ii) East–West. If we do (i), there is a $2 \times (n - 1)$ path left to pave, and that can be done in $F(n - 1)$ ways. If we do (ii), then we are forced to put another block with its long side going East–West and are left with a $2 \times (n - 2)$ path, which can be paved in $F(n - 2)$ ways. We have obtained the recurrence formula

$$F(n) = F(n - 1) + F(n - 2).$$

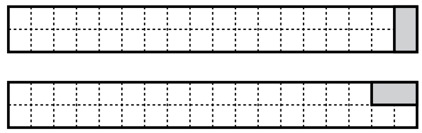

Figure 6.3: Paving a Path

Note that $F(0) = F(1) = 1$. (The $F(0)$ is not quite a joke. There is exactly one way to pave a path of length 0, namely do nothing!) If we feel uncomfortable with $F(0)$, we can instead observe that $F(2) = 2$.

By the recurrence formula, $F(3) = 3$, $F(4) = 5$, and so on. The next numbers are 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181. So $F(18) = 4181$.

Comment. Once again we have bumped into the *Fibonacci sequence,* which shows up surprisingly often in various areas of mathematics, among others geometry, number theory, the theory of algorithms, and mathematical biology. It all started with a harmless little puzzle about rabbits in Leonardo of Pisa's *Liber Abbaci* (1202).

We can do the same sort of computation if we want to pave a $3 \times n$ path with 1×3 blocks. The recurrence relation is $F(n) = F(n-1) + F(n-3)$, with $F(0) = F(1) = F(2) = 1$.

VI-46. Find integers m and n with $m < n$ such that

$$\frac{1}{(m)(m+1)} + \frac{1}{(m+1)(m+2)} + \cdots + \frac{1}{(n-1)(n)} = \frac{1}{47}.$$

Hint: $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.

Solution. For concreteness, look for example at

$$\frac{1}{20 \cdot 21} + \frac{1}{21 \cdot 22} + \cdots + \frac{1}{36 \cdot 37} + \frac{1}{37 \cdot 38}.$$

By the formula of the hint, the above sum is equal to

$$\left(\frac{1}{20} - \frac{1}{21}\right) + \left(\frac{1}{21} - \frac{1}{22}\right) + \cdots + \left(\frac{1}{36} - \frac{1}{37}\right) + \left(\frac{1}{37} - \frac{1}{38}\right).$$

Note the wholesale cancellation: the sum is $1/20 - 1/38$. In the general case the sum is $1/m - 1/n$. This kind of series is called a *telescoping series.* They like to come to mathematics contests, but there are also many genuine applications. This chapter is full of telescoping series.

So we are looking for integers m and n such that

$$\frac{1}{m} - \frac{1}{n} = \frac{1}{47}.$$

In the relation $1/k(k+1) = 1/k - 1/(k+1)$, set $k + 1 = 47$. We obtain $1/47 = 1/46 - 1/(46 \cdot 47)$. So we can take $m = 46$ and $n = 2162$.

Comment. To find all solutions, rewrite the equation as $mn + 47m - 47n = 0$ (but remember that the variables can't be 0). In a move reminiscent of completing the square, we get equivalently

$$(m - 47)(n + 47) = -47^2.$$

So $m - 47$ and $n + 47$ must be factors of -47^2 with $m < n$. Also, we can't have m negative and n positive, for then our sum would involve division by 0. Setting $m-47 = -1$ and $n+47 = 47^2$ gives the solution already found, and $m-47 = -47^2$, $n + 47 = 1$ gives $m = -2161$, $n = -46$. If instead of 47 we use a number with several divisors, such as 48, the number of solutions jumps.

VI-47. For any real number x, let $\lfloor x \rfloor$ be the largest integer less than or equal to x. Find

$$\lfloor 2 \rfloor + \lfloor \log_2 3 \rfloor + \lfloor \log_2 4 \rfloor + \cdots + \lfloor \log_2 1022 \rfloor + \lfloor \log_2 1023 \rfloor.$$

Solution. Note that $\lfloor \log_2 2 \rfloor = \lfloor \log_2 3 \rfloor = 1$, and $\lfloor \log_2 4 \rfloor = \lfloor \log_2 5 \rfloor = \lfloor \log_2 6 \rfloor = \lfloor \log_2 7 \rfloor = 2$, and so on, with finally $\lfloor \log_2 512 \rfloor = \cdots = \lfloor \log_2 1023 \rfloor = 9$. So we want to compute the sum

$$1 \cdot 2 + 2 \cdot 4 + 3 \cdot 8 + \cdots + 9 \cdot 512.$$

We could just use the calculator, but maybe it is easier to use a general technique. Let S be the above sum. Then $2S = 1 \cdot 4 + 2 \cdot 8 + \cdots + 9 \cdot 1024$. Thus $2S - S = 9 \cdot 1024 - (2 + 4 + \cdots + 512)$. Finally, $2 + 4 + \cdots + 512 = 1024 - 2$, so $S = 8194$.

VI-48. (a) Suppose that $x \neq 1$. Find a simple expression for $P_n(x)$, where

$$P_n(x) = (1 + x)(1 + x^2)(1 + x^4) \cdots (1 + x^{2^n}).$$

(b) Suppose that $|x| < 1$. What happens to $P_n(x)$ as n grows without bound?

Solution. (a) Calculate. We get $P_0(x) = 1 + x$, $P_1(x) = (1 + x)(1 + x^2) = 1 + x + x^2 + x^3$, and $P_3(x) = 1 + x + x^2 + \cdots + x^7$. A clear pattern is emerging. We want to convince ourselves, and others, that the pattern really is there. Calculate the next product. To multiply the product of the first three terms by $1 + x^8$, multiply by 1, also by x^8, and add. We get

$$1 + x + x^2 + \cdots + x^7 + x^8(1 + x + x^2 + \cdots + x^7).$$

The first half gives the powers of x up to x^7, the second half gives the powers from x^8 to x^{15}, so the sum gives the powers up to x^{15}. The general situation is now clear:

$$P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^{2^{n+1}-1}.$$

This is a geometric series. If $x \neq 1$, the sum is

$$\frac{1 - x^{2^{n+1}}}{1 - x}.$$

Another way: Calculate $(1 - x)P_n(x)$. We have $(1 - x)(1 + x) = 1 - x^2$. But $(1 - x^2)(1 + x^2) = 1 - x^4$, and $(1 - x^4)(1 + x^4) = 1 - x^8$, and so on. Thus

$$(1 - x)P_n(x) = 1 - x^{2^{n+1}}.$$

(b) If $|x| < 1$, then as n grows without bound, $x^{2^{n+1}}$ approaches 0, and therefore $P_n(x)$ approaches $1/(1 - x)$. So if $|x| < 1$ then the *infinite product* $(1 + x)(1 + x^2)(1 + x^4)\cdots$ is equal to $1/(1 - x)$.

Comments. 1. The problem is harder if a particular x is used, such as $3/\pi$. If we compute the first few terms of the sequence to 9 decimal places, we may lose sight of the structure. A calculator is undeniably useful, but it can be confusing to use it too soon.

2. The result can be connected with *binary expansions.* Every non-negative integer can be expressed uniquely as a sum of distinct powers of 2. It follows that for every such integer k, a term x^k appears exactly once when we expand the infinite product.

Using the base 3 expansion, we find in a similar way that

$$(1 + x + x^2)(1 + x^3 + x^6)(1 + x^9 + x^{18})\cdots = \frac{1}{1 - x}$$

if $|x| < 1$. The idea can be substantially generalized. Problems of this type are connected to important ideas—generating functions, the zeta-function, and so on.

VI-49. Let $a_0 = 0$ and $a_1 = 1$. For $n \geq 0$, let $a_{n+2} = (a_n + a_{n+1})/2$. Show that a_n approaches $2/3$ as n becomes large. Hint: Let $d_n = a_{n+1} - a_n$.

Solution. Assume—this is not obvious—that the a_n do approach some number x. When n is large a_{n+2}, a_{n+1}, and a_n will be close to x, so $x = (x + x)/2$. Solve for x. Oops! We get $0 = 0$, true but not helpful. So let's take a look at the sequence, which is how we should have started anyway.

The first few terms are 0, 1, 1/2, 3/4, 5/8, 11/16, 21/32, 43/64, 85/128. Notice that each numerator is almost twice the previous one: $3 = 2 \cdot 1 + 1$, $5 = 2 \cdot 3 - 1$, $11 = 2 \cdot 5 + 1$, $21 = 2 \cdot 11 - 1$, and so on. Or maybe notice that if we multiply numerators by 3, the result is almost a power of 2. Or since the a_i (maybe) approach $2/3$, look in detail at the numbers $a_i - 2/3$. Patterns jump out, particularly if we don't use a calculator.

As the hint suggested, let $d_n = a_{n+1} - a_n$ (here d stands for *difference*). Calculate. We get $d_0 = 1$, $d_1 = -1/2$, $d_2 = 1/4$, $d_3 = -1/8$, $d_4 = 1/16$. The conclusion that $d_n = (-1)^n/2^n$ is irresistible and not hard to prove. Note that

$$d_{n+1} = a_{n+2} - a_{n+1} = \frac{a_n - a_{n+1}}{2} = -\frac{d_n}{2}.$$

Since $d_0 = 1$, it follows that $d_n = (-1/2)^n$.

Now that we know d_n, the rest is easy. We have

$$(a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) = a_n - a_0 = a_n,$$

and therefore $a_n = 1 + r + r^2 + \cdots + r^{n-1} = (1 - r^n)/(1 - r)$, where $r = -1/2$. It follows that as n gets large a_n approaches $2/3$.

Comments. 1. Suppose that (a_n) satisfy the recurrence and $a_0 = p$, $a_1 = q$, where $p \neq q$. Define a new sequence (b_n) by $b_n = (a_n - p)/(q - p)$. It is not hard to check that (b_n) satisfies the recurrence and $b_0 = 0$, $b_1 = 1$, so we know everything about (b_n), and therefore everything about (a_n).

The recurrence $a_{n+2} = (1 - t)a_{n+1} + ta_n$, where t is any real number, can be analyzed in the same way, for it can be rewritten as

$$a_{n+2} - a_{n+1} = -t(a_{n+1} - a_n).$$

If $|t| < 1$, then as n gets large, a_n approaches a number which we can compute if we know a_0 and a_1.

2. We flip a fair coin repeatedly, and get 2 dollars for every head and 1 dollar for every tail. Let p_n be the probability that we "hit" n dollars *exactly* at some time or other. How we get to exactly $n+2$ dollars? If we hit $n+1$ exactly and then toss a tail, or if we hit n exactly and then toss a head. So $p_{n+2} = (1/2)p_{n+1} + (1/2)p_n$, the recurrence of this problem.

VI-50. The sequence (e_n) obeys the recurrence $e_{n+2} = e_{n+1} - e_n + 1$. What are all the possible numbers of different terms that the sequence can have?

Solution. Let $e_1 = a$ and $e_2 = b$. The first few terms are a, b, $b-a+1$, $2-a$, $2-b$, $a-b+1$, a, b, ... Thus $e_7 = e_1$ and $e_8 = e_2$, there is cycling, and therefore there can't be more than 6 different terms.

There can be 6 terms. Indeed for "most" choices of a and b there are 6. Take for example $a = 0$ and $b = 10$. There can be 1 term. For that, we need $a = b = b-a+1$, that is, $a = b = 1$.

There *can't* be 2 terms. If consecutive terms are never equal and there are no more than 2 terms, then $a = b - a + 1 = -b + 2$, forcing $a = b = 1$. If some two consecutive terms are equal, then because of the cycling we may assume they are the first two terms. Then $e_3 = 1$. If $a \neq 1$, then since $e_4 = 2 - a$ we conclude that a, e_3, and e_4 are all different.

There can be 3 terms, for example if $a = 0$ and $b = 0$. There can be 4, for example if $a = 3$ and $b = 3$. But there can't be 5. For if two terms among the first 6 are equal, then because of the cycling we can assume that $e_i = a$ for some i with $2 \leq i \leq 6$. If $e_4 = a$, then $a = 1$ and therefore $e_3 = b$, so there are no more than 4 terms. If $e_2 = a$, then $e_4 = e_5$, and if $e_3 = a$ then $e_4 = e_6$, and so on, so again there are no more than 4 terms.

VI-51. Show that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Solution. Since the poser of the question kindly supplied a formula for the sum, we need only check that the formula is correct. Let $s_k = k(k+1)(k+2)/3$. Note that

$$s_k - s_{k-1} = \frac{k(k+1)(k+2)}{3} - \frac{(k-1)(k)(k+1)}{3} = k(k+1),$$

so our sum is equal to

$$(s_1 - s_0) + (s_2 - s_1) + (s_3 - s_2) + \cdots + (s_n - s_{n-1}).$$

Almost everything cancels and $s_0 = 0$, so the sum is s_n.

Another way: Here is a nice combinatorial argument. The number of ways of choosing 3 numbers from the collection $\{1, 2, 3, \ldots, n+2\}$ is

$$\binom{n+2}{3}, \qquad \text{that is,} \qquad \frac{n(n+1)(n+2)}{6}.$$

We now calculate the number of choices another way.

Imagine having chosen the 3 numbers, and let x be the largest of the 3 chosen numbers. Maybe $x = 3$. Then there are $\binom{2}{2}$ possibilities for the other two. Maybe $x = 4$. Then there are $\binom{3}{2}$ possibilities for the other two. If $x = 5$, there are $\binom{4}{2}$ possibilities for the other two. Go on in this way. Finally, if $x = n+2$ there are $\binom{n+1}{2}$ possibilities for the other two. We conclude that

$$\frac{n(n+1)(n+2)}{6} = \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n+1}{2}.$$

Finally, note that $\binom{k}{2} = k(k-1)/2$, and multiply by 2 to get the desired result.

Comments. 1. This result is mentioned by Āryabhata (sixth century).

2. The sum of the problem can be rewritten as

$$(1^2 + 1) + (2^2 + 2) + \cdots + (n^2 + n).$$

We conclude that

$$(1^2 + 2^2 + \cdots + n^2) + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{3}.$$

Now a little manipulation gives a formula for $1^2 + 2^2 + \cdots + n^2$.

3. The idea generalizes. For example, we can show in the same way that

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

Chapter 7

Number Theory

Introduction

This chapter has questions about the remainder when an integer, usually quite large, is divided by another integer. There are a few questions about the shape of the decimal expansion of certain numbers. The chapter also contains a number of variants of the popular question about the number of consecutive 0's at the right of the decimal representation of $n!$. Many of the remaining problems involve *Diophantine Equations*, that is, equations in which we look for *integer* solutions. The restriction to integers radically alters the nature of the arguments, although familiar tools can still be helpful—see VII-50 for a number-theoretic instance of completing the square.

Number-theoretic problems are not part of the senior secondary curriculum. Indeed, apart from calculation of greatest common divisors, least common multiples, and prime factorization, number theory—that is, the study of the properties of the *integers*—is not studied in the schools. Number-theoretic questions occur more frequently in mathematics contests. And number theory remains a central branch of mathematics, ever lively, ever new—witness the recent proof by Andrew Wiles of Fermat's Last Theorem, a result that was sought for more than three centuries.

Some of the problems in this chapter will seem difficult, mainly because of lack of familiarity. It can be hard to know how to start. But one should remember that the integers are concrete, certainly far more concrete than electrons. We can often discover what is going on by experimenting. The integer solutions of an equation can often be found by a systematic search; such a search can at least yield important clues.

We have tried to introduce many of the fundamental concepts of number theory through problems. For example, there are three proofs of the fact that there are infinitely many primes—see VII-8, VII-13, and VII-56.

Problems and Solutions

VII-1. Find all pairs (x, y) of non-negative integers such that $x + y = 768$ and the greatest common divisor of x and y is 64.

Solution. If the greatest common divisor of x and y is 64, then in particular $x = 64u$ and $y = 64v$ for some integers u and v. Then $x + y = 768$ if and only if $u + v = 12$.

We need to make sure that the greatest common divisor of u and v is 1. That leaves the possibilities $u = 1$, $v = 11$; $u = 5$, $v = 7$; $u = 7$, $v = 5$; and $u = 11$, $v = 1$.

VII-2. Let $P(x) = ax^2 + bx + c$. Suppose that $P(n)$ is an integer for every integer n. Show that a, b, and c needn't all be integers. What can be said about a, b, and c?

Solution. Put $n = 0$. Then $P(n) = c$, so c is an integer. But a and b needn't be integers. For example, for any integer n one of n or $n+1$ is even, so n^2+n is always even. It follows that $(1/2)(n^2 + n)$ is an integer for any n. So if we take $a = 1/2$, $b = 1/2$, and $c = 0$, then $P(n)$ is always an integer. (To save words, the letter n always represents an integer.)

In general, note that since c is an integer, $an^2 + bn$ is an integer. Put $n = 1$. We conclude that $a + b$ is an integer. Put $n = -1$. Thus $a - b$ is an integer. Add: $2a$ is an integer.

If a is an integer, then since $a+b$ is an integer, so is b. The only other possibility is that a is half of an odd integer. Then since $a + b$ is an integer, b is also half of an odd integer. Finally, we need to show that if c is an integer and $a = s/2$ and $b = t/2$ with s and t odd, then $an^2 + bn + c$ is always an integer.

We need to show that $(1/2)(sn^2 + tn)$ is an integer. This is obvious if n is even. If n is odd, then sn^2 and tn are odd and therefore their sum is even, so $(1/2)(sn^2 + tn)$ is an integer.

Comment. Using the same ideas, we can solve the following problem. Let $P(x) = ax^2 + bx + c$, where this time a, b and c are integers. Suppose that $P(n)$ is divisible by 15 for every integer n. Show that a, b, and c are divisible by 15.

If instead of 15 we use an even number, for example 42, we can conclude that 42 divides c. But 42 needn't divide a and b: they could each be an odd multiple of 21.

VII-3. Let a, b, and c be positive integers. Show that if $a^3 + b^3 + c^3$ is divisible by 7 then at least one of a, b, or c is divisible by 7. Hint: Find the remainders when $(7k + 1)^3$, $(7k + 2)^3, \ldots, (7k + 6)^3$ are divided by 7.

Solution. Any positive integer n can be expressed as $7q + r$, where the remainder r is 0, 1, 2, 3, 4, 5, or 6. Imagine expanding $(7q + r)^3$. We get something divisible by 7 (the details don't matter) plus r^3. So the remainder when n^3 is divided by 7 is the same as the remainder when r^3 is divided by 7. For $r = 1, 2, \ldots, 6$ these remainders are quickly computed. They are 1, 1, 6, 1, 6, and 6, so if n is not divisible by 7 the only possibilities are 1 and 6.

If none of a, b, or c is divisible by 7, the possible remainders when $a^3 + b^3 + c^3$ is divided by 7 are obtained by taking three numbers each of which is 1 or 6, and finding the remainder when their sum is divided by 7. There are only four cases to look at. The possible remainders are 1, 3, 4, and 6. In particular, a remainder of 0 can't happen.

Comment. A number which is not divisible by 7 is of the form $7k \pm 1$ or $7k \pm 2$ or $7k \pm 3$. The cubes of these are respectively of the form $7k \pm 1$, $7k \pm 1$, and $7k \mp 1$. So the issue is whether we can find three numbers, each of which is 1 or -1 and whose sum is divisible by 7. Clearly we can't.

This way of looking at things has greater symmetry, and symmetry is always useful. For example, we can see at once that if we have *five* numbers, none of which is a multiple of 7, then the sum of their cubes can't be a multiple of 7.

VII-4. Show that 13 divides $a^{13} - a$ for any integer n. Hint: Look at $(13q + r)^{13} - (13q + r)$.

Solution. Any integer a can be written in the form $a = 13q + r$, where $0 \le r \le 12$. (Just divide a by 13; q is the quotient and r is the remainder.) *Imagine* expanding

$$(13q + r)^{13} - (13q + r)$$

by using the Binomial Theorem, or by just multiplying out. The result is $r^{13} - r$ plus the sum of a bunch of terms with 13's in them. It follows that 13 divides $a^{13} - a$ if and only if 13 divides $r^{13} - r$.

So we only need to show that $r^{13} - r$ is divisible by 13 for any integer r from 0 to 12. The task has been cut down from considering infinitely many a to looking at 13 numbers only. That still takes some work, and numbers like 10^{13} are beyond the comfort zone of the calculator, so we simplify things a bit first.

Any integer a can be written in the form $a = 13k + r$ where $-6 \le r \le 6$. Thus we need only show that $r^{13} - r$ is divisible by 13 for $-6 \le r \le 6$. But since $(-x)^{13} - (-x) = -(x^{13} - x)$, we only need to show that $r^{13} - r$ is divisible by 13 for $r = 0, 1, \ldots, 6$. That's easy to do with a calculator.

Comment. There is nothing very special about 13. It can be shown that if p is *any prime,* then p divides $a^p - a$ for every integer a. This important fact is usually called *Fermat's Theorem,* after its seventeenth-century discoverer. For proofs of Fermat's Theorem, see any basic Number Theory book.

VII-5. Find a multiple of 999 whose decimal representation is made up entirely of 1's.

Solution. We experiment with smaller numbers to get some insight. So let's first find a multiple of 9 of the shape $11\ldots1$. Recall that a number is divisible by 9 if and only if the sum of its digits is divisible by 9. So stringing together 9 1's works.

Now look at 99. A number is divisible by 99 if it is divisible by 11 and the quotient is divisible by 9. A number made up of an even number $2k$ of 1's is divisible 11—just think about the ordinary "long division" process. The quotient looks like $1010\ldots101$, where the block 10 is repeated $k-1$ times, and at the end there is a 1. So there are k 1's, and therefore the quotient is divisible by 9 if k is a multiple of 9. We conclude that stringing together 18 1's works.

The same idea works with 999. Our number should be divisible by 111, and the quotient divisible by 9. A number made up of $3k$ 1's is divisible 111—again just think about the ordinary division process. The quotient looks like $100100\ldots1001$, where the block 100 is repeated $k-1$ times. So again there are k 1's, and we conclude that stringing together 27 1's works.

Another way: Here is a more algebraic-looking version of the same argument. A number made up of all 1's must be $(10^n-1)/9$ for some positive n. We need an n such that 999 divides $(10^n-1)/9$. Equivalently, $9(10^3-1)$ should divide 10^n-1.

We use the identity

$$x^m - 1 = (x^{m-1} + \cdots + x + 1)(x-1).$$

Let $x = 10^3$. The identity shows that 10^3-1 divides $10^{3m}-1$. We need another factor of 9, so 9 should divide $1 + x + \cdots + x^{m-1}$.

But 1, x, x^2, …, x^{m-1} are powers of 10, so each is 1 more than a multiple of 9. We can force their sum to be divisible by 9 by letting $m = 9$. It follows that $(10^{27}-1)/9$ is a multiple of 999 of the right shape.

Comment. Let k be a positive integer which is divisible neither by 2 nor by 5. We sketch a proof of the surprising fact that there is a multiple of k whose decimal representation is made up entirely of 1's. Let $m = 9k$. It is enough to show that there is a multiple of m which is made up entirely of 9's.

For $n = 0$, 1, 2, and so on, imagine computing the remainder when 10^n is divided by m. There are at most $m-1$ possible remainders, so by the time n reaches $m-1$ we must have seen some remainder twice. Suppose that 10^s and 10^t, where $s < t$, give the same remainder on division by m. Then $10^t - 10^s$ is a multiple of m. But

$$10^t - 10^s = 10^s(10^{t-s} - 1),$$

and since m has no common factor with 10^s, it follows that m divides $10^{t-s}-1$. The decimal expansion of this last number is made up entirely of 9's.

VII-6. When 800 is divided by the positive integer n, the remainder is 5. List all possible values of n.

Solution. Let q be the quotient when 800 is divided by n. The remainder is 5 if $n > 5$ and $800 = qn + 5$, or equivalently $795 = qn$. So all we need is the divisors of 795 greater than 5.

Since $795 = 3 \cdot 5 \cdot 53$, the divisors of 795 that qualify are 15, 53, 159, 265, and 795.

VII-7. Let $\tau = (\sqrt{5}+1)/2$. Then τ is a root of $x^2 - x - 1 = 0$. Find integers a and b such that $\tau^8 = a\tau + b$.

Solution. The exponent 8 is intimidating, so let's first solve the same problem for τ^2. Because τ is a root of $x^2 - x - 1 = 0$,

$$\tau^2 = \tau + 1.$$

Now solve the problem for τ^3. Multiply both sides of the preceding equation by τ. We obtain $\tau^3 = \tau^2 + \tau$. Substitute $\tau + 1$ for τ^2. We conclude that $\tau^3 = 2\tau + 1$.

Multiply both sides of the preceding equation by τ, and substitute $\tau + 1$ for τ^2. We obtain $\tau^4 = 3\tau + 2$. Repeat. In quick succession we get

$$\tau^5 = 5\tau + 3, \quad \tau^6 = 8\tau + 5, \quad \tau^7 = 13\tau + 8, \quad \text{and} \quad \tau^8 = 21\tau + 13.$$

Thus $a = 21$ and $b = 13$. The pattern is clear and we could easily continue.

Another way: Square both sides of the relation $\tau^2 = \tau + 1$, then substitute $\tau + 1$ for τ^2. We obtain $\tau^4 = 3\tau + 2$. Square again, substitute, and we are done.

Another way: Compute τ^8 quickly by squaring three times. The result is $(21\sqrt{5} + 47)/2$. Set this equal to $a\tau + b$. After simplifying a bit, we obtain

$$21\sqrt{5} + 47 = a\sqrt{5} + a + 2b.$$

It follows that $a = 21$ and $a + 2b = 47$, giving $b = 13$.

Comments. 1. The first two solutions work with essentially no change for any root of $x^2 - px - q = 0$, where p and q are integers. The case $p = q = 1$ was chosen because the familiar Fibonacci numbers show up.

The idea works with polynomials of higher degree. For example, let λ be a root of $x^3 = px^2 + qx + r$. Given a positive integer n, we can compute integers a_n, b_n, and c_n such that

$$\lambda^n = a_n\lambda^2 + b_n\lambda + c_n$$

by closely imitating the argument that was used for τ.

2. The first solution got to τ^8 slowly, in seven steps. In the second and third solutions, τ^8 was reached quickly by squaring twice. For any a, and any positive integer n, we can get to a^n quickly. See VII-49 and IX-28 for other examples.

VII-8. Note that exactly one prime divides $2^2 - 1$ and exactly two divide $2^4 - 1$. Show that $2^8 - 1$ has three prime factors, $2^{16} - 1$ has four, $2^{32} - 1$ has at least five, and $2^{64} - 1$ has at least six.

Solution. Note that $2^8 - 1 = (2^4 - 1)(2^4 + 1)$. Since $2^4 - 1$ has two prime factors, while $2^4 + 1 = 17$ and 17 is prime, it follows that $2^8 - 1$ has three prime factors. We could instead note that $2^8 - 1 = 255$, then factor 255, an easy task. But when a number has structure, it is usually a mistake wantonly to destroy that structure by going to the decimal expansion.

Note that $2^{16} - 1 = (2^8 - 1)(2^8 + 1)$. We know that $2^8 - 1$ has three prime factors. Also, $2^8 + 1 = 257$, and a fairly quick check shows that 257 is prime.

We now deal with $2^{32} - 1$. Factor it as $(2^{16} - 1)(2^{16} + 1)$. We saw that $2^{16} - 1$ has four prime factors. Since $2^{16} + 1$ is an odd number greater than 1, it has at least one odd prime divisor p.

We show that p can't be any of the prime factors of $2^{16} - 1$. Suppose to the contrary that p divides $2^{16} - 1$. Since p also divides $2^{16} + 1$, it follows that p divides $(2^{16} + 1) - (2^{16} - 1)$, that is, p divides 2, which is impossible since p is odd.

We conclude that $2^{32} - 1$ has at least five prime factors. In fact, it has exactly five, for $2^{16} + 1$ turns out to be prime.

Finally, look at $2^{64} - 1$. We know that $2^{32} - 1$ has at least five prime factors. And $2^{32} + 1$ has at least one (odd) prime factor p. No such p can divide $2^{32} - 1$, for if it did it would divide $(2^{32} + 1) - (2^{32} - 1)$, namely 2. Thus $2^{64} - 1$ has at least six prime factors.

Comments. 1. In the same way, we can argue that $2^{128} - 1$ has at least 7 prime factors, $2^{256} - 1$ has at least 8, and in general $2^{(2^n)} - 1$ has at least n prime factors. It follows that there are infinitely many primes. This result, with a different proof, can be found in Euclid's *Elements.*

2. We show that $2^{16} + 1$ is prime. If a number N is composite, then it can be expressed as xy, where $2 \le x \le y$. It follows that $x \le \sqrt{N}$. Note that x has a prime factor, possibly equal to x.

Thus if *none* of the primes up to $\sqrt{N}$ divide N, we can conclude that N is prime. Let $N = 2^{16} + 1$. Then $\sqrt{N}$ is a bit bigger than 2^8, so we only need to test whether p divides N for primes p less than 256.

For each prime p less than 256, test whether p divides 65537. Finally we get to use the calculator. The work is repetitive but doesn't take long. None of these primes divides 65537, so $2^{16} + 1$ is prime.

3. It turns out that $2^{64} - 1$ has more than 6 prime factors, for $2^{32} + 1$ is not prime. In 1640, Fermat conjectured that $2^{(2^n)} + 1$ is prime for every non-negative integer n. The number is prime for $n = 0$, 1, 2, 3, and 4. Around 1750, Euler showed that Fermat's conjecture is false by observing that

$$2^{32} + 1 = 641 \cdot 6700417$$

(these two factors are prime). It is not known whether there is *any* $n > 4$ for which $2^{(2^n)} + 1$ is prime!

VII-9. Express 159999 as a product of primes.

Solution. We could use a calculator to hunt for prime factors—it wouldn't even take long. But let's exploit *structural* information. Note that

$$159999 = 160000 - 1 = 400^2 - 1 = (400 - 1)(400 + 1).$$

Factor the relatively small numbers 399 and 401. It turns out that $399 = 20^2 - 1 = 3 \cdot 7 \cdot 19$, and that 401 is prime.

VII-10. Find positive integers a and b such that $9 < (a/b)^2 < 10$ and b is as small as possible.

Solution. The fraction a/b should be between 3 and $\sqrt{10}$, which is about 3.1622. Note that 31/10 is in the right region. The only question that remains is whether we can find a fraction with a smaller denominator.

Take successively $b = 2, 3, 4, \ldots, 10$. For any b, we need to check whether any of $1/b$, $2/b$, $3/b$, $\ldots$, $(b-1)/b$ lies in the interval $(0, \sqrt{10} - 3)$. Since $b \le 10$, only $1/b$ has any chance of landing in the interval, so the checking goes quickly. After a short while we arrive at the answer $b = 7$, $a = 22$.

Another way: When we are trying to solve a problem about integers, it is usually a good idea to work as much as possible with integers rather than with alien quantities like $\sqrt{10}$, or even fractions.

The inequality $9 < (a/b)^2 < 10$ can be rewritten as $9b^2 < a^2 < 10b^2$, so we are looking for a perfect square between $9b^2$ and $10b^2$.

The first perfect square after $9b^2$ is $(3b + 1)^2$. We want $(3b + 1)^2 < 10b^2$, or equivalently $b^2 > 6b + 1$. It is clear that 7 is the smallest positive integer that satisfies this inequality. Thus $b = 7$ and $a = 3b + 1 = 22$.

Comment. There is nothing really special about 9 and 10. Given integers m and n with $0 < m < n$, we can use the idea of the second solution to find efficiently the fraction a/b with smallest denominator such that $m < (a/b)^2 < n$.

VII-11. Let x, y, and z be integers such that $x^3 + 2y^3 = 4z^3$. Show that *every* power of 2 divides x, and conclude that $x = y = z = 0$.

Solution. Since $x^3 = 4z^3 - 2y^3$, it follows that x^3 is an even number, and therefore x is even. Let $x = 2x_1$. Substitute $2x_1$ for x in the given equation. We conclude that $8x_1^3 + 2y^3 = 4z^3$ and therefore $4x_1^3 + y^3 = 2z^3$.

By the same reasoning, y is even, say $y = 2y_1$. If we substitute and simplify, we obtain $2x_1^3 + 4y_1^3 = z^3$. But then z must be even, say $z = 2z_1$. If we substitute and simplify, we obtain $x_1^3 + 2y_1^3 = 4z_1^3$. Note that this equation has the same shape as the original equation.

We repeat the argument, and conclude that $x_1 = 2x_2$, $y_1 = 2y_2$, and $z_1 = 2z_2$ for some integers x_2, y_2, and z_2, and that $x_2^3 + 2y_2^3 = 4z_2^3$. And we can continue in this way *forever.*

From the fact that $x = 2x_1$ and $x_1 = 2x_2$, we conclude that $x = 4x_2$. But $x_2 = 2x_3$, so $x = 8x_3$. Similarly, $x = 16x_4$, and so on.

The only integer that is divisible by every power of 2 is 0. Similarly we conclude that y and z are divisible by every power of 2, and are therefore also equal to 0.

VII-12. Call a year *lucky* if the sum of the digits in the decimal representation of the year is divisible by 7. Find the shortest possible gap between lucky years. Find the longest gap.

Solution. To counter pessimists, we show that it is possible to have two lucky years in a row. If n is a positive integer whose decimal representation doesn't end in a 9, then the digit sum of $n+1$ is 1 more than the digit sum of n. Thus if one digit sum is divisible by 7 then the other can't be.

Now examine years n whose decimal representation ends in $d9$, where d is a digit other than 9. Then the digit sum of n is 8 more than the digit sum of $n+1$, so the digit sums can't be both divisible by 7. We can deal similarly with years n whose decimal representation ends in $d99$ or $d999$ where d is a digit other than 9.

But if the decimal representation of n ends in $d9999$, where d is a digit other than 9, then the digit sum of n is 35 more than the digit sum of $n+1$, and if one digit sum is divisible by 7, so is the other. The first year n such that n and $n+1$ are both lucky is 69999—a long wait.

Unless n ends in 9, the digit sum of $n+1$ is 1 more than the digit sum of n. To get the longest conceivable gap between lucky years, we need a lucky n that ends in a 3, and such that the digit sum of $n+7$ leaves a remainder of 1 on division by 7. That way, the digit sums of $n+1$, $n+2$, …, $n+6$ leave remainders of 1, 2, …, 6 on division by 7, and just when we are ready for a lucky year, the remainder starts at 1 again.

Note that 993 is lucky and 1000 has digit sum that leaves a remainder of 1 on division by 7, so between the years 993 and 1006 there were 12 unlucky years, the maximum possible. There will be 12 unlucky years between 7993 and 8006.

VII-13. Let $a_0 = 2$ and let $a_{n+1} = a_0a_1a_2\cdots a_n + 1$ when $n \geq 0$. (a) Show that if $m \neq n$, then a_m and a_n have no common factor greater than 1. (b) Conclude that there are infinitely many primes.

Solution. (a) We may assume that $m < n$. Let d be a positive integer that divides both a_m and a_n. Since $m \leq n-1$, we conclude that d divides $a_0a_1\cdots a_{n-1}$. But since

$$a_n = a_0a_1\cdots a_{n-1} + 1,$$

and d divides a_n, it follows that d must divide 1, so $d = 1$.

(b) For any integer n, let p_n be a prime that divides a_n, say the smallest such prime if there is more than one. From part (a), we conclude that if $m \neq n$ then $p_m \neq p_n$. So p_0, p_1, p_2, … are all different, and therefore there are infinitely many primes.

Comment. Calculate the first few a_k. We get $a_0 = 2$, $a_1 = 3$, $a_2 = 7$, $a_3 = 43$, and $a_4 = 1807$. The first four are prime, but $1807 = 13 \cdot 139$.

Around -300, Euclid used an argument similar in spirit to prove that there are infinitely many primes.

VII-14. Find all positive integers n such that the decimal expansion of $1/n$ has the shape $.abcabcabcabc\ldots$, where a, b, and c are digits.

Solution. Let $s = 100a + 10b + c$. Then

$$\frac{1}{n} = 0.abcabcabcabc\cdots = \frac{s}{10^3} + \frac{s}{10^6} + \frac{s}{10^9} + \frac{s}{10^{12}} + \cdots.$$

The infinite geometric series on the right has first term $s/1000$ and common ratio $1/1000$, so it has sum $s/999$. Alternately, if we multiply both sides by 1000, we obtain $1000/n = s + 1/n$. We conclude that the decimal expansion of $1/n$ has the required shape precisely if $1/n = s/999$ for some s, that is, if $999 = sn$.

That happens if and only if n is a positive divisor of 999. Since $999 = 3^3 \cdot 37$, the decimal expansion of $1/n$ has the required shape when $n = 1$, 3, 9, 27, 37, 111, 333, or 999.

VII-15. (a) Let $\mathcal{C}$ be the infinite set

$$\left\{2^1,\, 2^3,\, 2^5,\, \ldots,\, 2^{2n+1},\, \ldots\right\}.$$

Explain why the sum of one or more different members of $\mathcal{C}$ can never be a perfect square.

b) Let $a_1 = 2$, and for any n let $a_{n+1} = (a_1 + a_2 + \cdots + a_n)^2 + 1$. So $a_2 = 5$, $a_3 = 50$, and $a_4 = 3250$. Let $\mathcal{D}$ be the infinite set

$$\{a_1,\, a_2,\, a_3,\, \ldots,\, a_n,\, \ldots\}.$$

Explain why the sum of one or more different members of $\mathcal{D}$ can never be a perfect square.

Solution. We give an informal explanation instead of flooding the page with symbols. Could $2^{471} + 2^{499} + 2^{1945}$ be a perfect square? The terms have the common factor 2^{471}. So our sum is equal to $2^{471}(1 + 2^{28} + 2^{1446})$, that is, 2^{471} times an odd number. The largest power of 2 that divides our sum is therefore 2^{471}. But the largest power of 2 that divides a perfect square has the shape 2^k where k is *even*, and certainly 471 is not even.

The same idea works in general. Suppose that N is the sum of some elements of $\mathcal{C}$. Let 2^{2a+1} be *smallest* among the numbers that were added together. So the other numbers, if any, involve higher powers of 2. It follows that N is 2^{2a+1} times an odd number, so the largest power of 2 that divides N is 2^{2a+1}. Since $2a + 1$ is odd, it follows that N can't be a perfect square.

(b) Each number in $\mathcal{D}$ is 1 more than a perfect square. Note that the numbers in $\mathcal{D}$ grow extremely fast. Let N be a sum of elements of $\mathcal{D}$, and let a_n be the largest number used in making N. We have

$$a_n = q^2 + 1, \qquad \text{where} \qquad q = a_1 + a_2 + \cdots + a_{n-1}.$$

The other numbers used in making N add up to at most q. It follows that

$$q^2 + 1 \leq N \leq q^2 + q + 1.$$

Since the first perfect square after q^2 is q^2+2q+1, the number N can't be a perfect square.

Comments. 1. The *Fermat numbers* $2^{2^0}+1$, $2^{2^1}+1$, $2^{2^2}+1$, ... also grow extremely fast, and each is 1 more than a perfect square. Using an argument similar to that of part (b), we could show that no sum of distinct Fermat numbers can be a perfect square.
2. A better but harder problem is to ask whether there is a set of 1000 numbers such that the sum of one or more of them is never a perfect square. To solve that problem, we have to think about what might prevent sums from being perfect squares. The sets $\mathcal{C}$ and $\mathcal{D}$ supply two such obstructions, divisibility considerations and very fast growth.

VII-16. For any positive integer n, let $f(n)$ be the integer closest to $(\sqrt{n+1}+\sqrt{n})^2$. Find a simple formula for $f(n)$.

Solution. We compute for a while to get our bearings. It turns out that $f(1) = 6$, $f(2) = 10$, $f(3) = 14$, and $f(4) = 18$. It looks as if the answer could be $4n+2$. We show that this is correct by showing that $f(n)$ is below $4n+2$, but not by much.

Wherever $\sqrt{n+1}+\sqrt{n}$ goes, $\sqrt{n+1}-\sqrt{n}$ likes to tag along. Note that

$$(\sqrt{n+1}+\sqrt{n})^2 + (\sqrt{n+1}-\sqrt{n})^2 = 4n+2,$$

and therefore

$$(\sqrt{n+1}+\sqrt{n})^2 = 4n+2-(\sqrt{n+1}-\sqrt{n})^2 = 4n+2-\frac{1}{(\sqrt{n+1}+\sqrt{n})^2}.$$

Since $1/(\sqrt{n+1}+\sqrt{n})^2$ decreases as n increases, it is always less than or equal to $1/(\sqrt{2}+1)^2$, which is roughly 0.172, and in particular much less than 1/2. It follows that $f(n) = 4n+2$.
Another way: Since

$$(\sqrt{n+1}+\sqrt{n})^2 = 2n+1+\sqrt{4n^2+4n}$$

we need to find the integer nearest to $\sqrt{4n^2+4n}$. Since $4n^2 < 4n^2+4n < (2n+1)^2$, the only candidates are $2n$ and $2n+1$, and surely $2n$ is not in the running. To make sure, look at $2n+1-\sqrt{4n^2+4n}$ and rationalize the numerator. We get $1/D$, where $D = 2n+1+\sqrt{4n^2+4n}$. In particular, $D > 4n+1$, so the integer closest to $\sqrt{4n^2+4n}$ is indeed $2n+1$.

Note that if we had written $2\sqrt{n^2+n}$ instead of $\sqrt{4n^2+4n}$, the result would be less obvious.

VII-17. Find all possible values of the leftmost digit in the decimal expansion of 2^n.

Solution. If we compute the first few powers of 2, we quickly obtain 1, 2, 3, 4, 5, 6, and 8 as possible initial digits, but 7 and 9 are reluctant to appear. We can either persist in our computations or try to see what's going on.

Here is a list of the first 20 powers of 2, deliberately arranged in two rows of 10 each.

1	2	4	8	16	32	64	128	256	512
1024	2048	4096	8192	16384	32768	65536	131072	262144	524288

Note that for any number a in the first row, the number b below it starts with the same digit as a does. If we made a third row, and even a fourth, the pattern would continue. The explanation is clear: $b = 1024a$, and 1024 is nearly equal to 1000.

This observation points to a solution. Since 1024 is a bit larger than 1000, we can get an initial digit of 7 if we multiply 64 by 1024 the right number of times. For example, we can multiply by $(1024)^k$, where k is the smallest integer such that $(1024)^k > 70/64$. It turns out that $k = 4$. So 7 is the first decimal digit of 2^{46}. It is also the first decimal digit of 2^{56}, 2^{66}, ..., 2^{96}.

The same idea works for 9. Multiply 8 by $(1024)^k$ where k is the smallest integer such that $(1024)^k > 9/8$. It turns out that $k = 5$, so 9 is the first decimal digit of 2^{53}.

Comment. The solution depended on the "accidental" (but very useful) fact that 1024 is close to a power of 10. We sketch an alternative approach that uses logarithms. Let $f(n) = n \log 2 - \lfloor n \log 2 \rfloor$. So $f(n)$ is the fractional part of $\log(2^n)$.

By manipulating logarithms, we can see that 2^n has leftmost digit equal to 7 precisely if $7 < 10^{f(n)} < 8$. So we want to find an integer n such that the fractional part of $n \log 2$ lies between $\log 7$ and $\log 8$. For assurance that we can closely specify the fractional part of $n \log 2$, we could use ideas like those in IX-3, but here we just play with the calculator.

The calculator says that we want the fractional part of $(0.30129995)n$ to lie roughly between 0.8451 and 0.903. The number n shouldn't be too large, to make sure that the calculator approximations are close enough to the truth.

The eye focuses on the "01" in the decimal expansion of $\log 2$. Multiply $\log 2$ by 10, and then by an integer k such that $(0.0129995)k$ is in the right range. It is easy to see that $k = 65$ works. We get $n = 650$, a number much larger than the 46 we found earlier. A more sophisticated version of this approach, using *continued fractions,* will get us our 46.

VII-18. The positive integer N is *perfect* if the sum of the positive divisors of N is equal to $2N$. For example, 6 is perfect, since the positive divisors of 6 are 1, 2, 3, and 6, and $1 + 2 + 3 + 6 = 12$. Show that if $2^n - 1$ is a prime number then $2^{n-1}(2^n - 1)$ is perfect.

Solution. Let $q = 2^n - 1$. Since q is prime, the positive integers that divide N are $1, 2, 4, \ldots, 2^{n-1}$ and $q, 2q, 4q, \ldots, 2^{n-1}q$.

The sum of the positive divisors of N is therefore

$$(1+q) + 2(1+q) + 4(1+q) + \cdots + 2^{n-1}(1+q).$$

But $1 + q = 2^n$, and the geometric series $1 + 2 + 4 + \cdots + 2^{n-1}$ has sum $2^n - 1$, so the sum of the positive divisors of N is equal to $2^n(2^n - 1)$, that is, $2N$.

Comment. The result above can be found in Euclid's *Elements* (ca. −300). Books 7, 8, and 9 of the *Elements* deal with Number Theory, not Geometry. The result about perfect numbers is the culminating theorem of Book 9.

There is much non-mathematical speculation associated with perfect numbers: God created the world in 6 days because 6 is perfect, or maybe the other way around; the lunar month lasts 28 days because 28 is perfect. Number mysticism motivated some of the early mathematics, just as astrology motivated parts of astronomy.

Around 1750, Euler proved that every even perfect number *must* have shape $2^{n-1}(2^n - 1)$, where $2^n - 1$ is prime. It is still not known whether there are any odd perfect numbers, but if there is one it has at least 420 prime factors and is greater than 1.9×10^{2550}.

Primes of the form $2^n - 1$ are called *Mersenne primes.* As of now, 38 Mersenne primes are known, and hence 38 perfect numbers. The largest currently known Mersenne prime is $2^{6972593} - 1$. It was found on June 1, 1999 by Nayan Hajratwala. Several Mersenne primes, including this one, have been discovered through the *Great Internet Prime Search.* The task of searching is distributed among a large number of personal computers, which in effect form an enormously powerful supercomputer. Anyone can play.

VII-19. It is easy to verify that $3 \cdot 4 = 12$, $33 \cdot 34 = 1122$, and $333 \cdot 334 = 111222$. Does this pattern continue forever?

Solution. The decimal representation of $10^k - 1$ consists of k 9's, so the representation of $(10^k - 1)/3$ consists of k 3's, and the representation of $(10^k + 2)/3$ has $(k - 1)$ 3's followed by a 4. We have

$$\left(\frac{10^k - 1}{3}\right)\left(\frac{10^k + 2}{3}\right) = \frac{10^{2k} + 10^k - 2}{9} = \frac{10^{2k} - 1}{9} + \frac{10^k - 1}{9}.$$

The representation of $(10^{2k} - 1)/9$ consists of $2k$ 1's, while the representation of $(10^k - 1)/9$ consists of k 1's. Add. The sum has k 1's followed by k 2's.

VII-20. Find the first three positive integers a such that $a^{999} + 1$ is a multiple of 128.

Solution. If n is odd, then

$$x^n + 1 = (x+1)(x^{n-1} - x^{n-2} + \cdots - x + 1).$$

Let $n = 999$. For any integer a, the sum $a^{998} - a^{997} + \cdots - a + 1$ is odd. This is obvious if a is even. And if a is odd we are adding together an odd number of odd numbers.

Thus the only way to make $a^{999} + 1$ divisible by 128 is to make sure that 128 divides $a + 1$. The first positive a that works is therefore 127, the second is 255, and the third is 383.

VII-21. Show that the only integer solutions of $x \log 2 = y \log 3$ are $x = y = 0$. Here log means logarithm to the base 10.

Solution. Suppose that $x \log 2 = y \log 3$. Then

$$10^{x \log 2} = 10^{y \log 3}.$$

But $10^{x \log 2} = (10^{\log 2})^x = 2^x$ and similarly $10^{y \log 3} = 3^y$, so

$$2^x = 3^y.$$

That can't happen if x and y are positive integers, since 2^x is divisible by 2 but 3^y isn't. If x (and therefore y) is negative, then since $(-x)\log 2 = (-y)\log 3$, we reach the same conclusion.

Comment. The same argument works for logarithms to any base a. But we get nothing new: for any fixed base a, $\log_a u$ is a constant multiple of $\log u$, and therefore the assertion $x \log_a 2 = y \log_a 3$ is equivalent to the assertion $x \log 2 = y \log 3$.

VII-22. Let $N = 1! + 2! + 4! + 8! + 16! + 32! + 64!$. Find the remainder when N^2 is divided by 7.

Solution. The terms from 8! on are divisible by 7, and $1!+2!+4!$ leaves a remainder of 6 on division by 7, so $N = 7k + 6$ for some integer k that we don't need to know. We want the remainder when $(7k+6)^2$ is divided by 7. Since

$$(7k+6)^2 = 7(7k^2 + 12k) + 6^2,$$

the remainder when $(7k+6)^2$ is divided by 7 is the same as the remainder when 6^2 is divided by 7, namely 1.

Comment. Suppose that we are only interested in the remainder on division by m. Then in calculations that involve *only addition, subtraction, and multiplication* we can replace at will any number x by any y that differs from x by a multiple of m. In particular, any multiple of m can be replaced by 0, in a certain sense it *is* 0.

Here is a right way to think while figuring out the remainder when N^2 is divided by 7. In N, the stuff past 4! is basically the same as 0, and $1! + 2! + 4!$ "is" 6, so

N^2 "is" 36, which "is" 1. If our calculator is broken and we can't find $6 \cdot 6$, note that 6 "is" -1 (they differ by a multiple of 7), so $6 \cdot 6$ "is" $(-1)(-1)$.

The replacements described above work when we add or multiply, but not when we work with exponents. For example, 1 and 8 leave the same remainder on division by 7, but 2^1 and 2^8 do not. (There *are* regularities for exponentiation, but they are not the same as the ones for addition and multiplication.)

The above informal reasoning can be made fully precise by using the notion of *congruence* developed around 1800 by Carl Friedrich Gauss. Congruences have been an indispensable tool in number theory ever since.

VII-23. The integer a is said to be *relatively prime* to b if 1 is the only positive integer that divides both a and b. For example, 14 is relatively prime to 15. How many positive integers less than 162 are relatively prime to 162?

Solution. Since $162 = 2 \cdot 3^4$, we need to count the integers from 1 to 161 which are divisible neither by 2 nor by 3. Listing is easy: 1, 5, 7, 11, 13, ... Counting isn't too bad, even if we don't stop to think.

But we can save work if we notice that a is relatively prime to 162 if and only if $a + 6$ is relatively prime to 162. So the chunks 1 to 6, 7 to 12, 13 to 18, and so on each have the same number of members of our list. There are 162/6 of these chunks, and each chunk contains two numbers relatively prime to 162, for a total of (2)(162/6).

Another way: There are 162 numbers from 1 to 162. The 81 even numbers are *not* relatively prime to 162. Throw them away. That leaves 81 odd numbers. Every third one of them is divisible by 3. Discard these 27 numbers. That leaves the 54 numbers that are relatively prime to 162.

Comment. Let $\phi(n)$ be the number of integers from 1 to n that are relatively prime to n. The calculation above shows that $\phi(162) = 54$. The function ϕ is usually called the *Euler phi-function.* The phi-function plays an important role in number theory.

It is not hard to find $\phi(n)$ if we know the prime factorization of n. Calculating $\phi(p^n)$ when p is prime makes for a nice accessible problem. We need to count the integers from 1 to p^n which are relatively prime to p^n. There are p^n numbers from 1 to p^n. The only ones that are *not* relatively prime to p^n are the multiples of p. There are p^{n-1} multiples of p in our interval, and therefore $\phi(p^n) = p^n - p^{n-1}$.

VII-24. Find the least positive integer n such that 12^{12} divides $n!$.

Solution. Since $12^{12} = 2^{24} \cdot 3^{12}$, we want (the prime factorization of) $n!$ to "contain" at least twenty-four 2's, and at least twelve 3's.

Divide the numbers from 1 to n into groups of three, with maybe one or two left over. Every group of three consecutive numbers contributes at least one 3 (it can contribute more). So to take care of the 3's, we don't need more than twelve groups of three. And every group of four consecutive numbers contributes at least three 2's, so we don't need more than eight such groups. It follows that $n \leq 36$.

Because 9, 18, and 36 each contribute an extra 3, while 27 contributes two, 36! has five more threes than we need. Thus there are still enough 3's if we step back all the way to 27!. Is 27! good enough? It is not hard to check that there are only twenty-three 2's in 27!, so $n = 28$.

VII-25. Find the least positive integer N such that 243 divides N and the decimal digits of N are all the same.

Solution. Since $243 = 3^5$, N must have at least five 3's in its prime factorization. Let U_k be the number whose decimal expansion consists of exactly k 1's. Then $N = dU_k$, where d is a digit. We can get a couple of 3's by taking $d = 9$, but that still leaves us three 3's short. So 27 must divide U_k.

Recall that 9 divides x if and only if 9 divides the sum of the decimal digits of x. It follows that k must be a multiple of 9. The first possibility is thus $k = 9$. That produces a number small enough to fit into a calculator. We find that $U_9 = 9 \cdot 12345679$—pretty! But 3 doesn't divide 12345679, so U_9 doesn't have enough 3's.

What about $k = 18$? Divide U_{18} by U_9. We can do the division in our heads by thinking about the ordinary paper and pencil division process. We get $U_{18} = qU_9$, where q has decimal expansion that looks like 1, then a bunch of 0's, then a 1. The sum of the digits of q is not divisible by 3, so U_{18} has no more 3's in it than U_9 did.

What about $k = 27$? Again, divide by U_9. We get $U_{27} = qU_9$, where the decimal expansion of q has exactly three 1's and quite a few 0's. Thus q is divisible by 3 but not by 9. We conclude that $9U_{27}$ is the smallest positive integer that works.

Comment. By induction or otherwise one can show that 3^k is *always* the highest power of 3 that divides U_{3k}.

VII-26. Find the remainder when $2^{400} + 1$ is divided by $2^{40} - 1$.

Solution. We can use formal long division or a trick. Here's the trick. For brevity, let $x = 2^{40}$. Then

$$\frac{2^{400} - 1}{2^{40} - 1} = \frac{x^{10} - 1}{x - 1} = x^9 + x^8 + \cdots + x + 1.$$

Note that $2^{400} + 1 = (2^{400} - 1) + 2$. So if we divide $2^{400} + 1$ by $2^{40} - 1$, the quotient is $x^9 + x^8 + \cdots + 1$ and the remainder is 2.

Comment. What we did probably shouldn't be called a trick. There is an old joke that a trick used more than once is a Method, and the idea that we used crops up often.

The formal long division argument should also be carried out. It is not difficult, if we use the abbreviation $x = 2^{40}$. As a variant, we could look at the same problem with $2^{40} - 1$ replaced by $2^{40} + 1$.

VII-27. Find the rightmost non-zero digit in the decimal expansion of 49!.

Solution. Since 49! is fairly large, maybe we should think a bit before picking up the calculator. If we were dealing with 2000!, thinking first would be almost necessary, unless we were willing to write a computer program to do the calculation.

How is the last non-zero digit of a product ab related to the last non-zero digit of a and of b? With *last* digits, the answer is clear: to find the last digit of ab, take the last digits of a and b, multiply, and if necessary take the last digit of the product. That's not quite true for last non-zero digits: $6 \cdot 15 = 90$, and yet the last non-zero digit of $6 \cdot 5$ is 3. But everything is fine as long as we avoid multiplying by numbers whose last non-zero digit is 5.

The last digit of 4! is 4. So is (for good reason) the last digit of $6 \cdot 7 \cdot 8 \cdot 9$. And last digits of 11, 12, ... match the last digits of 1, 2, ... So if we multiply together all the numbers from 1 to 49 which are *not* divisible by 5, the result r has last digit 6.

We now deal with the multiples of 5. The product of these is

$$(1 \cdot 2 \cdot 3 \cdot 4)(6 \cdot 7 \cdot 8 \cdot 9)(5^{10}).$$

Multiplication by the first two parts leaves us with a last digit of 6. We must now deal with 5^{10}. Divide by 2 ten times, then multiply by 10^{10}. Since r is divisible by many 2's, the final digits when we divide r repeatedly by 2 are 8, 4, 2, 6, 8, 4, 2, 6, 8, 4. Thus the last non-zero digit of 49! is 4.

Comment. We can ask the closely related but harder: for what n does the decimal expansion of $n!$ end in a 6 followed by ten 0's? The answers are $n = 45$, 46, and 48. The ideas used in dealing with 49! can be developed into general machinery.

VII-28. Find an integer n such that 63 divides $100n + 81$.

Solution. Note that $100n + 81$ is divisible by 63 if and only if $100n + 81$ is divisible by 9 and by 7.

Since 9 divides 81, it follows that 9 divides $100n + 81$ if and only if 9 divides n. Let $n = 9q$. Then $100n + 81 = 9(100q + 9)$. We need to choose q so that 7 divides $100q + 9$.

But $100q + 9 = (98q + 7) + (2q + 2)$, and 7 divides $98q + 7$, so 7 should divide $2q + 2$, or equivalently 7 should divide $q + 1$. It is impossible not to spot the answer $q = 6$.

Since $n = 9q$, we can take $n = 54$. There are infinitely many other solutions. For 63 divides $100n + 81$ if and only if 63 divides $100(n + 63k) + 81$. Thus other possibilities for n are 117, 180, -9, and so on.

Comment. The numbers were small, so we used ad hoc methods. Note the use of *Divide and Conquer:* a problem about the medium-sized number 63 was solved by splitting the problem into two subproblems involving the smaller numbers 9 and 7.

There are efficient general procedures for answering questions like this even for very large numbers. Look for the *Extended Euclidean Algorithm* in Number Theory books or on the Internet.

VII-29. Find the least positive integer n such that 99 divides $2^n - 1$.

Solution. We could start calculating. There are difficulties with that, since the problem poser could have sent us on a wild goose chase—is there really such an n? Also, we could soon be looking at numbers too large to fit in a calculator.

We show how to avoid large numbers. We want to find an n such that the remainder when $2^n - 1$ is divided by 99 is 0, or equivalently the remainder when 2^n is divided by 99 is 1. Let r be the remainder when 2^k is divided by 99. We find the remainder when 2^{k+1} is divided by 99.

We have $2^k = 99q + r$ for some quotient q, and therefore

$$2^{k+1} = 2(2^k) = 2(99q + r) = 198q + 2r.$$

Since 99 divides 198, the remainder when 2^{k+1} is divided by 99 is the same as the remainder when $2r$ is divided by 99, and that's easy to find.

Make a list of remainders when 2^k is divided by 99. We are looking for a remainder of 1, so start say at $k = 7$, since nothing below 7 has a chance. We get remainders 29, 58, 17, 34, and so on. If we have a programmable calculator or a computer, we can write a program to do the work. But we can also grind things out. The least positive n that gives a remainder of 1 is 30.

Another way: We need the smallest positive n such that $2^n - 1$ is divisible by 9 and by 11. These are a lot smaller than 99, and that may make up for having to look at two problems.

First deal with 9. Just as in the first solution, note that if r is the remainder when 2^k is divided by 9, then the remainder when 2^{k+1} is divided by 9 is the same as the remainder when $2r$ is divided by 9. We get the sequence of remainders 2, 4, 8, 7, 5, 1, 2, ...

The remainder is 1 at $n = 6$. Note also that there is cycling, so the remainder is 1 at $n = 6$, $n = 12$, $n = 18$, and so on, and nowhere else.

Repeat the computation for 11. The remainders on division by 11 are 1 for $n = 10$, 20, 30, and so on. The smallest n which is in both lists is 30.

Comment. We show that without doing any calculation, we can be completely confident that *there is* an n such that 99 divides $2^n - 1$.

Imagine computing the remainders when $2^n - 1$ is divided by 99, for $n = 1$, 2, ..., 99. The remainder when an integer N is divided by 99 must be one of 0, 1, ..., 98. So in our computations, either we bump into a remainder of 0, or some remainder occurs two or more times.

Suppose that $2^s - 1$ and $2^t - 1$ give the same remainder on division by 99, where $1 \le s < t \le 99$. Then 99 divides the difference between the numbers, namely $2^t - 2^s$. Since that difference is $2^s(2^{t-s} - 1)$, we conclude that 99 divides $2^n - 1$, where $n = t - s$. So we were sure to bump into a suitable n sooner or later.

Exactly the same argument shows that if a and m are positive integers which have no common divisor greater than 1, then there is a positive integer n, with $n \le m - 1$, such that m divides $a^n - 1$.

VII-30. Find the smallest positive integer n such that $2n$ is a perfect square, $3n$ is a perfect cube, and $5n$ is a perfect fifth power.

Solution. The positive integer x is a perfect square precisely if any prime p occurs to an even power in the prime factorization of x. More generally, x is a perfect k-th power iff for any prime p, the number of p's in the prime factorization of x is a multiple of k.

Any perfect square that is divisible by 2 must be divisible by 2^2. So if $2n$ is to be a perfect square, 4 must divide $2n$, and therefore 2 must divide n. Similarly, we find that 3 and 5 must divide n. Thus $n = 2^a 3^b 3^c q$, where a, b, and c are positive and q is not divisible by 2, 3, or 5.

For $2n$ to be a perfect square, $a+1$, b, and c must be even. Similarly, for $3n$ to be a perfect cube, a, $b+1$ and c must be divisible by 3, and for $5n$ to be a perfect fifth power, a, b, and $c+1$ must be divisible by 5.

First we find a such that 2 divides $a+1$ and 3 and 5 divide a. So a must be odd and 15 must divide a. The smallest possible a is 15. Similarly, the smallest b is 21, and the smallest c is 24. So $n = 2^{15} 3^{21} 5^{24}$.

VII-31. Let $N = 3^{2003} + 5^{2003} + 7^{2003}$. Find the rightmost digit in the decimal expansion of N.

Solution. The rightmost digit in the decimal expansion of 5^n is 5 for every positive n. Look now at powers of 3, starting with 3^1. The rightmost digits are 3, 9, 7, 1, 3, 9, and so on. Note the cycling with period 4. Thus the rightmost digit of 2^{2003} is the same as the rightmost digit of 3^3, namely 7.

Repeat the calculation for powers of 7. We get 7, 9, 3, 1, 7, $\dots$, so the rightmost digit of 7^{2003} is 3. Since $7+5+3=15$, the required rightmost digit is 5.

Another way: The powers 3^n end in turn with 3, 9, 7, 1, $\dots$, and the powers 7^n end with 7, 9, 3, 1, $\dots$. Note that $3^n + 7^n$ ends in 0 for all odd n, so plays no role.

Comment. Look at the last digit of $a^n + (10-a)^n$, or more generally at the remainder when $a^n + (d-a)^n$ is divided by d. Imagine multiplying $d + (-a)$ by itself n times and expanding. We get a bunch of terms that contain d and a final term $(-a)^n$. If n is odd, this final term is $-a^n$. Thus $a^n + (d-a)^n$ is equal to $a^n - a^n$ plus something divisible by d, so it is divisible by d. In particular, because 2003 is odd, $3^{2003} + 7^{2003}$ is a multiple of 10.

VII-32. Find three consecutive integers greater than 20 such that the smallest is a multiple of 20, the second a multiple of 21, and the third a multiple of 22.

Solution. Let the smallest of the three numbers be $20a$. We want 21 to divide $20a+1$. Now $20a = 21a - a$, so 21 divides $20a+1$ if and only if 21 divides $a-1$, that is, iff $a = 21b+1$ for some b.

Finally, 22 should divide $20(21b+1)+2$; equivalently, 11 should divide b. Putting $b = 11$ gives $a = 232$, so the smallest numbers that work are 4640, 4641, and 4642.

In general, $b = 11t$ for some positive integer t, so the first number must have shape $4620t + 20$.

Another way: Let the numbers be $N + 20$, $N + 21$, and $N + 22$. Since 20 divides the first, 20 must divide N. Similarly, 21 and 22 divide N. The smallest positive integer N simultaneously divisible by 20, 21, and 22 is their least common multiple, namely 4620.

Comments. 1. In the first solution, the three divisibility conditions were handled one at a time. In the second solution, we took advantage of the symmetry and handled them all at once.

2. There are general techniques for finding a number x such that m_1 divides $x - a_1$, m_2 divides $x - a_2$, m_3 divides $x - a_3$, and so on. To find information, search in Number Theory books or on the Web for the phrase *Chinese Remainder Theorem*.

VII-33. A jar contains 1000 coins, worth altogether \$99.95. Any coin is either a five-cent piece, a dime, or a quarter. Find the largest number of quarters there could be in the jar.

Solution. Suppose that there are x five-cent pieces, y dimes, and z quarters. The coins are worth $5x + 10y + 25z$ cents, so $5x + 10y + 25z = 9995$. Equivalently, $x + 2y + 5z = 1999$. Since there are 1000 coins, $x + y + z = 1000$. From these two equations, we obtain $y + 4z = 999$. The largest multiple of 4 less than 999 is 996, so the largest possible z is 249.

Another way: We can proceed more informally, taking advantage of the fact that the numbers are easy to work with. One way of getting \$99.95 with 1000 coins is to use 999 dimes and 1 five-cent piece.

Trade in dimes for five-cent pieces and quarters, leaving the total value and the number of coins unchanged. The smallest possible trade is 4 dimes for 3 five-cent pieces and 1 quarter. Do this as often as possible. The largest multiple of 4 that is below 999 is 996, so we make 996/4 trades and end up with 249 quarters.

VII-34. A rooster is worth five ch'ien, a hen three ch'ien, and three chicks one ch'ien. With 100 ch'ien we buy 100 of them. How many roosters, hens, and chicks are there?

Solution. Suppose that there are x roosters, y hens, and z chicks. Then

$$x + y + z = 100 \qquad \text{and} \qquad 5x + 3y + \frac{z}{3} = 100.$$

Eliminate z by "multiplying" the second equation by 3, then subtracting the first equation. We get $14x + 8y = 200$, or equivalently $7x + 4y = 100$. There is no sense in developing general theory for such a simple equation. An obvious solution is $x = 0$, $y = 25$, which gives $z = 75$.

To get *all* of the solutions, note that if $7x + 4y = 100$ and $7x' + 4y' = 100$, then $7(x' - x) = 4(y - y')$, and therefore in particular 4 must divides $x' - x$. And every time that we increase x by 4, we must decrease y by 7.

Thus the general solution in integers is $x = 4t$, $y = 25 - 7t$, $z = 75 + 3t$, where t is an integer. But our numbers must be non-negative, so t only takes on the four values 0, 1, 2, and 3.

Comment. This problem comes from the fifth-century mathematical manual of Zhang Qiujian, but may be much older. "Hundred Fowl" problems occur repeatedly in the Chinese literature.

Somehow, the problem travelled to Western Europe, probably through Arab intermediaries. There are seven "Hundred Fowls" problems in *Propositiones ad acuendos juvenes* (Problems to Sharpen the Young), a collection attributed, probably incorrectly, to the late eighth-century scholar Alcuin of York. In one of these problems, 100 bushels of barley are distributed among 100 people, each man receives 3 bushels, each woman 2, and every two children get a bushel—in old problems, women always get less.

Mahāvīra (ninth century) has a problem in which pigeons are sold at 5 for 3 coins, *sārasa* birds at 7 for 5 coins, swans at 9 for 7 coins, and peacocks at 3 coins each. A man bought for the king's son 100 birds and paid 100 coins. In Euler's *Algebra,* someone buys 100 animals for 100 crowns, hogs at $3\frac{1}{2}$ crowns each, goats at $1\frac{1}{3}$ crowns each, sheep at $\frac{1}{2}$ crown each.

The ninth-century mathematician abū Kāmil tackles more elaborate "Hundred Fowls" problems. For example, he asks for the ways that one can buy 100 birds with 100 drachmas, if ducks cost 2 drachmas each, hens are 1 drachma, doves are 2 per drachma, ring-doves are 3 per drachma, and larks are 4 per drachma. Most mathematicians of the time were satisfied to find a single solution to a problem. Abū Kāmil took a more modern point of view, and wanted all solutions. He asserted that in this case there are 2696. There are in fact 2678.

VII-35. The arithmetic progression 9, 32, 55, 78, ... begins with the perfect square 9. Find the sum of the *next three* perfect squares in this arithmetic progression.

Solution. We could find the next three squares by playing with a calculator, or programming a computer to search. But thinking is faster and easier. The arithmetic progression has common difference 23, and therefore the terms are $9 + 23n$, for $n = 0$, 1, 2, and so on.

We want $9 + 23n$ to be a perfect square, say x^2. Rewrite $9 + 23n = x^2$ as $23n = x^2 - 9 = (x-3)(x+3)$. We need to make sure that 23 divides $x - 3$ or $x + 3$. The smallest x greater than 3 comes from putting $x + 3 = 23$; so $x^2 = 400$. The next x comes from putting $x - 3 = 23$; so $x^2 = 676$. Finally, the next x is obtained by putting $x + 3 = 2 \cdot 23$; so $x^2 = 1849$. The required sum is 2925.

VII-36. Find the largest integer k such that 3^k divides 100!.

Solution. We need to "count 3's" in the numbers from 1 to 100. Note that 3 contributes one 3, as does 6. Then 9 contributes two 3's, and 12 contributes another one. So in particular the largest k such that 3^k divides 12! is 5. We can continue all the way to 100. The work is straightforward, and if we are careful we should find the answer. But there are faster ways.

One number, namely 81, contributes four 3's. Two numbers, namely 27 and 54, contribute three each. The multiples of 9 apart from 81, 27, and 54 contribute two each. There are 11 multiples of 9 from 1 to 100, so 8 numbers contribute two 3's. Finally, the rest of the multiples of 3 contribute one each. There is a total of 33 multiples of 3 from 1 to 100, so 22 of them contribute one 3 each. Thus

$$k = (1 \cdot 4) + (2 \cdot 3) + (8 \cdot 2) + (22 \cdot 1) = 48.$$

Another way: There is a more efficient way of counting. Imagine that each number from 1 to 100 has to pay a one dollar fine for each 3 "in it." There are 33 numbers which are multiples of 3. Collect a dollar from each one. Then collect an additional dollar from each of the 11 multiples of 9. Then collect still another dollar from each of the 3 multiples of 27, and finally collect a dollar from the only multiple of 81. Each number has now paid its proper fine, and the fines total $33 + 11 + 3 + 1$.

Comment. The second argument generalizes nicely. Let p be a prime, and let p^k be the largest power of p that divides $n!$. Collect a dollar from each multiple of p in the interval from 1 to n. How many dollars do we get? There are $\lfloor n/p \rfloor$ multiples of p from 1 to n, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. So for example $\lfloor 100/3 \rfloor = 33$.

Now collect an additional dollar from each multiple of p^2 in our interval. There are $\lfloor n/p^2 \rfloor$ of these. Go on in this way. Ultimately we find that

$$k = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots .$$

The theme crops up often in mathematical contests. Usually the question is about the number of 0's that $n!$ ends in. So we are looking for the largest power of 10 that divides $n!$. Since 10 isn't prime, we can't apply the idea directly. But there are always more 2's than 5's in $n!$. Since 5's are the limiting resource, we can take $p = 5$.

VII-37. (a) The numbers 16000, 16001, 16002, ..., 16006 are divided by 7. Find the sum of the seven remainders so obtained. (b) Find the sum of the remainders when 16000, 16100, 16200, ..., 16600 are divided by 7.

Solution. (a) Divide 16000 by 7; the remainder is 5. We could repeat the division with 16001, but it is obvious that the remainder advances by 1 to 6. For 16002, the remainder again "advances by 1," to 0. The remainders are 5, 6, 0, 1, 2, 3, 4, and their sum is 21.

(b) The first remainder is 5. For the next, note that we are adding 100, which is $98+2$. Since 98 is a multiple of 7, it follows that the remainder of 16100 on division by 7 "advances" by 2 from 5, so it is 0. Add 100 again, and the remainder advances by 2 again. So the sequence of remainders is 5, 0, 2, 4, 6, 1, 3. Again the sum is 21.

Another way: In part (a), we don't need to calculate any remainders! For our sum is the sum of the remainders when 7 consecutive numbers are divided by 7. These remainders must be 0, 1, ..., 6 in some order, so the sum is 21.

The same idea works for (b). Look at the remainders when the numbers $16000 + 100t$ are divided by 7, where t ranges from 0 to 6. These remainders are all *different*. For if two of the numbers had the same remainder, their difference would be divisible by 7, which is clearly not the case. It follows that the remainders are again 0, 1, ..., 6 in some order.

Comment. Part (a) generalizes nicely. Let m be a positive integer. Add up the remainders when m consecutive numbers are divided by m. These remainders are 0, 1, 2, ..., $m-1$ in some order, so the sum is always $m(m-1)/2$.

For part (b), suppose that a and d have no common factor greater than 1. Then the remainders when a, $a+d$, $a+2d$, ..., $a+(m-1)d$ are divided by d are just 0, 1, ..., $m-1$ in some order. If m and d do have a common factor greater than 1, the situation is somewhat more complicated. This could be an interesting problem to explore.

VII-38. How many different positive integers divide $2^2 \cdot 3 \cdot 5 \cdot 7$?

Solution. Our number doesn't have many positive divisors, so we can list them all and then count. To make things interesting, look at a more general problem. Let $N = 2^a \cdot 3^b \cdot 5^c \cdot 7^d$, where a, b, c, d are positive integers. We calculate the number of positive divisors of N.

We first count the positive divisors of 2^a. Here is a complete list: 2^0, 2^1, ..., 2^a, altogether $a+1$ of them. Next count the positive divisors of $2^a \cdot 3^b$. We make all of them by multiplying any one of the $a+1$ positive divisors of 2^a by one of 3^0, 3^1, ..., 3^b. Thus there are $(a+1)(b+1)$ such divisors.

How many positive divisors does $2^a \cdot 3^b \cdot 5^c$ have? We make such a divisor by multiplying any one of the $(a+1)(b+1)$ positive divisors of $2^a \cdot 3^b$ by one of 5^0, 5^1, ..., 5^c. So there are $(a+1)(b+1)(c+1)$ such divisors. Similar reasoning shows that N has $(a+1)(b+1)(c+1)(d+1)$ positive divisors. In particular, 420 has 24 positive divisors.

Comment. There is a more picturesque way of describing the same idea. In front of us, we have a box that has a 2's in it, another box with b 3's, another with c 5's, and so on.

To make a positive divisor of $2^a \cdot 3^b \cdot 5^c \cdot 7^d$, first we decide how many 2's to grab from the first box. There are $a+1$ choices, since we might grab no 2's. For *every* choice from the first box, there are $b+1$ ways of deciding how many 3's to take from the second box, and so on.

VII-39. If we don't count order, the number 5 can be written as a sum of one or more positive integers in exactly seven different ways, namely 5, $4+1$, $3+2$, $3+1+1$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$. (Mathematicians say that 5 has seven *partitions*.) How many partitions does the number 7 have?

Solution. We list carefully the partitions and then count. It is convenient to first count the partitions of 6. Think of the partitions of 6 that have a 1 in them. Remove the last 1 from these partitions. We get all the partitions of 5, so there are 7 partitions of 6 that have a 1. The rest are so easy to list that maybe we shouldn't think too much: they are 6, $4+2$, $3+3$, and $2+2+2$, for a total of 11.

Now count the partitions of 7. By the same argument as above, 7 has 11 partitions that have a 1. The rest of the partitions can be listed explicitly: 7, $5+2$, $4+3$, $3+2+2$, for a total of 15. (The fact that we went up by 4 and then by 4 again is an accident that doesn't recur.)

VII-40. Find the largest integer n such that $n+7$ divides n^3+7.

Solution. Divide the polynomial x^3+7 by $x+7$. We get

$$\frac{x^3+7}{x+7} = x^2 - 7x + 49 - \frac{336}{x+7}.$$

Since $n^2-7n+49$ is an integer for every integer n, we only need to find the largest n such that $336/(n+7)$ is an integer. It is clear that $n = 329$.

Comment. Let $P(x)$ be a polynomial with integer coefficients, and let a be an integer. What is the largest integer n such that $n-a$ divides $P(n)$? The argument is essentially the same as in the numerical example above. *If* we divided $P(x)$ by $x-a$, we would get

$$\frac{P(x)}{x-a} = Q(x) + \frac{r}{x-a},$$

where $Q(x)$ is a polynomial with integer coefficients, and r is the remainder when $P(x)$ is divided by $x-a$. By the Remainder Theorem, $r = P(a)$.

We conclude that $P(n)/(n-a)$ is an integer precisely if $P(a)/(n-a)$ is an integer. If $P(a) = 0$, then $P(n)/(n-a)$ is an integer for any $n \neq a$. If $P(a) \neq 0$, then the largest n for which $P(n)/(n-a)$ is an integer is $a + |P(a)|$.

VII-41. Describe all of the ways to express 500 as a sum of consecutive positive integers.

Solution. The numbers are small enough that we can experiment. Does just 500 by itself qualify? That depends on how we use language. A mathematician would say that it does. Others might want to see a plus sign before calling an expression a sum.

We can try to use two numbers. They would have to average out to 250, and it is clear that nothing works. The same thing happens with three numbers and with four. We can do it with five: 98, 99, 100, 101, 102. We can push on, learning shortcuts along the way, and after a while find a complete answer, but there is a better way.

First we try to express 500 as a sum of an odd number $2k+1$ of positive integers. Let the middle number be m. So $(2k+1)m = 500$. Since the numbers should be positive, m should be bigger than k. We can have $2k+1 = 1$, giving a bare 500. Or $2k+1 = 5$, giving $m = 100$, and the numbers 98, 99, 100, 101, 102. Or $2k+1 = 25$, giving $m = 20$ and the numbers 13, 14, …, 25, …, 37. Putting $2k + 1 = 125$ makes m smaller than k, so it is only allowed if we remove the condition that the numbers have to be positive.

Now try to express 500 as a sum of an even number $2k$ of positive integers. Let the two middle numbers be m and $m + 1$. Then the sum of all the numbers is $k(2m + 1)$. In order to make all the numbers positive, we need $m \geq k$. The only possibility is $2m+1 = 125$, which gives the numbers 59, 60, 61, 62, 63, 64, 65, 66.

Comment. We could also have solved the problem by letting the numbers be a, $a+1$, …, $a+(k-1)$, and using the formula for the sum of an arithmetic progression. But the expressions we get are uglier—try it. Exploiting the symmetry by concentrating on the middle point makes things easier.

The requirement that the numbers be positive makes the problem more messy. It isn't hard to develop a general theory if numbers are allowed to be negative.

We can generate other problems that use the same ideas. For example, we can ask for all the ways of expressing a given integer as a sum of consecutive *odd* positive integers, or as a sum of consecutive integers in the sequence 2, 5, 8, 11, ….

VII-42. Call a set of integers *good* if there are three numbers in the set whose sum is divisible by 3. Find the smallest n such that every set with n or more elements is good.

Solution. How can $a + b + c$ be divisible by 3? After a little playing we can see that the remainders when we divide a, b, and c by 3 must either be all the same, or they must be 0, 1, and 2 in some order.

If there are seven or more numbers, then some remainder occurs at least three times, so any set with seven or more elements is good. What about sets of six? If some remainder occurs three or more times, the set is good. If none does, then two each of the remainders must be 0, 1, and 2, so the set is good.

What about sets of five? Again, if some remainder occurs three or more times, then the set is good. And if all remainders occur two or fewer times, then each of 0, 1, and 2 must occur, and again the set is good. What about sets of four elements? The set $\{0, 1, 3, 4\}$ is not good, so $n = 5$.

Comment. If we define n as above, but with three replaced everywhere by four, the problem becomes more difficult. If a set has thirteen or more elements, then some remainder occurs at least four times, so $n \leq 13$. But finding n exactly takes a longer argument.

More generally, we can call a set m-good iff there are m numbers in the set whose sum is divisible by m. It turns out that every set of $2m-1$ numbers is m-good, but some sets of $2m-2$ elements are not m-good. We do not know a simple proof.

VII-43. Let $P(x) = (x-1)(x-2)(x-3)\cdots(x-19)$. When we multiply this out, the result has shape $a_0x^{19} + a_1x^{18} + \cdots + a_{19}$. How many of the a_i are odd?

Solution. We could actually multiply, but that sounds like hard work.

Let $Q(x)$ be any polynomial with integer coefficients, and look at $(x+e)Q(x)$, where e is even. The coefficients of $eQ(x)$ are all even. So a coefficient of $xQ(x) + eQ(x)$ is odd iff the corresponding coefficient of $xQ(x)$ is odd. Thus $(x+e)Q(x)$ has just as many odd coefficients as $xQ(x)$, which in turn has just as many odd coefficients as $Q(x)$.

If we use the above idea repeatedly, we see that $(x-1)(x-2)\cdots(x-19)$ has the same number of odd coefficients as $(x-1)(x-3)(x-5)\cdots(x-19)$. We can do better. Note that $x-3 = (x-1)-2$, $x-5 = (x-1)-4$, and so on. It follows that $P(x)$ and $(x-1)^{10}$ have the same number of odd coefficients.

By the Binomial Theorem, the coefficient of x^k in $(x-1)^{10}$ is (apart from sign, which we don't care about) $\binom{10}{k}$, that is, $(10!)/(k!)(10-k)!$. There is symmetry about $k=5$, so calculation is short. There are 4 odd coefficients.

Another way: To find the number of odd coefficients of $(x-1)^{10}$, or of much larger powers of $x-1$, use the following idea. Replace five of the $x-1$ terms by $x+1$. That doesn't change the parity of the coefficients. So we want the number of odd coefficients of $(x^2-1)^5$, that is, of $(x^2-1)(x^2-1)^4$. Replace two of the x^2-1 terms by x^2+1. The parity of the coefficients does not change, so we want the number of odd coefficients of $(x^2-1)(x^4-1)^2$, which is the same as the number of odd coefficients of $(x^2-1)(x^8-1)$. Finally, we calculate! The answer is 4.

Comment. The idea of the second method generalizes. We can use it to show that the number of odd coefficients in the expansion of $(x-1)^n$ is simply the number of 1's in the binary expansion of n. For example, the binary expansion of 119 is 1110111, and exactly 6 of the numbers $\binom{119}{k}$ are odd.

VII-44. Let b and c be odd integers. Show that $x^2 + 2bx + 2c = 0$ does not have integer solutions.

Solution. Let r be one of the roots. We show that r can't be an integer by showing that r can't be odd and r can't be even.

If r were odd, then since the sum of the roots is $-2b$, the other root would be an odd integer. Then the product of the roots would be odd. But this product is the even number $2c$. We have reached a contradiction, and conclude that r can't be odd.

If r were even, then since the sum of the roots is $-2b$, the other root would be an even integer. Then the product of the roots would be divisible by 4. But this product is $2c$, twice an odd number, so it is not divisible by 4. So again we have reached a contradiction, and conclude that r can't be even.

Another way: The roots of our polynomial are $-b \pm \sqrt{b^2 - 2c}$. To show that the roots are not integers, we show that $b^2 - 2c$ can't be a perfect square.

Note that b^2 is the square of an odd number. The square of the odd number $2k+1$ is $4k^2+4k+1$, and so is of the form $4x+1$ for some integer x. Another way of putting it is that when we divide the square of an odd number by 4, the remainder must be equal to 1.

Also, c is odd, say $c = 2u+1$. Then $2c = 4u+2$—equivalently, $2c$ leaves a remainder of 2 when we divide it by 4. Thus $b^2 - 2c$ is of the form $4n+3$, and therefore can't be a perfect square.

Comment. The square of $2k+1$ is $4(k^2+k)+1$. But k^2+k is always even, so actually the square of an odd integer has shape $8n+1$. As an illustration, we can tell instantly that 1234567837 isn't a perfect square: that number is 37 more than a multiple of 8, so it is of the shape $8n+5$. A few other facts of this type are occasionally useful in contests. For example, any perfect square which is not divisible by 3 is of the shape $3n+1$, and any perfect square which is not divisible by 5 is of the form $5n \pm 1$.

VII-45. Find an integer n such that $5n+12$ and $8n+11$ have a common divisor greater than 1.

Solution. We can find such an n by trying 0, 1, 2, and so on until we bump into something that works. But maybe there is no such n, the poser of the problem made a mistake, and we will calculate forever. We need an idea.

Suppose that d divides $5n+12$ and $8n+11$. Then d divides $a(5n+12)-b(8n+11)$ for any integers a and b. To make things look nice, take $a = 8$ and $b = 5$. So d divides 41. Since 41 is prime, if $d > 1$, then d must be 41.

We are looking for an integer n such that 41 divides $5n+12$. Let $5n+12 = 41q$. We want $41q - 12$ to be divisible by 5. The smallest positive q that works is 2. So if $n = 14$, then $5n+12$ is divisible by 41.

If $n = 14$, then $8n+11$ is also divisible by 41. We can show this by simply dividing, or else note that since $8(5n+12)-5(8n+11) = 41$, and 41 divides $5n+12$, it follows that 41 *must* divide $8n+11$.

VII-46. Find all *non-negative* integers x, y such that $5x+7y = 200$.

Solution. Let (x, y) be a solution. Since 5 divides 200 and 5 divides $5x$, it follows that 5 must divide $7y$, and therefore 5 must divide y. Let $y = 5u$. Substitute and simplify. We get $x+7u = 40$. Note that u can be any of 0, 1, 2, 3, 4, or 5. The corresponding x are 40, 33, 26, 19, 12, and 5.

Another way: One solution is easy to pick out: $x = 40$, $y = 0$. Shrink x by an amount a, and compensate by increasing y by b. Then $5x + 7y$ changes by $5a - 7b$. We want this change to be 0. So $5a = 7b$, and therefore $a = 7t$ for some t, and $b = 5t$. Thus the general solution of our equation in integers is $x = 40 - 7t$, $y = 5t$, where t is any integer. For non-negative solutions, we need $0 \le t \le 5$.

Comment. Many "word problems" come down to finding all non-negative integer solutions of $ax + by = c$, where a, b, and c are positive integers. A generic example looks like this: "Mangoes cost 49 cents each, and papayas \$1.19. I bought some of each and paid \$11.83. How many of each did I buy?" This sort of problem can be solved by an efficient systematic procedure like the *Extended Euclidean Algorithm* or an inefficient systematic procedure (try everything) or something in between. Sometimes we can exploit special properties of the numbers (49 and 119 are both divisible by 7) to shorten the calculation.

VII-47. When we divide 5500, 6070, and 6469 by the positive integer m, the remainders are all equal. Find all possible values of m.

Solution. The remainders are equal if and only if m divides both $6070 - 5500$ and $6469 - 6070$. So we want m to divide 570 and 99. The numbers are small, so we can identify m without much trouble. Note that $399 = 3 \cdot 7 \cdot 19$, while $570 = 2 \cdot 3 \cdot 5 \cdot 19$. Thus the largest integer that divides both is $3 \cdot 19$, and the values of m are the positive divisors of 57, namely 1, 3, 19, and 57.

Comment. If the numbers were much larger, factorization might take impractically much time. Here is an idea that works even for numbers much larger than the ones we are dealing with. The common divisors of 570 and 399 are the same as the common divisors of 399 and $570 - 1 \cdot 399$, that is, of 399 and 171. And the common divisors of 399 and 171 are the same as the common divisors of 171 and $399 - 2 \cdot 171$, that is, of 171 and 57. Finally, the common divisors of 171 and 57 are the same as the common divisors of 57 and $171 - 3 \cdot 57$, that is, of 57 and 0. And the greatest common divisor of 57 and 0 is obviously 57. This procedure for finding greatest common divisors is called the *Euclidean Algorithm.* It is of great theoretical and practical importance.

VII-48. A child brought some pennies to the bank, one-third as many 5-cent coins as pennies, one-third as many dimes as 5-cent coins, and somewhere between 75 and 100 quarters. In exchange, she received a \$100 bill. How many quarters did she bring?

Solution. Let d be the number of dimes and q the number of quarters. Then there are $3d$ five-cent pieces and $9d$ pennies. Thus

$$9d + 15d + 10d + 25q = 100 \cdot 100,$$

which simplifies to $34d + 25q = 10000$. Since 25 divides two of the terms, it must divide d; Let $d = 25x$. Thus $34x + q = 400$, so $400 - q$ must be a multiple of 34.

Now look for multiples of 34 between $400-100$ and $400-75$. There is exactly one, namely 306, so $q = 94$.

VII-49. Find the remainder when 2^{524} is divided by 144.

Solution. Imagine that the division has been done. Then $2^{524} = 144q + r$, where q is the quotient and r the remainder. Since 16 divides both 2^{524} and 144, it must divide r. Let $r = 16s$; then $2^{520} = 9q + s$. It remains to compute the remainder s when 2^{520} is divided by 9.

Find the remainders when successive powers of 2, starting with 2^0, are divided by 9. The first few are 1, 2, 4, 8, 7, 5, 1, 2, 4. We assume that the apparent cycling with period 6 is real. Then the remainder when 2^n is divided by 9 is 1 whenever n is a multiple of 6. The largest multiple of 6 up to 520 is 516. Four more steps get us to 520, so the remainder s is 7. It follows that $r = 16 \cdot 7 = 112$.

To prove that there is indeed cycling, we show that the remainder when 2^{n+6} is divided by 9 is equal to the remainder when 2^n is divided by 9, that bumping the exponent up by 6 makes no difference. All we need to do is to show that 9 divides $2^{n+6} - 2^n$. That's easy, for $2^{n+6} - 2^n = 2^n(2^6 - 1) = 63 \cdot 2^n$.

Comments. 1. Let a and m be integers, with m positive. Look at the remainders when a^0, a^1, a^2, and so on are divided by m. There are at most m possible values of the remainder, so by the time we reach a^m we must have seen one remainder at least twice. Suppose that a^s and a^t give the same remainder, where $s < t$.

We show that the "next" remainders are equal. We know that m divides $a^t - a^s$, and want to show that m divides $a^{t+1} - a^{s+1}$. This is easy, for

$$a^{t+1} - a^{s+1} = a(a^t - a^s).$$

Since the remainders for a^{s+1} and a^{t+1} are equal, the remainders for a^{s+2} and a^{t+2} are equal, and so on. Thus there is cycling from a^s on.

2. We show how to compute directly the remainder when a^{524} is divided by 144, for *any* integer a. Find the remainder when a^2 is divided by 144. Use that to find the remainder when a^4 is divided by 144, use that to find the remainder when a^8 is divided by 144, and so on up to a^{512}. Then use the fact that $a^{524} = a^{512} \cdot a^8 \cdot a^4$.

Call the process of multiplying two numbers and finding their remainder on division by 144 a *step.* Then we find our remainder in 11 steps. That's far better than the 519 steps we would take if we multiplied by a, then by a again, and so on.

In the real world of *public key cryptography*, computers use essentially the same idea with enormously large exponents. The method is called the *Binary Method for Exponentiation.* It has many uses. For other examples, see VII-7 and IX-28.

VII-50. For what integers n is $n^2 + n + 17$ a perfect square?

Solution. When we work with quadratics, it is often useful to complete the square. But in number-theoretic questions, it is wise to avoid fractions. So first we multiply by 4. Note that

$$n^2 + n + 17 = y^2 \quad \text{iff} \quad 4n^2 + 4n + 68 = 4y^2 \quad \text{iff} \quad (2n+1)^2 + 67 = 4y^2.$$

We need to find integers y, n such that $67 = (2y)^2 - (2n+1)^2$. Let $2y = u$, $2n+1 = v$. We need to have $(u-v)(u+v) = 67$, with u even (and v odd, but that's automatic if u is even).

Luckily, 67 has very few factors. There are four possibilities: $u - v = 1$, $u + v = 67$; $u - v = -1$, $u + v = -67$; $u - v = 67$, $u + v = 1$; and $u - v = -67$, $u + v = -1$. (Actually, the last two must give the same n and y as the first two.)

For each possibility, find v, and then n. We get $n = 16$ or $n = -17$.

Comment. "Completing the square" is a simple but powerful idea which is underused in high school, while factoring by inspection is visited repeatedly, from grade 9 all the way to grade 12. Factoring expressions like $a^2 - b^2$ and $a^3 - b^3$ is indeed important, but factoring by inspection expressions like $x^2 - 6x - 91$ has almost no uses in real mathematics.

VII-51. Can a triangle with sides 13, 14, 15 be split into two right-angled triangles with integer sides?

Solution. Let the corners of the original triangle be A, B, and C, with $AB = 13$, $AC = 15$, and $BC = 14$. If the task can be done, then one side will be cut into two parts, and the other two will not. The right angles are created in the process of splitting, and each uncut side becomes a hypotenuse.

So we ask: Which of 13^2, 14^2, 15^2 can be expressed as the sum of two non-zero squares? The numbers are small, so we can find the answer by inspection: 13^2 and 15^2 can be, and 14^2 cannot. Thus *if* the splitting can be done, we have to split the side of length 14. Drop a perpendicular from A to the point P on BC.

Let $h = AP$, $x = BP$ and $y = PC$. By the Pythagorean Theorem, we want

$$h^2 = 13^2 - x^2 = 15^2 - y^2 \qquad \text{and} \qquad y + x = 14.$$

So we need $y^2 - x^2 = 56$. If we divide the left hand side by $y + x$, and the right by 14, we find that $y - x = 4$, and therefore $x = 5$, $y = 9$, and $h = 12$, so indeed the two triangles have integer sides.

VII-52. Find all eight-digit numbers that are divisible by 7, 11, and 13, and whose first four digits are 1234. Hint: Find $7 \cdot 11 \cdot 13$.

Solution. The number n is divisible by 7, 11, and 13 if and only if n is divisible by 1001.

We find the first multiple of 1001 which is greater than 12340000. That can be done by hand or by calculator. With the calculator, divide 12340000 by 1001; the quotient is about 12327.67. Multiply 1001 by 12328; the result is 12340328.

Now keep adding 1001 until the numbers get too big. There are 10 numbers that work. They have decimal expansion 1234 followed by 0328, 1329, ..., 8336, and 9337.

VII-53. The positive integers x, y, z are in arithmetic progression. Their product is 20 times their sum, and $x < z$. What can we say about z?

Solution. Let the numbers be $a-d$, a, and $a+d$. Their sum is $3a$, their product is $a(a-d)(a+d)$, so $60a = a(a-d)(a+d)$. Since $a \neq 0$, we may divide both sides by a, so $(a-d)(a+d) = 60$.

The sum of $a-d$ and $a+d$ is the even number $2a$, so $a-d$ and $a+d$ are both odd (impossible, their product is 60) or both even. The only possibilities are $z = a+d = 10$ and $z = 30$, for $z = 2$ and $z = 6$ yield negative x.

Comment. We could work directly with x, y, and z. The resulting solution is equally short, and in some sense simpler. But the symmetrization $x = a-d$, $y = a$, $z = a+d$ should probably be almost a reflex action.

VII-54. How many different ordered pairs (x, y) of positive integers are there such that the greatest common divisor of x and y is 24 and their least common multiple is 14400?

Solution. Let (x, y) be such a pair, and let $x = 24a$ and $y = 24b$. Then a and b have no factor greater than 1 in common, and therefore the least common multiple of x and y is $24ab$. But this is 14400, so $ab = 600$.

Conversely, suppose that a and b have no factor greater than 1 in common and $ab = 600$. If we set $x = 24a$ and $y = 24b$, we obtain a solution to our problem. So we need to count the number of pairs (a, b) such that a and b have no common factor greater than 1 and $ab = 600$.

Note that $600 = 2^3 \cdot 3 \cdot 5^2$. First decide whether a gets the 2's, yes or no. If the answer is no, give the 2's to b. (Since a and b have no factor greater than 1 in common, either a gets all the 2's or b does.)

Then decide whether a gets the 3, yes or no, then whether a gets the 5's. At every prime, there are two choices, so the total number of choices is 2^3.

VII-55. Find all integers n such that $(31.5)^n + (32.5)^n$ is an integer.

Solution. Let $x = (31.5)^n + (32.5)^n$. If $n \leq -1$, then $0 < x < 1$, so x can't be an integer. If $n = 0$, then x is an integer. Now look at positive n.

Note that $x = (63^n + 65^n)/2^n$. If n is even, then 63^n and 65^n are both perfect squares. The square of an odd integer $2t+1$ is $4(t^2+t)+1$, one more than a multiple of 4. Thus $63^n + 65^n$ is of the form $4k+2$, and in particular is not divisible by 4. But 2^n *is* divisible by 4 for $n \geq 2$, so if n is even and positive then x is not an integer.

Let n be positive and odd, let $a = 63$ and $b = 65$. Then

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \cdots - ab^{n-2} + b^{n-1}).$$

In particular, $63^n + 65^n$ is divisible by $a+b$, that is, by 128. The second term in the product above is a sum of an odd number of odd terms, so it is odd. Thus $x = (128/2^n)q$, where q is odd. So if n is positive and odd, then x is an integer precisely if 2^n divides 128, that is, when $n = 1$, 3, 5, or 7.

Comment. Calculator exploration can be helpful in number-theoretic problems, but it can also mislead. After some experimentation, we might well have conjectured that $(31.5)^n + (32.5)^n$ is an integer whenever n is odd and positive. The first counterexample, namely $n = 9$, involves numbers beyond the reach of standard calculators.

VII-56. Define the sequence (a_n) by

$$a_1 = 2 \quad \text{and} \quad a_{n+1} = a_n^2 - a_n + 1 \quad \text{when} \quad n \geq 1.$$

(a) Suppose that $d > 1$ and d divides a_n. Find the remainder when a_{n+1} is divided by d. (b) Suppose that the remainder when a_n is divided by d is 1. Find the remainder when a_{n+1} is divided by d. (c) Show that no integer greater than 1 divides two different terms of the sequence. (d) Conclude that there are infinitely many primes.

Solution. (a) If d divides a_n, then d divides $a_n^2 - a_n$, and therefore the remainder when we divide $a_n^2 - a_n + 1$ by d is 1.

(b) If the remainder when we divide a_n by d is 1, then $a_n = qd + 1$ for some q. But then

$$a_{n+1} = a_n^2 - a_n + 1 = q^2d^2 + qd + 1,$$

and therefore a_{n+1} leaves a remainder of 1 on division by d.

(c) Suppose that an integer d greater than 1 divides some term(s) of the sequence. Let a_k be the *first* term divisible by d. Then by part (a), a_{k+1} leaves a remainder of 1 on division by d. But then by part (b), all subsequent terms of the sequence also leave a remainder of 1, and in particular are not divisible by d. So no integer greater than 1 can divide more than one term of the sequence.

(d) Finally, the payoff! The sequence is growing, indeed growing very rapidly. For each term a_n of the sequence, there is at least one prime p_n that divides a_n. But no integer greater than 1 can divide two distinct terms, so all the p_n are different, and therefore there are infinitely many primes.

VII-57. Find all right-angled triangles with integer sides such that one of the legs—not the hypotenuse—is 100.

Solution. We need to find positive integers y and z such that $100^2 + y^2 = z^2$, that is, such that $(z - y)(z + y) = 100^2$. In particular, $z - y$ must be a divisor d of 100^2. Then $z + y = 100^2/d$. Since $z - y < z + y$, we conclude that $d < 100$.

Note that $z = [(z - y) + (z + y)]/2$, so d and $100^2/d$ must be of the same parity—both even or both odd—and it is clear they can't be both odd. Conversely, suppose that d is a positive divisor of 100^2 with $d < 100$, and that d and $100^2/d$ have the same parity. Then by setting $z = (100^2/d + d)/2$ and $y = (100^2/d - d)/2$ we obtain a solution.

Since $100^2 = 2^4 \cdot 5^4$, the possibilities for d are $d = 2$, 4, 8, 10, 20, 40, and 50. The rest is straightforward. For example, $d = 40$ yields $z = 145$ and $y = 105$.

VII-58. How many integers x are there, with $0 \leq x < 300$, such that $2^x - x^2$ is a multiple of 5?

Solution. Examine first the remainder when 2^x is divided by 5. For $x = 0$, 1, 2, 3 the remainders are 1, 2, 4, and 3. Then at $x = 4$ the remainder is 1, and the pattern 1, 2, 4, 3 repeats. The remainder when x^2 is divided by 5 exhibits the pattern 0, 1, 4, 4, 1, and this pattern repeats. The cycle length is 4 in the first case, 5 in the second, so if x is stepped up by 20, both patterns repeat. (For a detailed justification of why we can be confident that the patterns continue, see the comments after VII-49.) For $0 \leq x \leq 19$ we get

1	2	4	3	1	2	4	3	1	2	4	3	1	2	4	3	1	2	4	3
0	1	4	4	1	0	1	4	4	1	0	1	4	4	1	0	1	4	4	1

There are matches at $x = 2$, 4, 16, and 18, so there are 4 matches in each cycle of 20. There are 15 such cycles from 0 to 299, so $2^x - x^2$ is a multiple of 5 for 60 integers in our interval.

VII-59. The number $1/n$, where n is a positive integer, is called a *unit fraction*. Express 1 as a sum of seven different unit fractions. Hint: The fact that $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$ may be useful, or the more general $\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$.

Solution. There are several solutions. We have $1 = 1/2 + 1/3 + 1/6$. But then

$$\frac{1}{6} = \frac{1}{6}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) = \frac{1}{12} + \frac{1}{18} + \frac{1}{36}.$$

Using the same idea, we find that $1/36 = 1/72 + 1/108 + 1/216$, and therefore 1 is the sum of unit fractions with denominators 2, 3, 12, 18, 72, 108, and 216.

Or start with $1 = 1/2+1/2$. Let $n = 2$. We get $1/2 = 1/3+1/6$. Now let $n = 3$. We get $1/3 = 1/4+1/12$, which gives the representation $1 = 1/2+1/4+1/6+1/12$. Take $n = 4$. We can replace $1/4$ by $1/5 + 1/20$. Take $n = 6$, and replace $1/6$ by $1/7 + 1/42$. Finally, replace $1/7$ by $1/8 + 1/56$.

VII-60. You want to tile a 15 foot by 15 foot room with square tiles, without cutting any tile. That can be done with the blue tiles. It can also be done with the red tiles, but that requires 72 more tiles than if the blue tiles are used. The red tiles are less than 1 foot in width. How big are the blue tiles?

Solution. Suppose that r red tiles laid end to end measure 15 feet. Then we need r^2 red tiles for the room. Similarly, if b blue tiles measure 15 feet, then we need b^2 blue tiles. We know that $r^2 - b^2 = 72$.

Thus $r + b$ and $r - b$ are positive integers whose product is 72. Could we have $r + b = 72$, $r - b = 1$? No, for then $r = (72 + 1)/2$, contradicting the fact that r is an integer. Similar reasoning shows that $r + b$ and $r - b$ must both be even.

The possibilities are (i) $r+b=36$, $r-b=2$; (ii) $r+b=18$, $r-b=4$; and (iii) $r+b=12$, $r-b=6$.

Possibilities (ii) and (iii) are ruled out by the fact that red tiles are less than 1 foot in size. For example, if $r+b=18$ and $r-b=4$, then $r=11$, giving red tile size 15/11. Only possibility (i) remains, so $b=17$, and the blue tiles have side length 15/17.

VII-61. The tens digit of a perfect square is 7. What can the units digit be?

Solution. A little doodling soon leads to the conjecture that 6 is the only answer. Let's prove it. We are interested in the last two digits of the perfect square a^2. Only the last two digits of a matter. For let $a = 100k + d$, where $0 \le d \le 99$. Then $a^2 = 100(100k^2 + 2kd) + d^2$, and therefore the last two digits of a^2 and d^2 are the same. So to answer our question, we only need to square the numbers from 0 to 99.

Before doing that, let's think some more. We only need to square the numbers from 0 to 49, because if d is the remainder when a is divided by 50, then $a = 50q+d$ for some quotient q. But then $a^2 = 100(25q^2 + qd) + d^2$, and therefore the last two digits of a^2 and d^2 agree.

Before squaring the numbers from 0 to 49, let's think some more. The numbers a from 25 to 49 are equal to $50 - b$ for some b between 1 and 25. Then $a^2 = 100(25-b) + b^2$, so a^2 and b^2 have the same last digit. Thus we need only examine the squares from 0^2 to 25^2. Now we can compute. Only the square of 24 has next to last digit equal to 7, so the only possible last digit is 6. In fact the units digit is 6 if and only if the tens digit is odd.

Comment. The solution shows that we shouldn't take the calculator out of its case too early. But we did! Let u be the units digit of a. Then the units digits of a^2 and u^2 are the same, and $a-u$ is a multiple of 10.

Now compare tens digits. Since $a^2 - u^2 = (a+u)(a-u)$, and $a+u$ is even, it follows that $a^2 - u^2$ is divisible by 20. Thus the tens digit of a^2 is odd iff the tens digit of u^2 is odd. The tens digit of u^2 is odd when $u = 4$ and when $u = 6$. In either case, the units digit of u^2 is 6.

The right balance between thinking and computation is not always obvious. As the cost of computation decreases, there is an inevitable tilt away from thinking.

VII-62. Show that if one entry in an infinite arithmetic progression is a perfect cube, then infinitely many entries are perfect cubes.

Solution. The result is obvious if all the entries are the same. If they are not, let d be the common difference. By multiplying if necessary all entries by -1, we can ensure that d is positive. Let x^3 be a perfect cube in the progression. We show there is a positive t such that $(x+t)^3$ is in the progression.

We need to make sure that d divides the difference $(x+t)^3 - x^3$. This difference is equal to $3x^2t + 3xt^2 + t^3$, so we can for example take $t = d$.

VII-63. Let $N = 501 \cdot 502 \cdot 503 \cdots 999 \cdot 1000$. Find the largest power of 2 that divides N.

Solution. We could simply count. There are 250 odd numbers in our list, and they supply no 2's. So we only need to look at the remaining 250 numbers! It's not as bad as it sounds. For example, 502 only has one 2. The same is true of 506, 510, and so on up to 998. So 125 numbers each supply one 2. We need to examine the remaining 125 numbers. We could continue, learning a few shortcut tricks along the way, and after a while find the answer, which turns out to be 2^{500}. The simplicity of the answer is a sign that we should look for

Another way: Let $N = (n+1)(n+2)\cdots(2n)$. We find the largest power of 2 that divides N. First calculate the number of 2's "in" N for small n.

Start with $n = 1$. The highest power of 2 that divides 2 is 2^1. Let $n = 2$. The highest power of 2 that divides $3 \cdot 4$ is 2^2. The highest power of 2 that divides $4 \cdot 5 \cdot 6$ is 2^3. For $5 \cdot 6 \cdot 7 \cdot 8$, it is 2^4. Interesting!

We have $N = (2n)!/n!$. In $(2n)!$, separate the even and the odd numbers. We get

$$(2n)! = [1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)]\,[2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)]\,.$$

But $2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n) = (2^n)(1 \cdot 2 \cdot 3 \cdot \cdots \cdot n)$, so

$$N = \frac{(2n)!}{n!} = (2^n)(1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1))$$

and therefore 2^n is the largest power of 2 that divides N.

Another way: Suppose that with hard work we have calculated the highest power of 2 that divides say $42 \cdot 43 \cdot 44 \cdot \cdots \cdot 82$. What happens when we go to $43 \cdot 44 \cdot 45 \cdots 84$? We "lose" the 2's in 42, and "gain" the 2's in 84, for a net gain of one 2.

In general, every time that we increase n by 1, we lose the 2's in $n+1$ and gain the 2's in $2n+2$, for a net gain of one 2. Since when $n = 1$ there is one 2, it follows that there are n 2's in $(n+1)\cdots(2n)$.

Comments. 1. In the comments that follow VII-36, it is shown that the highest power of the prime p that divides $m!$ is p^e, where

$$e = \left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \left\lfloor \frac{m}{p^3} \right\rfloor + \cdots .$$

The number of 2's in $(2n)!/n!$ is the number of 2's in $(2n)!$ minus the number of 2's in $n!$. In the formula above, let $p = 2$. Put $m = 2n$, and subtract the result of putting $m = n$. Almost everything cancels. All that we are left with is $\lfloor 2n/2 \rfloor$, which is n.

2. In the third solution, we avoided using the word "induction," but in fact gave a full induction argument. This is a quite clean instance of induction. Students sometimes find the induction proof of a formula like the one for $1 + 2 + \cdots + n$ confusing. And indeed induction is probably the worst way of proving that this sum is $n(n+1)/2$.

VII-64. Let $x = 997756 - 300720\sqrt{11}$. Find integers a and b such that $x = (a - b\sqrt{11})^2$.

Solution. Expand $(a - b\sqrt{11})^2$. We conclude that if a and b are integers then

$$a^2 + 11b^2 = 997756 \qquad \text{and} \qquad 2ab = 300720.$$

Substitute $300720/2a$ for b in the first equation and simplify. We get

$$a^4 - 997756a^2 + 11 \cdot 150360^2 = 0.$$

This is a quadratic in a^2. The numbers are large but not impossibly large—though most calculators display at most 10 digits, they keep (barely) enough guard digits to do the job. The discriminant turns out to be 27556^2. Only one of the roots of the quadratic is a perfect square, namely the square of 716. So $a = \pm 716$. Finally, from $2ab = 300720$ we conclude that $b = \pm 105$.

Another way: We make greater use of the calculator. Let $y = (a + b\sqrt{11})^2$. Note that $y = 997756 + 300720\sqrt{11}$.

Using the calculator, we find that $\sqrt{x}$ is about 19.508794 and $\sqrt{y}$ is about 1412.4912, and therefore $\sqrt{x} + \sqrt{y}$ is very near to 1432. This sum ought to be $2a$, so we conclude that a is probably 716. We know that $a - b\sqrt{11}$ is about 19.508794. Put $a = 716$ and solve for b. To calculator accuracy, $b = 105$. We should now check that these values of a and b work *exactly*—there might not be integers a and b that satisfy our condition. They do.

VII-65. A high school has somewhere between 200 and 400 lockers, numbered 1, 2, 3, and so on. The sum of the locker numbers greater than mine is equal to the sum of the locker numbers less than mine. What is my locker number?

Solution. If we knew the number of lockers, we could quickly zoom in on the answer. But we don't, and examining the various possibilities from 200 to 400 would take a long time.

Let n be my locker number, and let the number of lockers be N. The locker numbers less than mine are $1, 2, \ldots, n-1$, and they add up to $(n-1)(n)/2$. The locker numbers greater than mine add up to $N(N+1)/2 - n(n+1)/2$. The two sums are equal. After simplifying, we obtain the equation $2n^2 = N^2 + N$.

For $N = 200$ to 400, we could calculate $N^2 + N$, divide the result by 2, and take the square root of the result, looking for an integer answer. It is not hard to write a program to do this.

Instead, note that exactly one of N or $N+1$ is even. Suppose it is N. Then $n^2 = (N/2)(N+1)$. But $N/2$ and $N+1$ have no common divisor greater than 1, and their product is a perfect square, so each is a perfect square. Let $N/2 = y^2$ and $N + 1 = x^2$. We reach the equation $x^2 - 2y^2 = 1$. Similarly, if $N+1$ is even, then $(N+1)/2 = y^2$ and $N = x^2$ for some integers x and y, and we get $x^2 - 2y^2 = -1$.

Since $N = 2y^2$ or $N = 2y^2 - 1$, and N lies between 200 and 400, the number y must be between 10 and 14. It is easy to examine $2y^2 \pm 1$ for these five values of y, looking for a perfect square. We get a "hit" at $y = 12$ and nowhere else. That gives $N = 288$ and $n = 204$.

Comment. Equations of the type $x^2 - ay^2 = \pm 1$, where a is a positive integer which is not a perfect square, are called *Pell equations,* after an English amateur mathematician who had little or perhaps nothing to do with them. Usually we are interested only in *integer* solutions.

The special case $a = 2$ occurs in early Greek mathematics, and other cases were studied by Diophantus. In India, there were contributions by among others Brahmagupta in the seventh century, Mahāvīra in the ninth, and most importantly Bhāskara in the twelfth century.

In Europe, they come up seriously again only in the seventeenth century with the work of Fermat and Brouncker. A beautiful complete theory was developed by Lagrange a century later.

VII-66. Express $3/7$ as a sum of different fractions of type $1/n$, where n is an integer. The following identity may be useful:

$$\frac{1}{k} = \frac{1}{k+1} + \frac{1}{k(k+1)}.$$

Solution. The given identity is easy to verify. So $1/7 = 1/8 + 1/56$, and $1/8 + 1/56 = 1/9 + 1/72 + 1/57 + 1/3192$. It follows that

$$\frac{3}{7} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = \frac{1}{7} + \frac{1}{8} + \frac{1}{56} + \frac{1}{9} + \frac{1}{72} + \frac{1}{57} + \frac{1}{3192}.$$

Comment. It may be interesting to look for simpler solutions. For instance, $3/7 = 1/3 + 1/14 + 1/42$. We can also use the so-called *greedy algorithm*, in which we subtract repeatedly the largest $1/n$ that is less than or equal to our current number. So $3/7 - 1/3 = 2/21$, $2/21 - 1/11 = 1/231$, and therefore $3/7 = 1/3 + 1/11 + 1/231$.

Fractions of type $1/n$ are called *unit fractions*, or *Egyptian fractions*, because the ancient Egyptians didn't have symbols for other fractions, except for $2/3$, and for some peculiar reason expressed other fractions by summing distinct unit fractions. There are many unsolved problems about unit fractions. For example, does the equation $4/n = 1/x + 1/y + 1/z$ have a solution in positive integers x, y, z for every $n > 1$? It does for $n < 10^8$.

VII-67. Find the number of solutions of $\frac{1}{x} + \frac{1}{y} = \frac{1}{60}$ in positive integers.

Solution. As long as we remember that x and y can't be 0, we can rewrite the equation as $xy - 160x - 160y = 0$, or equivalently $(x-160)(y-160) = 160^2$. Note that x and y must each be larger than 160, for if one or both is less than or equal to 160, then $1/x + 1/y > 1/160$. Let $u = x - 160$ and $v = y - 160$. Then u and v are positive, and $uv = 160^2$. Conversely, if u is a positive divisor of 160^2, then $x = 160 + u$, $y = 160 + 160^2/u$ is a solution of the original equation.

It follows that there are just as many solutions as there are positive divisors of 160^2. We could count these divisors by making a list, but there is a better way. We have $160^2 = 2^{10} \cdot 5^2$. To make a positive divisor u of 160^2, we need to decide how many 2's u will have (0, or 1, or 2, ..., or 10, eleven choices in all). For every such choice, we must decide how many 5's u will have (three choices in all). So there are $11 \cdot 3$ ways of making u, and therefore our equation has 33 solutions.

Another way: Start from $xy - 160x - 160y = 0$ and solve for y. We get

$$y = \frac{160x}{x-160} = 160 + \frac{160^2}{x-160}.$$

Integer solutions are obtained by making sure that $x - 160$ divides 160^2. The rest of the analysis goes as before.

Comment. The strategy we used to count the number of positive divisors of 160^2 can be used to compute quickly the number of positive divisors of any integer N once we know the prime factorization of N. See VII-38 for a more leisurely discussion.

VII-68. Find the solutions of $xy = x^2 + 97x + 3y - 213$ in positive integers.

Solution. The equation is linear in y. Solve for y. We get

$$y = \frac{x^2 + 97x - 213}{x-3} = x + 100 + \frac{87}{x-3}.$$

(the last expression was obtained using ordinary polynomial division). It follows that $x - 3$ must divide 87 exactly. Luckily, 87 doesn't have many integer divisors: they are ± 1, ± 3, ± 29, and ± 87. Since x is positive, we get the possibilities $x = 4$, 2, 6, 32, 90. Note that $x = 2$ yields a positive y, and the others obviously do.

VII-69. Find all integers a such that $1 + a + a^2 + a^3$ is a power of 2.

Solution. It is easy to see that $a = 0$ and $a = 1$ work. We will show that nothing else does. Note that

$$1 + a + a^2 + a^3 = 1 + a + a^2(1+a) = (1+a)(1+a^2).$$

In order for the product of $1 + a$ and $1 + a^2$ to be a power of 2, both $1 + a$ and $1 + a^2$ must be powers of 2.

When is $a^2 + 1$ a power of 2? Except when $a = 0$, $a^2 + 1$ must be even, so a must be odd. Calculate $a^2 + 1$ for the first few positive odd a to get some insight.

We get 2, 10, 26, 50, 82. Except for the first entry, these are certainly not powers of 2, in fact they each only have one 2 in them.

That's true in general. Let a be odd, say $a = 2k+1$. Then $a^2+1 = 4k^2+4k+2 = 2(2k^2 + 2k + 1)$. Since $2k^2 + 2k + 1$ is always odd, 4 can't divide $a^2 + 1$. So the only way $a^2 + 1$ can be a power of 2 for odd a is if $a = \pm 1$. But $a = -1$ is not a solution of the original problem, since in that case $a + 1$ is not a power of 2.

VII-70. If eggs in a (large) basket are taken out 2, 3, 4, 5, or 6 at a time, there are 1, 2, 3, 4, and 5 eggs left over respectively. If they are taken out 7 at a time, there are no eggs left over. Find the least number of eggs that can be in the basket.

Solution. If we use other numbers, the problem can be a lot harder. But the first five conditions collapse into one. If N is the number of eggs, then $N+1$ is divisible by 2, 3, 4, 5, and 6. That's equivalent to asking that $N + 1$ be divisible by 60.

Thus $N + 1 = 60k$ for some positive integer k. We are told that N is divisible by 7, that is, $60k - 1$ is divisible by 7. It's sensible now just to try various k. But first we make a move that could be useful with uglier numbers. Note that $60k - 1 = 56k + (4k - 1)$, so 7 divides $60k - 1$ if and only if 7 divides $4k - 1$. And it is easy to see that 2 is the smallest positive k for which 7 divides $4k - 1$. Thus $N = 119$.

Comment. This problem, with changes of wording but unchanged numbers, is ancient. It occurs in the work of Bhāskara I (sixth century), al-Haitham (eleventh century), Fibonacci (thirteenth century), and has been a staple of puzzle collections ever since. The main result needed to deal with generalizations of this problem is called the *Chinese Remainder Theorem,* because questions like this were first studied in China. There is also a sustained history of interest in such problems in India, starting with Āryabhata (sixth century) and Brahmagupta (seventh century).

VII-71. Call an angle *regular* if it is $360°/n$ for some integer n. Find all triangles whose angles are regular.

Solution. Let the angles be $360/x$, $360/y$, and $360/z$, where x, y, and z are integers. From the fact that the angles of a triangle add up to 180°, we obtain, after dividing by 360, the equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{2}.$$

Without loss of generality we may take $x \le y \le z$. Note that $3 \le x \le 6$, for if $x \le 2$ then $1/x + 1/y + 1/z > 1/2$, while if $x > 6$, then $1/x + 1/y + 1/z \le 3/7$.

If $x = 6$ then, because $x \le y \le z$, we must have $x = y = z = 6$, so there is a single shape, the equilateral triangle, which we list as $(60, 60, 60)$. If $x = 5$, we are trying to solve the equation $1/y + 1/z = 1/2 - 1/5$. Now do a trial and error search. Could $y = 5$? Yes; then $z = 10$, and we get $(72, 72, 36)$. Could $y = 6$? No, then z is not an integer. Could $y \ge 7$? No, for then $1/y + 1/z \le 2/7 < 3/10$.

If $x = 4$, we are looking at $1/y + 1/z = 1/4$. Again a quick search would yield the solutions, but for variety we take a more algebraic approach. The equation simplifies to $4y + 4z = yz$, which can be rewritten as $(y - 4)(z - 4) = 16$. Thus each of $y - 4$ and $z - 4$ divides 16. Since $4 \leq y \leq z$, we get $y = 5$, $z = 20$, giving $(90, 72, 18)$; $y = 6$, $z = 12$, giving $(90, 60, 30)$; and $y = 8$, $z = 8$, giving $(90, 45, 45)$.

If $x = 3$, we are looking at $1/y + 1/z = 1/6$, or equivalently $(y - 6)(z - 6) = 36$. The possible values of $y - 6$ are 1, 2, 3, 4, and 6. This yields the shapes $(120, 360/7, 60/7)$, $(120, 45, 15)$, $(120, 40, 20)$, $(120, 36, 24)$, and $(120, 30, 30)$. So altogether there are 10 different shapes.

Comment. All but one of the shapes probably can be found by unsystematic search, but we are unlikely to just bump into $(120, 360/7, 60/7)$. Sometimes a patient organized breakdown into cases is the best way to solve a problem.

Chapter 8

Probability

Introduction

Probability and statistics are playing an increasing role in the curriculum. Most of the material that senior high school students cover is of the "pre-calculus" type in which there is seldom any question about what method to apply. Probability has a greater apparent diversity of problems, and through lack of practice it can seem more difficult than it is.

A majority of the problems are about simple gambling games, mostly with coins and dice. The motivation for this is partly historical—the first results of the seventeenth century founders of probability theory were about dice games. Also, dice games are concrete, but give rise to a rich variety of questions.

The two most interesting problems in the chapter are well-known. Problem VIII-18, about an interrupted game, was a matter of controversy at the beginnings of mathematical probability. And VIII-34, which deals with "non-transitive" dice, shows that it is possible to rationally prefer B over A, and C over B, but A over C.

Problems and Solutions

VIII-1. There are 24 points on the periphery of a circle, 15° apart. Alfred picks two of the points, say A and B, at random. Then Cecilia picks two points C and D at random from the 22 points that remain. Alfred wins if the line segment AB meets the line segment CD, and Cecelia wins otherwise. Find the probability that Alfred wins.

Solution. There is less to the problem than meets the eye. Imagine that the picking has been done and let A be the first point chosen by Alfred. Let P, Q, and R be the other three chosen points, listed counterclockwise from A. Then Alfred wins if and only if $B = Q$, and this has probability 1/3. The number of points and their spacing are irrelevant.

VIII-2. Two different numbers are chosen at random from the numbers 1, 2, 3, ..., 100. Find the probability that their sum is even.

Solution. There are $\binom{100}{2}$ ways of choosing two numbers from our collection, and all these ways are equally likely. The sum of the numbers is even if either (i) both numbers are even or (ii) both numbers are odd. There are $\binom{50}{2}$ ways of choosing two even numbers, and $\binom{50}{2}$ ways of choosing two odd numbers. The required probability is therefore

$$\frac{\binom{50}{2} + \binom{50}{2}}{\binom{100}{2}}.$$

Now we can compute. The probability is 49/99.

It is marginally easier to compute the number of ways that the sum can be odd, for then one number must be even and the other odd, and there are 50^2 ways of selecting such a pair of numbers.

Another way: Choose a number, then choose another. Whatever number we first picked, there are 49 left that will produce an even sum, so the probability is 49/99.

VIII-3. (a) Ninety-nine fair coins are flipped. Eve wins if the number of heads is even, and Odin wins if the number is odd. Find the probability that Eve wins. (b) What about if 100 fair coins are flipped?

Solution. (a) Since 99 coins were tossed, the number of heads is even if and only if the number of tails is odd. But by symmetry, the probability that the number of tails is odd is the same as the probability that the number of heads is odd. So the probability that Eve wins is the same as the probability that she loses, and therefore each is 1/2.

Another way: The same argument can be reworded. Imagine that the tossing is done onto a glass-topped coffee table. Suppose that Eve is looking down at the table and sees an even number of heads. If instead she were lying under the table, she would see an odd number of heads, so the two events are equally likely.

(b) The symmetry argument of part (a) breaks down, but calculation with 2 coins or 4 coin suggests that the probability is still 1/2. Imagine that of the 100 coins, one is gold and 99 are copper. Look first at the gold coin. If it shows tails, the probability that there is an even number of heads among the copper coins, and therefore among all the coins, is 1/2. And if it shows heads, then since with probability 1/2 the number of heads among the copper coins is odd, it follows that with probability 1/2 the number of heads among all the coins is even. Note that the gold coin doesn't even have to be fair as long as the copper coins are.

Comment. It can be shown that if n fair coins are tossed, the probability that there are exactly k heads is $\binom{n}{k}/2^n$. So the probability that the number of heads is even is

$$\frac{1}{2^n}\left(\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots\right).$$

Recall that by the Binomial Theorem

$$(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^n.$$

Put $x=-1$ in the above identity, and rearrange. We obtain

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots.$$

Thus an even number of heads and an odd number of heads are equally likely. Too fancy, of course, but the general idea has many uses.

VIII-4. A box-like apartment building has a square base of side 30 meters. Point P is chosen at random on the circle of radius 60 meters whose center is the middle of the base of the building. Find, correct to two decimal places, the probability that two walls of the building are visible from P.

Solution. First we compute the probability that we can only see the East wall. That happens if P lies on the arc between A and B in Figure 8.1. The picture shows that $\angle AMB$ is twice the angle whose sine is $15/60$. The calculator says that $\angle AMB$ is about 28.955 degrees. Call this number x.

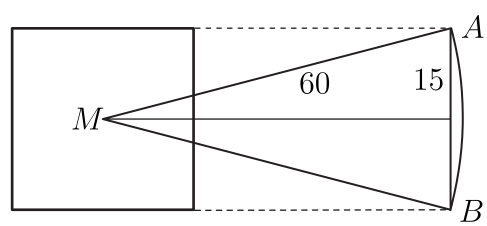

Figure 8.1: The Probability Two Walls are Visible

The probability that we can only see the East wall is $x/360$. By symmetry, the probability that we can only see the North wall is also $x/360$, and so on. Thus the probability that we can only see one wall is $4x/360$, and therefore the probability that we can see two walls is $1-4x/360$, approximately 0.68.

VIII-5. An integer n is chosen at random in the interval from 0 to 10^k-1. Find the probability that the decimal expansion of n contains one or more 5's.

Solution. For uniformity of notation, make sure that any number in our interval has exactly k digits by, if necessary, putting a padding of extra 0's on the left of the usual decimal representation. For example, if $k = 5$ then the number 123 is denoted instead by 00123.

There are 10^k integers in our interval. "At random" is ambiguous: we interpret it to mean that all numbers in the interval are equally likely to be chosen.

We count the number of integers in the interval with *no* 5's in their decimal expansion. The first digit of such a number can be chosen in 9 ways—any digit but 5. For each such choice, there are 9 ways of choosing the second digit, so there are 9^2 ways of choosing the first two digits. We conclude after a while that there are 9^k numbers with no 5's.

Thus the probability that n has no 5's is $9^k/10^k$. It follows that the probability p_k that n has at least one 5 is given by $p_k = 1 - (9/10)^k$.

Comment. As k gets large, $(9/10)^k$ approaches 0, and therefore p_k approaches 1. So in a sense "almost every" positive integer has a 5 in its decimal expansion. In a similar way, we can show that for example almost every integer has the string 1066 as part of its decimal expansion.

VIII-6. Two circles have the same center, one has radius 1 and the other has radius 2. Points P and Q are chosen independently and at random on the boundary of the outer circle. Find the probability that the line PQ passes through the inner circle.

Solution. By a rotation of a circle, any point on the circle can be carried to any other point. Place P at the top of the circle of radius 2. The line PQ crosses the small circle precisely if Q lies on the smaller arc that joins the points labelled A and B in Figure 8.2.

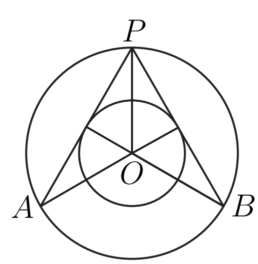

Figure 8.2: Two Circles

The probability that Q lies on arc AB is the ratio of $\angle AOB$ to a full rotation. The angle APO is easy to find, for its sine is r/R, where r and R are the radii of the two circles. Thus $\angle APB$ is twice the angle whose sine is r/R. By quoting a general result, or after some angle chasing, we find that $\angle AOB$ is twice $\angle APB$, so $\angle AOB$ is four times the angle whose sine is r/R.

In our case, $r = 1$ and $R = 2$, so $\angle APO$ is 30°, and therefore $\angle AOB$ is 120°. Thus the required probability is 120/360, or more simply 1/3.

Comment. When this problem was done in a school workshop, the presenter—not a student—argued that if P and Q are chosen at random, the perpendicular distance from O to PQ is just as likely to be between 0 and 1 as it is to be between 1 and 2, so the required probability is 1/2. This is plausible-sounding but incorrect. Geometric probability can be tricky!

VIII-7. Alphonse tosses 12 fair coins, and Beth tosses 13. Find the probability that Beth gets more heads than Alphonse.

Solution. Let p be the probability that when each tosses 12 coins, Beth gets more heads. Then by symmetry p is the probability that Alphonse gets more heads, and so the probability that they get equal numbers is $1 - 2p$.

Now Beth tosses her last coin. She gets more heads than Alphonse if (i) she was already leading after 12 tosses (probability p) or (ii) they were tied after 12 tosses and she got a head on the 13th (probability $(1 - 2p)(1/2)$). The two probabilities add to 1/2.

Another way: Such a simple answer deserves a simpler solution. If Alphonse is leading after 12 tosses, he has won, while if Beth is leading she has won. And if they are tied after 12, she has a 50–50 chance of winning.

VIII-8. A fair die was tossed until a 6 turned up. What is the probability that in the meantime a 5 was not thrown?

Solution. The events "5 occurs before 6" and "6 occurs before 5" are equally likely, so the probability is 1/2.

VIII-9. A group of twelve physicists, including Alpher, Bethe, and Gamow, get seated at random around a circular banquet table for twelve. Find the probability that Alpher, Bethe, and Gamow are next to each other, not necessarily in that order.

Solution. Choose at random the three chairs our trio are to occupy, without specifying who sits where. The number of ways to choose 3 objects from 12 is $\binom{12}{3}$, which turns out to be 220. Now count the number of ways to choose a set of three neighbouring chairs. Such a set is completely determined once we have chosen the middle chair. There are 12 ways of doing this, so the required probability is 12/220.

Comment. There were once three Cornell University physicists with these names; Bethe and Gamow were famous, and Alpher not so famous. They co-authored an important joint paper, and people joked that Bethe and Gamow had done the work, and added Alpher to make the author list sound like alpha, beta, and gamma, the first three letters of the Greek alphabet. Actually, Alpher had made a genuine contribution to the paper.

VIII-10. A class of 24 students is split at random into 4 volleyball teams. Find the probability that Alphonse and Beth are on the same team.

Solution. We can assume that the coach starts off by saying "You, you, you, you, and you, go with Alphonse." There are 23 students in addition to Alphonse, and 5 of these will join Alphonse, so the probability that Beth is one of them is 5/23.

Another way: There are more complicated ways to solve the problem. Call the teams I, II, III, and IV. There are $\binom{24}{6}$ ways of choosing who is to be on team I. If Alphonse and Beth are to be on team I, there are $\binom{22}{4}$ ways to choose the rest of the team. So the probability that Alphonse and Beth are both on team I is $\binom{22}{4}/\binom{24}{6}$. This is also the probability they are both on team II, that they are both on III, that they are both on IV. So the required probability is $4\binom{22}{4}/\binom{24}{6}$. After some work this simplifies to 5/23.

VIII-11. Seven physicists get seated in a row at random. Find the probability that Bethe sits directly between Alpher and Gamow.

Solution. Let Alpher and Gamow be the first to sit down. Their chairs can be chosen in $\binom{7}{2}$ ways, if we do not specify who sits where. These 21 ways are equally likely.

Now we count the number of ways that these chairs can have exactly one chair between them. Of the two chairs, the one on the left can be chosen in 5 ways. Once it is chosen, the other is determined, so the probability there is exactly one chair between Alpher and Gamow is 5/21.

Suppose there is exactly one chair between Alpher and Gamow. Now Bethe sits down. The probability he sits in the chair between his two colleagues is 1/5. So the required probability is (5/21)(1/5).

Another way: There are 7! ways of seating the seven people, all equally likely. We count how many of these ways have Bethe directly between Alpher and Gamow. The three adjacent chairs they are to occupy can be chosen in 5 ways. For each of these ways, there are 2 ways of seating our trio—Alpher is either on the left or on the right. And once the trio is seated, the remaining four chair can be filled in 4! ways. So there are (5)(2)(4!) ways of doing the seating. Thus our probability is (5)(2)(4!)/7!.

VIII-12. In front of you are two urns. One of them—you don't know which—contains 2 twenty-dollar bills and 5 five-dollar bills. The other contains 3 twenty-dollar bills and 11 five-dollar bills. You are given the following options: (i) Pick an urn, reach in and pick out a bill at random or (ii) Pour the contents of one urn into the other, then pick a bill at random. Which option is better? How much better?

Solution. If we opt for (ii), we are choosing from an urn that holds 21 bills, of which 5 are twenties. So the probability of winning twenty dollars is 5/21.

Option (i) is more difficult to evaluate. The probability of choosing the 7-item urn is $1/2$; once it has been chosen, the probability of getting twenty dollars is $2/7$. So the probability that we pick the 7-item urn *and* get twenty dollars is $(1/2)(2/7)$. Similarly, the probability we pick the 14-item urn and then get twenty dollars is $(1/2)(3/14)$. Add and simplify. The result is $1/4$. Since $1/4 > 5/21$, option (i) is slightly better.

How much better? The usual way of comparing is to look at the average winnings under each option. Under option (ii), on average we win $(5)(16/21) + (20)(5/21)$, that is, $180/21$. Under option (i), on average we win $5(3/4) + 20(1/4)$. The ratio of the expected winnings is about 1.021, so option (i) is about 2.1% better.

VIII-13. A point P is chosen at random in an $a \times b$ rectangle $\mathcal{R}$. Find the probability that P is closer to the center of the rectangle than to any corner.

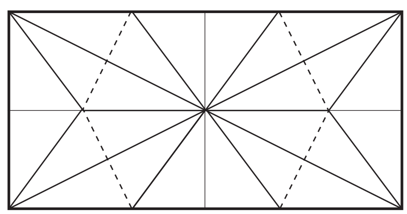

Figure 8.3: The Probability of Being Closer to the Center

Solution. We need to think about the meaning of "at random." If you buy a lottery ticket, then whether you win or not is a random event, but winning and not winning are not equally likely. In this problem, we might select P by throwing a dart at $\mathcal{R}$. If we are good at darts and aim at the center, then a little disk of radius r near the periphery has much less chance of being landed in than a disk of radius r near the center. Yet the point is chosen, in a reasonable sense of the term, at random.

For this problem, take "at random" to mean that given a region $\mathcal{S}$ in $\mathcal{R}$, the probability that P is in $\mathcal{S}$ is proportional to the area of $\mathcal{S}$. If the rectangle has corners $(0,0)$, $(a,0)$, (a,b), and $(0,b)$, we could choose P as follows. Use an ordinary computer random number generator to generate random numbers u and v between 0 and 1. Then let $P = (x,y)$ where $x = au$ and $y = bv$.

We now have a purely geometric problem: Find the area of the part of $\mathcal{R}$ made up of the points that are closer to the center than to any vertex. If A and B are distinct points, let ℓ be the perpendicular bisector of the segment AB. Then the points in the plane that are closer to A than to B are the points on the same side of ℓ as A is.

Look at Figure 8.3. The four dashed lines play, one after the other, the role of ℓ. Concentrate on the north-east quarter $\mathcal{Q}$ of the rectangle. The points of $\mathcal{Q}$ closer to the north-east corner than to the center consist of half the rhombus bisected by the dashed line, together with the triangle at the eastern edge of $\mathcal{Q}$. So exactly half of $\mathcal{Q}$ is closer to the north-east corner than to the center.

Similar reasoning applies to the other three quarters of $\mathcal{R}$. We conclude that the required probability is 1/2.

Comment. For students with not much experience in geometry, one might ask first about an $a \times a$ square—the geometry is much simpler. We can ask similar questions about parallelograms, or regular polygons. A point chosen at random in an equilateral triangle has probability 2/3 of being closer to the center than to any corner.

VIII-14. Alicia brought eight CDs to a party, each in its own case. Afterwards, the CDs were put back into the cases at random. (a) Find the probability that every CD but one is in the right case. (b) Find the probability that all but two are in the right case. (c) Find the probability that all but three are.

Solution. (a) If a CD is in the wrong case, then the CD that should be there is also misfiled. So the probability that all but one is right is 0.

(b) Imagine that the CD cases are lined up in a row, say alphabetically, and we fill them at random, starting on the left. The first case can be filled in 8 ways. For *each* of these ways, the second case can be filled in 7 ways. So the first two cases can be filled in $8 \cdot 7$ ways. For *each* of these ways, the third case can be filled in 6 ways, and so on. So there are 8! ways of filling the cases, all equally likely.

How many of these ways have exactly two mistakes? Exactly as many as there are of *choosing* the two CDs that are to be in each other's case, that is, $\binom{8}{2}$. So the probability of exactly two mistakes is $\binom{8}{2}/8!$, that is, 1/1440.

c) The three CDs which will be in the wrong cases can be chosen in $\binom{8}{3}$ ways. Say the CDs are P, Q, and R. If P is in Q's case, then Q must be in R's (if Q were in P's , then R would be in its own case, and there would be only two mistakes).

Similarly, there is only one admissible pattern in which P is in R's case. So there are $2\binom{8}{3}$ ways to have exactly three mistakes. Divide by 8! to find the probability. The answer is 1/360.

Comment. What is the probability that *no* CD is in the right case? That important problem was first solved by the Chevalier de Montmort in the early eighteenth century. Many famous mathematicians gave solutions, including Nicholas Bernoulli, de Moivre, Laplace, and Euler, and generalizations are still being studied. De Montmort was led to the problem by a popular card matching game.

The problem has many names, including *Problème des Rencontres* and *Problem of Derangements.* An old-fashioned "word problem" version goes like this: Ten gentlemen go to a party, checking their hats at the entrance. They drink a little too much, and when they leave they pick up a hat at random. What is the probability that no one gets his own hat?

If there are n gentlemen, the probability that no one gets his own hat turns out to be

$$\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}.$$

There are many proofs, but none is very easy. As n increases, the probability doesn't change much, and approaches e^{-1}, where e is the base for natural logarithms. Everything is connected to everything else!

VIII-15. Which is more likely: at least one six in 4 tosses of a fair die, or at least one double-six in 24 tosses of two fair dice?

Solution. Call getting something other than six when you toss a die a *failure.* The probability of a failure is 5/6. So the probability of 4 failures in a row is $(5/6)^4$. The probability of getting a six at least once is therefore $1 - (5/6)^4$, about 0.5177.

Similarly, the probability that we get something other than a double-six when we toss two dice is 35/36, so the probability we get a double-six at least once in 24 tosses is $1 - (35/36)^{24}$, roughly 0.4914. Thus getting at least one six in 4 tosses is the more likely event.

Another way: We solve the single die problem. The result of tossing four times can be represented as a "word" of length 4 made up using "letters" 1, 2, 3, 4, 5, and 6. There are 6^4 such words, all equally likely. There are 5^4 words that don't use the letter 6, so there are $6^4 - 5^4$ words that use at least one 6. Hence the required probability is

$$\frac{6^4 - 5^4}{6^4}.$$

Comment. This problem is often (wrongly) attributed to the Chevalier de Méré. He is said to have been a gambler who believed that there is a better than fifty-fifty chance of getting at least one double-six in 24 tosses of two dice. Some improve the tale and add that he gambled away his fortune by acting on this belief.

Nice story, nice moral. In fact de Méré was a *philosophe* associated with the court of Louis XIV, and disapproved of gambling. De Méré had calculated that there is a slightly *less* than fifty-fifty chance of getting at least one double-six in 24 throws. He was concerned because Cardano's *Liber de Ludo Aleae* (Book of Games of Chance), written about 100 years earlier, gave a rule of thumb that suggested the opposite.

De Méré asked Pascal, who confirmed de Méré's calculation. By the way, the famous *Pascal Triangle* that gives a way of generating the binomial coefficients appears in a book by Zhu Shijie, published in 1303, 320 years before Pascal was born. There are earlier references to the method by Jia Xian and Yang Hui. And it was known in India by the end of the ninth century, and possibly 1000 years before. It was known even in Europe long before Pascal. The "Pascal" triangle is on the title page of a 1527 book by Petrus Apianus.

The first systematic work in probability is Huygens' *De Ratiociniis in Ludo Aleae* (1657). Huygens also solves the problem of determining the smallest number of times one should toss one die (two dice) so as to have a better than fifty-fifty chance of getting at least one six (a double-six).

We describe Huygens' analysis for one six. Suppose player A wagers an amount t that he will get at least one six in n tosses. Let his expected winnings be e_n. It is clear that $e_1 = (1/6)t$. What happens in $n+1$ tosses breaks down into two cases: (i) The $(n+1)$-th toss results in a six and (ii) it doesn't. The probability of (i) is 1/6, giving a contribution to the expected winnings of $(1/6)t$. The probability of (ii) is 5/6, giving a contribution of $(5/6)e_n$. Thus Huygens obtains the recurrence formula $e_{n+1} = (1/6)t + (5/6)e_n$. Now he computes until he finds the first n, in this case 4, for which e_n climbs above $t/2$.

Huygens doesn't notice that e_n has a simple explicit formula. The combinatorial approach that now dominates first presentations of probability is completely absent from Huygens' work! His recurrence formula approach is in fact more powerful.

In the sixteenth and seventeenth centuries, there was some confusion between probability and mean. Note that the mean number of sixes in four tosses of a fair die is 4/6, while the mean number of double-sixes in thirty-six tosses of two dice is 24/36, exactly the same.

VIII-16. Alicia drives to work every morning on deserted streets, just keeping to the speed limit. The city recently installed a stoplight, which shows green for 45 seconds, amber for 3 seconds, and red for 24 seconds. She never goes through a red light, but if she can get through before it turns red, she always does. Assume that she can go from rest to her usual driving speed instantly. How much longer on average is her trip because of the stoplight?

Solution. With probability $(45+3)/(45+3+24)$, or more simply 2/3, the light shows green or amber when Alicia reaches it, and the delay is 0. With probability 1/3, the light shows red. When the light shows red, the average delay is 24/2 seconds. Thus 1/3 of the time the trip takes on average an extra 12 seconds, so the average increase is 4 seconds.

Comment. We assumed that Alicia arrives at the light at a time which is random with respect to the colour cycle of the light, so the probability she faces a red light is the fraction of the time that the light is red.

This assumption sounds reasonable, but if the timer in the light works perfectly, and if Alicia is a time fanatic, always leaving at exactly the same time and always driving at the same speed, then the assumption is wrong, and she never (or always) faces a red light. The assumption is also wrong if there are two synchronized stoplights not far from each other and we are studying the delay at the second.

VIII-17. With two dice, the probability that a sum of 3 is tossed is 2/36, the probability that the sum is 8 is 5/36, and so on. For a certain game, you need sums 1, 2, ..., 11, 12 to be equally likely. Can you relabel the faces of the dice so as to obtain this?

Solution. There are several ways to do it. We can leave one of the dice alone and label three of the faces of the other 0 and the other three 6.

Or else label the faces of one die 0, 2, 4, 6, 8, 10 and three faces of the other 1, and the remaining three 2. There are other possibilities.

Comment. Let $P(x) = x + x^2 + \cdots + x^{11} + x^{12}$. The first solution is associated with the factorization $P(x) = (x + x^2 + \cdots + x^6)(1 + x^6)$, and the second with the factorization $P(x) = (1 + x^2 + x^4 + \cdots + x^{10})(x + x^2)$. Everything is connected to everything else!

VIII-18. A and B are playing a coin-tossing game. A wins a point for each head tossed, and B wins a point for each tail. Whoever first gets 6 points wins the game. (a) After six tosses, A is leading 4 to 2. Find the probability that B comes back to win the game. (b) The players had agreed that the loser would pay the winner 100 gold coins. Unfortunately, when A was leading 4 to 2, B dropped dead. How much can A legitimately claim from B's estate?

Solution. (a) The easiest approach is to revise the rules somewhat: even after a player gets to 6 points, the tossing continues for the full 11 tosses, and whoever gets 6 *or more* points wins. The revised game is equivalent to the old, in the sense that a player wins it if and only if she wins the old one.

In the revised game, B wins precisely if there are 4 or more tails in the last 5 tosses. There are 2^5 possible head/tail patterns for the last 5 tosses, all equally likely. Of these, 6 lead to a win for B: HTTTT, THTTT, TTHTT, TTTHT, TTTTH, and TTTTT. So the probability is 6/32.

(b) By part (a), when the game was interrupted by death A had probability 26/32 of winning 100 gold coins, while B had probability 6/32. Imagine that the interruption is only temporary, that the game resumes, and that the interrupted game scenario takes place many times. Then on average A wins (100)(26/32) each time, and loses (100)(6/32), so A's average net gain is (100)(20/32). This is the fair assessment of what B's estate owes.

Comment. This is a modified version of the famous *Problem of Points,* which was studied by a number of Renaissance mathematicians, including Pacioli, Tartaglia, and Cardano. They all gave solutions, and all the solutions were wrong! The problem was discussed—and solved—in 1654 in a series of letters between Fermat and Pascal.

VIII-19. In a certain North American country, a letter reaches its intended destination with probability 4/5. Whenever B gets a letter, she replies immediately. If A sent B a letter and didn't get a reply, what is the probability that B actually got the letter?

Solution. The analysis will be a bit sloppy. A proper argument requires the notion of *conditional probability.* Imagine that the experiment of sending a letter is repeated a large number N of times. Then in the long run B receives the letter about $4N/5$ times, and doesn't receive it about $N/5$ times.

About $(4N/5)(1/5)$ times the reply doesn't get back to A. So A fails to get a letter about $N/5+4N/25$ times. Among these failures, B got the letter $4N/25$ times but her reply got lost. So the required probability should be $(4N/25)/(4N/25 + N/5)$, that is, $4/9$.

VIII-20. Xavier and Yolande pick, independently and at random, a digit from 0 to 9. Find the average of the absolute value of the difference between these digits.

Solution. Record their choices as a pair (x, y), where x is Xavier's choice and y is Yolande's. There are 100 pairs. We interpret "at random" to mean that all pairs are equally likely. We use the 10×10 dot array of Figure 8.4. The pairs (x, y)

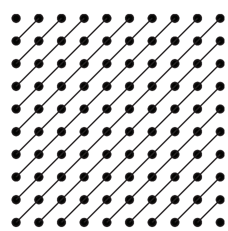

Figure 8.4: Xavier and Yolande

with difference 0 are just the points on the south-west to north-east diagonal of the square. One look shows there are 10 dots on this diagonal.

The pairs that differ by 1 are just the points on the two lines on either side of the diagonal, so there are 18 such pairs. Similarly, there are 16 pairs with difference 2, and so on. Thus the average absolute difference is

$$\frac{(0 \cdot 10) + (1 \cdot 18) + (2 \cdot 16) + \cdots + (8 \cdot 4) + (9 \cdot 2)}{100}.$$

The calculation is fairly short, so it may not be worthwhile to develop general tools. The answer is 3.3.

Another way: The differences when Xavier picks 0 add up to $1+2+\cdots+9$, namely 45. This is also true when he picks 9. When he picks 1 or 8, the differences add up to 37. When he picks 2 or 7, they add up to 31, when he picks 3 or 6, they add to 27, when he picks 4 or 5 they add to 25. Add up. We get

$$2(45 + 37 + 31 + 27 + 25).$$

To find the mean, divide by 100 as before.

Comment. Now that we have solved the problem for the numbers 0 to 9, we are ready to look at 0 to n. See VI-51 for ideas on how to tackle the final addition.

VIII-21. (a) A teacher had 100 students and 100 prizes to distribute. She chose a student at random and gave that student a prize. Again she chose a student at random, possibly the same one as before, and gave that student a prize. This process was repeated 100 times. Find the probability that Johnny didn't get a prize. (b) Do the problem again for 1000 students and 1000 prizes.

Solution. (a) The probability that Johnny is passed over in the first round is $1 - 1/100$. So the probability that he is passed over for 100 rounds in a row is $(1 - 1/100)^{100}$. The calculator says that this is about 0.36603. (b) Using the same idea as in part (a), we find that the answer is $(1 - 1/1000)^{1000}$, about 0.36696.

Comment. The remarkable thing is that the two answers are nearly equal. As n becomes large, $(1 - 1/n)^n$ approaches $1/e$, where e is the very important base for natural logarithms (e is roughly 2.71828). Please see the comments after VIII-14 for a discussion of a related problem.

VIII-22. Assume that in basketball games the better team always wins. Eight teams reach the last phase of a high school tournament. After that, if a team loses a game, it is eliminated. So four teams are left after the quarterfinal, and two get into the final. The *runner-up* is the team that played in the final but lost. Find the probability that the runner-up is in fact the second-best team.

Solution. There are complicated ways to find the answer, but also this simple one. Think of the eight teams as divided by the tournament organizers into two pools of four. The teams play in their pools, then the pool winners meet in the final. This is how things really are done, but that's not essential for the argument.

The second-best team is the runner-up precisely if it is not placed in the best team's pool. There are 3 places in the best team's pool, and 4 not in its pool, so the probability that the second-best team reaches the final is $4/7$.

VIII-23. Five fair dice are tossed. Find the most likely number of 6's.

Solution. Let the dice be black, blue, green, red, and white. Make a detailed record of the numbers that come up. So for example the record $(5, 2, 5, 1, 3)$ means that 5 came up on the black, 2 on the blue, 5 on the green, and so on. There are 6^5 possible records, all equally likely.

Find first the probability of 0 sixes. How many records have 0 sixes? There are 5 possible values for the first entry. For each of these, there are 5 possible values for the second entry, and so on. It follows that there are 5^5 records with 0 sixes. Thus the probability of 0 sixes is $5^5/6^5$, about 0.4019.

It looks as if we have found the winner already. But let's find the probability of 1 six. How many records have exactly 1 six? The six can be in any one of 5 places. Once the six is placed, the remaining four spots can be filled in 5^4 ways. So there are $5 \cdot 5^4$ records with exactly 1 six. This is exactly the same as the number of records with 0 sixes, so the events are equally likely. Since the two probabilities add up to more than 80%, all other numbers are much less likely than 0 or 1.

Comment. It is easy to guess wrong, somehow thinking that since we toss 5 times, and there are only 6 possible outcomes each time, getting 1 six should be significantly more likely than getting 0 sixes. Intuition can require retraining for questions of probability.

VIII-24. There are 52 cards in a standard deck, among them 12 face cards—Jacks, Queens, and Kings. Three cards are dealt from a well-shuffled deck. Find the probability that among them there is at least one face card.

Solution. Deal the three cards out *in a row*, face up. There are $52 \cdot 51 \cdot 50$ possible outcomes, all equally likely. How many ways are there to have at least one face card? It is easier to count the ways in which we have *none*.

Forty cards are non-face cards, so there are $40 \cdot 39 \cdot 38$ ways of dealing three non-face cards. Thus the probability none of the cards is a face card is p, where

$$p = \frac{40 \cdot 39 \cdot 38}{52 \cdot 51 \cdot 50}.$$

The probability of at least one face card is $1 - p$, about 0.553.
Another way: There are $\binom{52}{3}$ ways of selecting 3 cards, all equally likely. But there are 40 non-face cards, so there are $\binom{40}{3}$ ways of selecting 3 non-face cards. Thus the probability that *none* of the cards is a face card is $\binom{40}{3}/\binom{52}{3}$, about 0.447. So the probability of at least one face card is about 0.553.

Comment. The probability turned out to be significantly greater than 50%. On average we would win money steadily by betting at even odds that there will be at least one face card. Many would guess that the probability is less than 50%, another example in which first intuitions about probabilities are some distance from the truth.

VIII-25. A bowl contains 49 tickets, numbered from 1 to 49. Two tickets chosen at random are removed. Find the probability that the numbers on these tickets are within 10 of each other.

Solution. Imagine that the tickets are drawn one after the other. The result of the draw is described by a pair (x, y) with $1 \le x, y \le 49$ and $x \ne y$. There are $49 \cdot 48$ such pairs, all equally likely. We count the pairs for which the two numbers are no more than 10 apart.

For $x = 1$ and for $x = 49$, there are 10 values of y that work, for a total of $2 \cdot 10$. For $x = 2$ and for $x = 48$, there are 11 values of y that work. Continue. For $x = 10$

and for $x = 40$, there are 19 values of y. And finally, for any x with $11 \le x \le 39$, there are 20 values of y. There are n pairs for which the numbers are no more than 10 apart, where

$$n = 2(10 + 11 + 12 + \cdots + 18 + 19) + (29 \cdot 20) = 870.$$

Thus the required probability is 870/2352.

Another way: The solution should have started with a picture like the one that goes with VIII-20. Since the numbers must be different, the points on the diagonal are now missing, and of course the square of dots is 49×49.

In the first solution, we counted the points within 10 of the (missing) diagonal. The picture shows that the points *more* than 10 from the main diagonal look nicer, since they form two triangles. Slide the upper triangle down and fit it next to the lower one. We get a 39×38 array of dots. The probability the two numbers are *not* within 10 of each other is therefore $(39 \cdot 38)/(49 \cdot 48)$.

VIII-26. In a certain society, childbearers keep having children until they have a girl and then stop. But if they get six boys in a row, they give up and stop. In the long run, what percentage of children will be girls? Make reasonable simplifying assumptions.

Solution. We model the process as follows. A large number of people each toss a fair coin that has "G" written on one side and "B" on the other until a G turns up, or until there are six B's, whichever comes first. We want to find the long-run proportion of G's.

Start calculating. With probability 1/2, we get G immediately, and we have one G. With probability 1/4, the game goes BG, and we have one B and one G. With probability 1/8, the game goes BBG, and there are two B and one G. Continue in this way. With probability $1/2^6$, there are five B and one G, and finally again with probability $1/2^6$ there are six B.

The rest of the argument will be sloppy. Imagine playing a large number of games, say $64N$ for some large N. About 1/2 of the time there will be one child, giving $32N$ children roughly. About 1/4 of the time there are two, for a total of $(2)(16N)$ children. Continue in this way. We get

$$(1)(32N) + (2)(16N) + (3)(8N) + (4)(4N) + (5)(2N) + 6N + 6N,$$

children, a total of $126N$. Among these there are about $32N + 16N + 8N + 4N + 2N + N$ girls, that is, $63N$. The proportion of girls is about $63N/126N$, that is, 1/2. With an answer like this, we must look for

Another way: Assume that children are always born on the same day, say Labour Day. We follow the history of a group of people who start off with no children. On the first Labour Day, all will have a child. On Labour Day of the second year, again there will be births, probably a lot fewer because many had a girl immediately and so stopped. Things go on like that for 6 years.

Note that for *every year,* and for any child born that year, there is a fifty-fifty chance that the child is a girl, so the overall probability is 1/2. (The preceding sentence is in fact a complete analysis.)

Another way: Instead of imagining that many people play the coin tossing game once, imagine that one person plays the game many times, maybe taking a brief rest between games. On any toss of the coin, the probability of G is obviously 1/2. The division into games is irrelevant.

Comment. Is the coin tossing model reasonable? We assumed that with probability 1/2 a birth is that of a girl. That's not exactly true, even if there is no interference with "nature." In North America, the proportion of male births is greater than 50%, but boys are more fragile, so there are more six-month old girls than six-month old boys.

A more subtle built-in assumption is that the sexes of successive children are *independent* of each other, like the tosses of a coin. That's not strictly speaking true. The sex of a child is influenced by some external factors. For instance, there is evidence that male deep sea divers are slightly more likely to father girls. Thus the fact that a childbearer produces a boy changes to a tiny degree our estimate of the probability that her next child is a boy.

VIII-27. Suppose that I and another player take turns in throwing with two dice on the condition that I win if I throw 7 points and he wins if he throws 6 points, and I let him have the first throw. To find the ratio of my chance to his.

Solution. With probability 5/36, we obtain a sum of 6 when throwing two fair dice. The probability of throwing a 7 is 6/36. Call the players Alphonse and Beth, where Alphonse has the first throw. Let A stand for "Alphonse throws a 6," and A' for "Alphonse doesn't throw a 6." Similarly, let B stand for "Beth throws a 7," and B' for "B doesn't."

We list all of the ways that Alphonse can win. They are A, $A'B'A$, $A'B'A'B'A$, and so on. The probability of A is 5/36. The probability of $A'B'A$ is (31/36)(30/36)(5/36). The probability of $A'B'A'B'A$ is obtained by multiplying the preceding number by (31/36)(30/36), and so on. The required probability is the sum $a + ar + ar^2 + \cdots$, where $a = 5/36$ and $r = (31/36)(30/36)$. By the usual procedure for summing a geometric progression, the result is 30/61.

Now we can view it as obvious that with probability 1 one or the other must win, and conclude that the probability Beth wins is 31/61, or else we can repeat the analysis, adding up the probabilities of $A'B$, $A'B'A'B$, and so on. The ratio of my (Beth's) chance to his is 31 : 30.

Another way: Let p be the probability that Alphonse ultimately wins the game. For him to win, he must either win (i) on the first throw or (ii) later. The probability of winning on the first throw is 5/36. In order for Alphonse to win later, both players must fail on their first attempts (probability (31/36)(30/36)), and then (ultimately) Alphonse must win. But the probability he wins given that he has reached the third throw is p, since in a sense the game has started again. Thus $p = 5/36 + (31/36)(30/36)p$. Solve: $p = 30/61$.

Comment. This is a verbatim translation of a problem from Christiaan Huygens' *De Ratiociniis in Ludo Alea* (1657). Huygens did not work directly with probabilities, but rather with what would be nowadays called expectations. He reasoned as follows. Let the stake (the amount bet) be t. Let A's expectation when it is B's turn to play be x, and A's expectation when it is A's turn be y.

When it is A's turn, then with probability 5/36 he wins t, and with probability 31/36 he misses, it's now B's turn, and A's expectation is x. Thus $y = (5/36)t + (31/36)x$. If we analyze A's expectation when it is B's turn, we get $x = (6/36)(0) + (30/36)y$. Solve: $x = (31/61)t$.

VIII-28. The school cafeteria offers three equally unpleasant lunch choices A, B, and C. Every day, Zoe remembers how awful the previous day's meal was, and flips a fair coin to decide between the other two alternatives. Suppose that on day 1 of school she had lunch A. (a) Find the probability that she has lunch A on day 9. b) Let p_n be the probability she has A on day n. Find an expression for p_n.

Solution. a) There is some symmetry in the problem: on any day, lunches B and C are equally likely. We don't have perfect three-fold symmetry because she had A on the first day.

Start calculating. We have $p_1 = 1$. On day 2, she has B or C, so $p_2 = 0$. On day 3, whether she had B or C the day before she chooses A with probability 1/2, so $p_3 = 1/2$. It follows that on day 3, the she has each of B and C with probability 1/4. After eating B or C, she next chooses A with probability 1/2, so $p_4 = 1/4$. A similar calculation gives $p_5 = 3/8$, $p_6 = 5/16$, $p_7 = 11/32$, $p_8 = 21/64$, and finally $p_9 = 43/128$.

b) If we stare at the numbers p_n for a while, we notice that they seem to approach 1/3. This is reasonable: the choice Zoe made on day 1 should exert less and less influence as time goes on. So let's subtract 1/3 from each number. We get 2/3, $-1/3$, 1/6, $-1/12$, 1/24, $-1/48$, and so on. The formula almost leaps out: p_n is (probably) $1/3 + (2/3)(-1)^{n-1}/2^{n-1}$.

Maybe we should *prove* that the formula is correct, although the pattern is so striking that it is probably churlish to doubt it. We can do it by mathematical induction, but it is better to use a recurrence formula argument. What is p_{n+1} in terms of p_n? On day n, the probability she *doesn't* eat A is $1 - p_n$, and so she eats A the next day with probability $(1-p_n)/2$. Thus $p_{n+1} = (1-p_n)/2$. Now we check whether the conjectured expression for p_n satisfies this recurrence and the initial condition $p_1 = 1$. It does.

Comments. 1. The distinction between *guessing* that a certain pattern continues and *seeing* that it does can be subtle. For the sequence of this problem, the structure jumps out if we compute half a dozen terms, specially if we don't simplify. Someone who says that the result is obvious is probably right—and so is someone who says it isn't.

2. There are general methods for solving reasonably simple recurrences. Look in books or on the Internet for information on *Difference Equations.*

3. We can look at the same problem in a slightly different way. Any day she had, say that she was in *state* A. Because of the symmetry between B and C, if she had one of these, say she was in state O (for "other"). We then have the following *transition probabilities.* If you are in state A, you switch to O with probability 1. If you are in state O, you switch to A with probability 1/2 and stay in O with probability 1/2. This is a simple example of a *Markov Chain.* Markov chains have many applications, from business to physics.

VIII-29. Toss a fair coin six times. You win if there is a run of three *or more* heads or tails in a row. Find the probability that you win.

Solution. Record the result of the tosses as a word of length 6 made up of the letters H and/or T. There are 2^6 such words, all equally likely. Count the words in which there is a run of length three or more. To simplify the count, we count the words for which three in a row happens *and* the first letter is H, then double the result.

The word could start with HHH. The last three letters can be anything, so there are 2^3 such words. The word could start with HHT. Then it either ends in HHH, or TT followed by H or T, for a total of 3 words.

The word could start with HT. The next letter is either H or T. If the third letter is H, we get three in a row in 3 ways: the word can end with TTT, HHH, or HHT. If the third letter is T, we are looking at HTT. Three in a row happens if the next letter is T (2^2 words) or if the word continues with three H.

Thus there are 19 words that start with H, and therefore 38 overall. The probability of a run of three or more is 38/64.

Comment. The answer is 0.59375. With less effort, we can show that a run of three or more in 5 tosses has probability 1/2. The result for either 6 tosses or 5 goes against the intuition of many people. Sports writers tend to underestimate the probability of streaks. It is perfectly normal for a team to win or lose a few games in a row, or for a basketball player to have a so-called "hot" or "cold" streak. There is no mysterious psychology involved—the same thing happens with coins.

Suppose that students are asked, for homework, to toss a coin 500 times and record the results. Some are too busy to toss, and quickly write down a string of H's and T's. Almost certainly the string will not have enough streaks. In 500 tosses there should be several streaks of 6 or more, but someone inventing data will probably not know that.

Since faking is easier and cheaper than running an experiment, and the results are more controllable, it is not uncommon for experimental data to be invented, specially in situations that involve many repetitions. Unless the faking is done by a statistician, one can, by using a *runs test,* show that with high probability the numbers are not genuine.

VIII-30. Six different numbers are chosen at random from the integers 1 to 49. Find the probability that the smallest number is less than 6.

Solution. Six numbers can be chosen from 49 in $\binom{49}{6}$ ways, all equally likely. We now count the number of ways that smallest number chosen is greater than or equal to 6. This is the number of ways of choosing six numbers from the 44 numbers 6, 7, ..., 49, so it is $\binom{44}{6}$

It follows that the probability that *none* of the numbers is less than 6 is

$$\binom{44}{6}\Big/\binom{49}{6}.$$

Note that $\binom{44}{6} = (44\cdot 43\cdot 42\cdot 41\cdot 40\cdot 39)/6!$ while $\binom{49}{6} = (49\cdot 48\cdot 47\cdot 46\cdot 45\cdot 44)/6!$. The ratio is about 0.5048. Thus the probability that at least one number is less than 6 is about 0.49522.

Comment. An avid lottery player noticed that numbers less than 6 seemed to come up often, and brought to the author his suspicions that the lottery was crooked. People notice small numbers more than numbers in the middle range. Consecutive numbers also happen more often than intuition might suggest. And the lottery doesn't need to be crooked. It already keeps a far higher proportion of bettors' dollars than organized crime does.

VIII-31. Find the smallest number of fair dice we should toss to have a better than 50–50 chance of getting two or more sixes.

Solution. Let this number be n. Instead of tossing n dice, imagine tossing one die n times. The probability of *not* getting a six when we toss a die is 5/6. So the probability we fail to get a six n times in a row is $(5/6)^n$. We now compute the probability of exactly 1 six.

We could get six on the first try, then non-sixes the rest of the way. This has probability $(1/6)(5/6)^{n-1}$. Or we could get a non-six, then a six, then non-sixes the rest of the way. This has probability $(5/6)(1/6)(5/6)^{n-2}$, that is, $(1/6)(5/6)^{n-1}$. We go on like this and conclude that the probability of exactly 1 six in n tosses is $n(1/6)(5/6)^{n-1}$.

Thus n is the smallest integer such that

$$\left(\frac{5}{6}\right)^n + n\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{n-1} < \frac{1}{2}.$$

We can't expect to solve this kind of inequality by "formula." But calculator experimentation gives the answer quickly. It turns out that $n = 10$.

Comments. 1. We could use a graphing calculator to find the appropriate n. Let $f(x) = (5/6)^{x-1}(5/6 + x/6)$. Graph this function, and read off from the display the first integer n for which $f(n)$ dips below 1/2.

2. The above problem, together with more sophisticated ones, was analyzed by Christiaan Huygens in his *De Ratiociniis in Ludo Aleae*, published in 1657. The book marked the beginning of systematic calculation of probabilities.

VIII-32. A box contains 10 tickets, numbered from 1 to 10. (a) Alphonse picks a ticket at random, and keeps it. Then Beth picks a ticket. Find the probability that Alphonse's number is less than Beth's. (b) How does the answer change if Alphonse puts his ticket back in the box before Beth picks?

Solution. (a) Record the result of the experiment as a pair (a, b), where a is Alphonse's number and b is Beth's. Since Alphonse kept his ticket, $a \neq b$. All pairs (a, b) with $a \neq b$ are equally likely. Since a can be anyone of the 10 numbers, and b can take on 9 values for each a, there are 90 pairs.

How many pairs have $a < b$? Start counting. When $a = 1$, there are 9 pairs, when $a = 2$ there are 8, and so on, so there are $9 + 8 + \cdots + 1$ pairs, that is, 45. Thus our probability is 45/90.

Another way: We don't need to count. For any pair, such as $(3, 5)$, in which Alphonse is below Beth, there is an associated pair, obtained by interchanging the numbers, in which Alphonse is above Beth, so the probability is 1/2. This answer is intuitively obvious. But probability can be tricky, and the experiment is described in an asymmetric way (Alphonse picks first), so perhaps a bit of caution is a good idea.

(b) Again we can count. This time, since $a = b$ is possible, there are 100 equally likely pairs. Just as in part (a), there are 45 pairs in which Alphonse's number is less than Beth's, so the probability is 0.45.

It is neater to calculate first the probability that the numbers are *equal*. Whatever Alphonse picks, Beth matches it with probability 1/10. Thus with probability 0.90 the numbers are unequal. By symmetry Alphonse's number is less than Beth's with probability 0.45.

VIII-33. Four fair coins are tossed (tossing cycle #1). Those that come up tails are removed. The ones that remain are tossed again (tossing cycle #2). Those that come up tails are removed, and so on. Find the probability that there are three or fewer tossing cycles.

Solution. We sketch a natural approach without completing the details. How can there be three or fewer tossing cycles? If we get all tails on the first toss, there is only one cycle, and that has probability $1/2^4$.

There could be exactly two cycles. That can happen in several ways: one coin survives the first cycle, then dies on the second; or two coins survive the first cycle, but both die on the second, and so on. The probability that exactly one coin survives the first cycle ("three tails") turns out to be $4/2^4$. Given that a coin survives the first toss, the probability that it dies on the second is 1/2, so the probability that one coin survives the first cycle but then dies is $(4/2^4)(1/2)$. There are three more cases of this kind to examine, and we haven't even started thinking about the third cycle. But with patience and care, we could compute the answer.

Another way: Change the game a little. If a coin shows tails in any cycle, instead of throwing it out and making it feel bad, put a small red dot on it and keep it in the game. We need to find the probability that after 3 tosses of the coins, every coin has some red on it, that is, has landed tails *at least once.*

The probability that coin A shows heads 3 times in a row is 1/8, so after 3 tosses it has some red on it with probability 7/8. Thus the probability that they *all* have some red is $(7/8)^4$.

Comment. Let N be the number of cycles that the game lasts, and let $\Pr(X)$ denote the probability that X happens. We have

$$\Pr(N \leq 3) = \Pr(N \leq 2) + \Pr(N = 3).$$

We have seen that $\Pr(N \leq 3) = (7/8)^4$. By a similar analysis, $\Pr(N \leq 2) = (3/4)^4$. Now we know $\Pr(N = 3)$. The ideas generalize to more coins, more tosses, and to consistently biased coins.

VIII-34. The faces of dice A, B, and C have been relabelled as follows. A has two 2's, two 4's, and two 9's; B has two 3's, two 5's, and two 7's; and C has two 1's, two 6's, and two 8's.

X selects one of the dice. Then Y selects one of the remaining dice. They each toss the die they picked. Whoever comes up with the bigger number wins. Show that whatever die X picks, Y can pick a die that has a better than 50% chance of beating X.

Solution. We show that B is "better" than A. Toss A and B, and record the result as an ordered pair (a, b). By symmetry, all 9 achievable ordered pairs are equally likely. We count the ordered pairs for which B "beats" A by listing them all: $(2, 3)$, $(2, 5)$, $(2, 7)$, $(4, 5)$, and $(4, 7)$. So B beats A with probability 5/9. If X picks die A, and Y responds by picking B, then Y has probability 5/9 of beating X.

A similar calculation shows that die C beats B with probability 5/9 (so if X chooses B, then Y should choose C) and die A beats C with probability 5/9. So whatever X picks, Y can select one of the remaining dice and win with probability 5/9.

Comment. Last week, the teacher announced that in a week there would be an in-class essay on a topic that she would choose at random from A, B, or C. Students X, Y, and Z have studied these topics to various degrees. Table 8.1 shows how their marks depend on the topic the teacher chooses. The calculation that was done for the dice shows that Y has probability 5/9 of getting a higher mark than X, and Z has probability 5/9 of getting a higher mark than Y, but X has probability 5/9 of getting a higher mark than Z. So Y is "better" than X, and Z is better than Y, but X is better than Z!

	A	B	C
X	40	90	20
Y	30	50	70
Z	80	10	60

Table 8.1: The In-class Essay

This paradoxical situation shows up in opinion polls. It can and does happen that given a choice between products or politicians A and B, a majority prefers B, and a majority prefers C over B, but a majority prefers A over C.

Note that the marks table is a *magic square.* Sums of the marks in each row, in each column, and indeed in the two diagonals, are all equal. Everything is connected to everything else!

VIII-35. A fair die is tossed six times. (a) Find the probability that each of the six faces turns up once. (b) Find the probability that exactly five different faces turn up.

Solution. (a) Record the results, in order. Each record is a six-letter word made up of "letters" chosen from 1, 2, 3, 4, 5, 6. There are 6^6 such words, all equally likely.

How many words are there in which the letters are all different? There are 6 ways to choose the first letter, and for each of these there are 5 choices for the second letter, and so on. Thus there are 6! words in which the letters are all different. For the probability, divide by 6^6. The result simplifies to 5/324: it is very unlikely that the faces are all different.

(b) We count the words that have exactly 5 different letters. The letter that gets left out can be chosen in 6 ways. For each such choice, the letter that gets to occur twice can be chosen in 5 ways. Once this is done, decide *where* the repeating letter occurs. That can be done in $\binom{6}{2}$ ways. And once the repeating letter is placed, there are 4! ways to place the rest of the letters. Multiply. There are $15 \cdot 6!$ words. For the probability, divide by 6^6.

Another way: The answer for part (b) was 15 times the answer in part (a). That's not an accident. Construct the words with exactly 5 different letters in two stages: (i) Make a word with 6 different letters and (ii) Take that word, pick two positions, and replace the letter in the second position with a copy of the letter in the first position.

In this way, we produce every word that we want once and only once. So to answer part (b), multiply the answer in part (a) by $\binom{6}{2}$.

VIII-36. Xavier rolls one fair die, then Yolande rolls two. Xavier wins unless *at least one* of Yolande's dice beats Xavier's. Find the probability that Xavier wins.

Solution. Colour one of Y's dice red and the other green. Record the result of the game as a triple (a, b, c), where a is the number thrown by X, b is the number on Y's red die, and c the number on Y's green die. There are 6^3 possible triples (a, b, c), all equally likely. We count the triples that result in a win for X.

First count the triples with $a = 1$. If X is to win, we must have $b = 1$ and $c = 1$, so there is only 1 possibility. Now count the winning triples with $a = 2$. We must have $b \leq 2$ and $c \leq 2$. Thus there are 2 choices for b, and for each of them 2 choices for c, giving 2^2 cases. Similarly, there are 3^2 cases with $a = 3$, and so on. The total is $1^2 + 2^2 + \cdots + 6^2$, so X wins with probability $91/216$.

Chapter 9

Problems in Computation

Introduction

These questions are of a kind not ordinarily asked in secondary school but that seem appropriate in view of the pervasive presence of calculators in the mathematics classroom. Many of the questions are about numbers too large fully to fit in a standard calculator.

The problems are meant to be done with a calculator that displays no more than 10 digits. There are tools that show many more digits. The calculator utility in Microsoft Windows shows 31. Programs such as Maple or Mathematica can operate to virtually unlimited precision. There are free programs available on the Internet, including PARI-GP, UBasic, C libraries for programmers, and many others.

In several problems, including IX-6, IX-7, IX-16, IX-25, and IX-26, the main issue is that, because of roundoff errors, the calculator can give grossly inaccurate answers. Problems IX-11, IX-29, and IX-28 give a glimpse of some algorithms that computers use to do arithmetic with large numbers. And IX-5 shows the startling effectiveness of the approximation method for square roots often called Heron's Method, while IX-4 begins the process of extending the method.

Problems and Solutions

IX-1. A calculator is defective: the only operation keys that work are $+$, $-$, and $1/x$. Show how to use this calculator to find (a) the square of a number; (b) the product of two numbers.

Solution. (a) With the broken calculator compute

$$\frac{1}{t-1} - \frac{1}{t+1}, \qquad \text{that is,} \qquad \frac{2}{t^2-1}$$

(unless $t = \pm 1$, when we shouldn't need a calculator to find t^2). Invert and add the result to itself to get $t^2 - 1$, then add 1.

Or calculate $1/(t-1) - 1/t$, that is, $1/(t^2 - t)$, invert, and add t. There are a number of other approaches.

(b) Use the identity $(x + y)^2 - (x - y)^2 = 4xy$. By part (a), we can square with the broken calculator, so we can find $4xy$. If $xy \neq 0$, get rid of the 4 by inverting, adding the result to itself twice, and inverting again.

We can save quite a few steps by using a variant of the idea of part (a). Note that $1/(t-2) - 1/(t+2) = 4/(t^2 - 4)$. Let $f(t) = (t^2 - 4)/4$. Then $f(t)$ is easy to compute, and $xy = f(x + y) - f(x - y)$.

Comment. It is hard to imagine a practical application for part (a), but ideas like those of part (b) have many uses. We give a historical example.

In the bad old days before calculators, adding by hand was reasonably fast, but multiplication took a while. So to multiply people used slide rules, or tables of logarithms.

They also multiplied by using *quarter-squares* tables that list $t^2/4$, say for t from 0.0000 to 2.0000. If x and y are between 0 and 1, to find xy calculate $x + y$ and $x - y$, look up $(x + y)^2/4$ and $(x - y)^2/4$ in the table and subtract! If x and y aren't in the right range, shift decimal points until they are, multiply as before, then shift back.

They could also multiply using *half-squares* tables, because of the identity

$$xy = \frac{(x+y)^2}{2} - \left(\frac{x^2}{2} + \frac{y^2}{2}\right).$$

To multiply with a half-squares table takes three table look-ups, two additions, and a subtraction. With quarter-squares there are two table look-ups, an addition, and two subtractions. For a number of reasons half-squares were more popular.

IX-2. Which is larger, $\sqrt{160001} + \sqrt{159999}$ or 800?

Solution. My calculators incorrectly claim that the difference between the numbers is 0. In general, which is larger, $\sqrt{x+1} + \sqrt{x-1}$ or $2\sqrt{x}$? Equivalently, we compare

$$(\sqrt{x+1} + \sqrt{x-1})^2 \qquad \text{and} \qquad 4x.$$

The expression on the left is equal to $2x + 2\sqrt{x^2 - 1}$, which is less than $4x$.

Comment. By rationalizing the numerator twice, we get

$$\left(\sqrt{x+1} + \sqrt{x-1}\right) - 2\sqrt{x} = \frac{-2}{\left(\sqrt{x+1} + \sqrt{x-1} + 2\sqrt{x}\right)\left(\sqrt{x^2-1} + x\right)}.$$

The expression on the right is calculator-friendly. When $x = 160000$, its absolute value is about 4×10^{-9}. Such a "large" difference ought to have been detected by the calculator, but it wasn't.

One calculator, a cheap one that displays a paltry 5 digits in scientific notation, has no trouble with $(\sqrt{90001}+\sqrt{89999})-600$. The other cost twice as much, displays 10 digits, and can't handle the computation. It's what's inside that counts.

IX-3. Show that there is an integer k such that $k\pi$ is within 10^{-10} of an integer. Hint: Look at π, 2π, ..., $N\pi$ where $N = 10^{10}$.

Solution. Use a calculator to find the distance of $n\pi$ from the nearest integer for $n = 1$, 2, 3, and so on. The first 7 results are roughly 0.14, 0.28, 0.42, 0.43, 0.29, 0.15, and 8.8×10^{-3}. The dramatic dip at $n = 7$ shows that 22/7 is a good approximation to π. We could continue to calculate—not with a calculator, that's hopeless, not enough digits—but with a program that can handle high precision arithmetic. Instead, we will *think* about the calculation without actually doing it.

For any positive integer n, let a_n be the *fractional* part of $n\pi$, that is, the part of $n\pi$ after the decimal point. So

$$n\pi = \lfloor n\pi \rfloor + a_n,$$

where $\lfloor x \rfloor$ is the greatest integer which is less than or equal to x. For example, $7\pi = 21 + a_7$, where a_7 is about 0.991.

Let $N = 10^{10}$, and look at the sequence $a_1, a_2, \ldots, a_{N+1}$. Divide the interval $[0, 1)$ into N subintervals each of length $1/N$. The first subinterval is $[0, 1/N)$, the second is $[1/N, 2/N)$, and so on. There are N subintervals, and the sequence has $N + 1$ terms, so at least two of the a_i lie in the same subinterval. We conclude that there are integers m and n, with $m < n \le N + 1$, such that $|a_n - a_m| < 1/N$.

Because $n\pi = \lfloor n\pi \rfloor + a_n$ and $m\pi = \lfloor n\pi \rfloor + a_m$,

$$(n - m)\pi = (\lfloor n\pi \rfloor - \lfloor m\pi \rfloor) + (a_n - a_m).$$

But $|a_n - a_m| < 1/N$, and therefore that $(n - m)\pi$ is within $1/N$ of an integer.

Comments. 1. The proof didn't produce an *explicit* k such that $k\pi$ is within $1/N$ of an integer. But it says there is such a k of the form $n-m$, where $1 \le m < n \le N+1$, so $1 \le k \le N$. Thus if we are patient and look in turn at π, 2π, ..., $N\pi$, we are certain to bump into a suitable k.

Exactly the same argument shows that if x is any real number and N is a positive integer, then there is an integer k with $1 \le k \le N$ such that kx is within $1/N$ of an integer. There is a large literature on approximating real numbers efficiently with fractions. Some of the most striking results in Number Theory have been obtained by studying such approximations.

2. In the fifth century, Zu Chongzhi somehow found the approximation 355/113 to π. It was rediscovered in Europe in the sixteenth century. The approximation is remarkably accurate: 113π differs from 355 by about 3×10^{-5}, so 355/113 is within

2.7×10^{-7} of π. Other good approximations with not too large denominator can be found by using *continued fractions* (see almost any Number Theory book or the Internet).

IX-4. Let x_n be a good approximation to $\sqrt{a}$. We can obtain a (usually) better approximation x_{n+1} as follows. Let $x_n + c_n = \sqrt{a}$ and square both sides:

$$x_n^2 + 2x_n c_n + c_n^2 = a.$$

If c_n is small compared to x_n, throw away the c_n^2 term and conclude that c_n is approximately $(a - x_n^2)/2x_n$. So if

$$x_{n+1} = x_n + \frac{a - x_n^2}{2x_n} = \frac{x_n}{2} + \frac{a}{x_n}.$$

then x_{n+1} should be an excellent approximation of $\sqrt{a}$.

Use a similar idea to show how to go from an approximation x_n of $\sqrt[3]{a}$ to the next approximation x_{n+1}. Then starting from the approximation $x_0 = 2$ for $\sqrt[3]{10}$, compute x_1, x_2, and x_3.

Solution. Let $x_n + c_n = \sqrt[3]{a}$. Cube both sides. We obtain

$$x_n^3 + 3x_n^2 c_n + 3x_n c_n^2 + c_n^3 = a.$$

If c_n is small compared to x_n, then the last two terms on the left-hand side may be neglected, and c_n is approximately equal to $(a - x_n^3)/3x_n^2$. Let

$$x_{n+1} = x_n + \frac{a - x_n^3}{3x_n^2} = \frac{2x_n}{3} + \frac{a}{3x_n^2}.$$

Let $a = 10$ and $x_0 = 2$. To calculator accuracy, $x_1 = 4/3 + 10/12 = 2.1666667$, $x_2 = 2.1545036$, and $x_3 = 2.1544347$.

Comment. Note that x_3 differs from $\sqrt[3]{10}$ by about 2.23×10^{-9}, so we arrived quickly at a very good approximation. The procedure for calculating square roots was discovered long ago. Problem IX-5 gives a way of measuring its efficiency.

Try to adapt the idea of the above problem to $x^3 + 4x - 18 = 0$. An analogous procedure can be used to compute roots of $f(x) = 0$ for well-behaved functions f. In calculus courses it is usually called the Newton Method, even though Newton only casually mentioned it as a tool for approximating the roots of polynomials. There is an extensive discussion of a similar method in Qin Jiushao's *Shushu jiuzhang* (Mathematical Treatise in Nine Sections, 1247).

IX-5. Define a procedure for approximating $\sqrt{a}$ as follows. Let x_0 be a reasonably good first estimate of $\sqrt{a}$. If x_n is the n-th estimate, the next estimate x_{n+1} is defined by $x_{n+1} = (1/2)(x_n + a/x_n)$. Define e_n by $x_n = (1 + e_n)\sqrt{a}$.

(a) Let $a = 10$ and $x_0 = 3$. Compute x_1, x_2, and x_3.

(b) Show that $e_{n+1} = e_n^2/(2 + 2e_n)$.

(c) Use part (b) to compute e_3.

Solution. (a) We have $x_1 = (1/2)(3 + 10/3) = 19/6$. Similarly, $x_2 = 721/228$ and $x_3 = 1039681/328776$.

(b) By the definition of the numbers e_k,

$$(2 + 2e_{n+1})\sqrt{a} = 2x_{n+1} = x_n + \frac{a}{x_n} = (1 + e_n)\sqrt{a} + \frac{a}{(1 + e_n)\sqrt{a}}.$$

Simplify. The $\sqrt{a}$ terms disappear and we get

$$2 + 2e_{n+1} = 1 + e_n + \frac{1}{1 + e_n}.$$

When we "move" the 2 to the right-hand side and simplify we obtain the desired result.

(c) Since $(1 + e_0)\sqrt{10} = x_0 = 3$, to calculator accuracy $e_0 = -0.051316701$. Now compute with the formula of (b). To two significant figures, $e_1 = 1.4 \times 10^{-3}$, $e_2 = 9.6 \times 10^{-7}$, and $e_3 = 4.6 \times 10^{-13}$.

Comment. The technique used to approximate $\sqrt{a}$ is variously called the Babylonian Method, Heron's Method, or the Newton Method. It is wonderfully efficient.

We saw that in three steps we can find $\sqrt{10}$ with (in principle) relative error less than 5×10^{-13}. We say in principle because a real calculator will make truncation errors. In principle four iterations would give a relative error of 10^{-25}. The number of decimal places of accuracy roughly speaking doubles with each application of Heron's Method.

Problem IX-4 gives a justification of the recurrence formula that was used, and extends the idea to cube roots. A more general procedure called the *Newton-Raphson Method* can be used to approximate the roots of $f(x) = 0$. Usually, but not always, the Newton-Raphson Method shares the computational efficiency of Heron's Method.

IX-6. The number $119071 - 30744\sqrt{15}$ is about 4×10^{-6}. Calculate it correct to five significant figures.

Solution. If we key in $119071 - 30744\sqrt{15}$, the calculator is likely to produce an answer of 0—both of mine do—or something like 0.00001, which is somewhat worse. We could solve the problem with the calculator that comes with Microsoft Windows, or with a sophisticated computational tool such as *Maple*, *Mathematica*, or MATLAB, but with a little thinking an ordinary calculator will do.

Call our number $a - b\sqrt{15}$. Then

$$a - b\sqrt{15} = \frac{(a - b\sqrt{15})(a + b\sqrt{15})}{a + b\sqrt{15}} = \frac{a^2 - 15b^2}{a + b\sqrt{15}}.$$

With some calculator work we get $a^2 - 15b^2 = 1$. The calculator says that $a + b\sqrt{15}$ is about 238142. To five significant figures, the ratio is 4.1992×10^{-6}.

Comment. If a calculator is used to find the difference between two nearly equal numbers, accuracy is lost, sometimes catastrophically so. Most calculators store numbers internally to at most twelve significant figures, so the calculator may well think that the two numbers are the same.

A calculator answer of 0 at least alerts us to the fact that something went wrong. A more insidious mistake arises when, as is often the case, the final digit in the internal representation of a number is in error. Then the calculator may report an incorrect non-zero answer that we treat as being right—machines never lie.

By looking at the above problem in the right way, instead of finding the difference between nearly equal imprecisely known reals, we found the difference between the known integers a^2 and $15b^2$.

IX-7. Angle θ has radian measure 0.000002. Find $\sqrt{1 - \cos\theta}$ correct to two significant figures.

Solution. Most calculators give the answer 0. Since $\sqrt{1 - \cos\theta} \neq 0$, the calculator's answer isn't correct to two significant figures. The problem is that $\cos\theta$ is close to 1, so the calculator thinks that $1 - \cos\theta$ is 0.

We can get around the limitations of the calculator by noting that

$$\begin{aligned}\sqrt{1 - \cos\theta} &= \sqrt{1 - \cos\theta}\,\frac{\sqrt{1 + \cos\theta}}{\sqrt{1 + \cos\theta}} \\ &= \frac{\sqrt{1 - \cos^2\theta}}{\sqrt{1 + \cos\theta}} = \frac{\sin\theta}{\sqrt{1 + \cos\theta}}.\end{aligned}$$

Now compute. To two significant figures, the answer is 1.4×10^{-6}.

Comment. It can be shown that if the angle x, measured in radians, is not far from 0, then $\sin x$ is very close to x, and $\cos x$ is even closer to $1 - x^2/2$. These estimates are far more reliable than calculator values.

IX-8. A hollow copper sphere with radius 1 meter floats in a lake, half-submerged. Find, correct to two significant figures, the thickness of the copper. The density of copper is 8.94 times the density of water.

Solution. The sphere has volume k, where k is a constant that happens to be $4\pi/3$. Let t be the thickness of the copper. Then the interior of the sphere has volume $k(1-t)^3$, so the copper has volume $k - k(1-t)^3$.

The sphere displaces a volume $k/2$ of water. The weight of this water is equal to the weight of the sphere. Assume that what's inside the sphere weighs nothing. Then

$$k/2 = \rho(k - k(1-t)^3),$$

where $\rho = 8.94$. Simplify. The k cancels, $(1-t)^3 = 1 - 1/2\rho$, so

$$t = 1 - \sqrt[3]{1 - 1/2\rho}.$$

The calculator says that the copper is about 1.9 cm thick.

Another way: It is interesting to see what happens if we simplify too much, by cubing $1 - t$. We obtain the equation

$$8.94t^3 - 26.82t^2 + 26.82t - 0.5 = 0.$$

or, if we didn't cancel k, something even uglier. Note how efficiently "simplifying" has destroyed the structure!

But all is not lost. The *Solve* key on the calculator works nicely. So do many programs—spreadsheets, *Maple, Mathematica,* and others. But with just a simple calculator, or even paper and pencil, we can reach an answer fairly quickly.

Let's see where t is, roughly. Since t is quite small, t^3 and $3t^2$ are *very* small. For a first approximation, throw them away. Our first estimate for t is then $t = 0.5/26.82$, about 0.01864. One way to refine this already very good estimate is to rewrite the equation for t as

$$t = \frac{0.5}{26.82} + \frac{26.82t^2}{26.82} - \frac{8.94t^3}{26.82}.$$

Plug our current estimate for t into the right-hand side of the above equation. We get about 0.018988. When we then plug in 0.018988 for t, there is barely any movement, so to two significant figures $t = 0.019$.

IX-9. Find an integer N such that the first three digits of $\sqrt{N}$ to the right of the decimal point are 3, 2, and 1, in that order.

Solution. The integer N has the required property if and only if

$$a + 0.321 \le \sqrt{N} < a + 0.322$$

for some integer a. Equivalently, we want

$$a^2 + (0.642)a + (0.321)^2 \le N < a^2 + (0.644)a + (0.322)^2.$$

So we need to make sure that there is an integer between the lower bound and the upper bound. That will be true if we pick $a = 500$. The left bound is then about 250321.1, the right bound is about 250322.1, so we can take $N = 250322$.

Another way: Note that the difference $\sqrt{N+1}-\sqrt{N}$ between square roots of consecutive numbers is equal to $1/(\sqrt{N+1}+\sqrt{N})$. By the time $N = 250000$, this difference is (barely) less than 0.001. At 250000 (and therefore 250001) the first three digits of $\sqrt{N}$ after the decimal point are 000. At 251000, they are 999. So as N travels from 250001 to 251000, the first three digits of $\sqrt{N}$ after the decimal point travel from 000 to 999. In particular, they reach 321 at 250322.

Comment. We certainly don't need to go as far as 250322. One way to tackle the problem is with a brute force search—even a programmable calculator can handle the job. The smallest N that works is 2054.

IX-10. Let $X = (100+\sqrt{10001})^3$. Find the first two non-zero digits after the decimal point in the decimal expansion of X.

Solution. Direct calculation with a standard calculator doesn't work, so we use a trick, well, not really a trick, more like a Method. Let

$$N = (100+\sqrt{10001})^3 + (100-\sqrt{10001})^3.$$

Imagine expanding each of the cubes. Note that the terms that involve $\sqrt{10001}$ cancel, and therefore N is an integer. We conclude that

$$X = N + (\sqrt{10001}-100)^3.$$

Thus X is an integer (which happens to be 8000600) plus the small correction term $(\sqrt{10001}-100)^3$.

We can use the calculator directly to estimate the correction term. It is safer to rationalize the numerator and compute $1/(\sqrt{10001}+100)^3$. Either way, the calculator says that the correction term is 0.000000124, so the first two non-zero digits of X after the decimal point should be 1 and 2.

Comments. 1. If we had wanted $(\sqrt{10001}-100)^3$ correct to three significant digits, the calculator's answer could be misleading. When one of my calculators is put into *scientific mode*, it reports that the answer is about 1.249906×10^{-7}. That's essentially right, so 0.000000124 is certainly not correct to three significant digits.

2. Let n be a positive integer, and let

$$N = (100+\sqrt{10001})^n + (100-\sqrt{10001})^n.$$

We describe a way of seeing, without computation, that N is an integer. *If we expanded,* we would get

$$N = (100+\sqrt{10001})^n + (100-\sqrt{10001})^n = a + b\sqrt{10001},$$

where a and b are integers that we do not compute.

In the first expression for N, replace $\sqrt{10001}$ everywhere by $-\sqrt{10001}$. We get N right back. If we do the same thing with $a+b\sqrt{10001}$, we get $a-b\sqrt{10001}$. But there is no change, so $b = 0$!

IX-11. Let $N = 12345678$. Calculate N^2 exactly.

Solution. The answer is greater than 10^{14}, and the usual scientific calculators do not display—or carry internally—enough digits. But we can work our way around this limitation. Since $N = (1234 \times 10^4) + 5678$,

$$N^2 = (1234)^2 \times 10^8 + 2(1234)(5678) \times 10^4 + (5678)^2.$$

Using the calculator, we find that

$$N^2 = (1522756 \times 10^8) + (14013304 \times 10^4) + 32239684.$$

Add (by hand). The result is 152415765279684.

Comment. Computer programs for high precision arithmetic do multiplication by using an elaborate variant of the same idea. For a discussion, see Volume 2 of Knuth's *The Art of Computer Programming.*

IX-12. Let $a = 99999$. Calculate $2a^3 + 3a^2$.

Solution. We can experiment with a calculator, using one 9, two 9's, and three 9's. The experimentation gives rise to a plausible conjecture which turns out to be right. In general,

$$2(x-1)^3 + 3(x-1)^2 = 2(x^3 - 3x^2 + 3x - 1) + 3(x^2 - 2x + 1) = 2x^3 - 3x^2 + 1.$$

Put $x = 10^5$. We conclude that $2a^3 + 3a^2 = 1999970000000001$.

IX-13. Object A is travelling at velocity u relative to the Earth, and B is travelling in the opposite direction at velocity v relative to the Earth. Newton's theory of motion says that the relative velocity of B with respect to A is $u + v$. Einstein's theory of Special Relativity says that the relative velocity is

$$\frac{u+v}{1+\dfrac{uv}{c^2}},$$

where c is the speed of light in a vacuum.

Suppose that A and B are airplanes travelling in opposite directions, each at 0.5 kilometers per second. By how much do Newton's and Einstein's figures for their relative velocities differ, say to 2 significant figures? Assume that the velocity of light is 300,000 kilometers per second.

Solution. The Newtonian prediction is 1. Special Relativity predicts that the relative velocity is $1/(1 + 0.25/c^2)$. If we put this directly into my cheap calculator and subtract 1, we unfortunately obtain an answer of 0, which can't be correct to even 1 significant figure.

We can use algebraic manipulation to put the problem into a form the calculator can handle. In general, the difference between the two predictions is

$$u + v - \frac{u+v}{1+\dfrac{uv}{c^2}}, \qquad \text{that is,} \qquad \frac{uv(u+v)}{c^2+uv}.$$

Now we can compute without trouble. To two significant figures, the answer is 2.8×10^{-12}.

IX-14. Find the solutions of the equation $x^2 - 10^8 x + 200 = 0$, correct to 3 significant digits.

Solution. Use the quadratic formula. The roots are $(10^8 \pm \sqrt{10^{16} - 400})/2$. It turns out that, to 3 significant digits, $(10^8 + \sqrt{10^{16} - 400})/2 = 1.00 \times 10^8$.

The other root is trouble. Ordinary calculators claim that the answer is 0. This is definitely *not* correct to 3 significant figures. What are we to do?

A simple way out is to use the fact that if r and s are the roots of $ax^2+bx+c = 0$, then $rs = c/a$. We know one root to high accuracy. The product of the roots is 200, and therefore, to 3 significant digits, the other root is 2.00×10^{-6}.

Another way: Start from the standard formula for the roots of $ax^2+bx+c = 0$ and multiply "top" and "bottom" by $(-b \mp \sqrt{b^2 - 4ac})/2a$. After some manipulation we find that the roots of the quadratic are

$$\frac{2c}{-b \mp \sqrt{b^2 - 4ac}}.$$

In particular, $(10^8 - \sqrt{10^{16} - 800})/2$, which fooled the calculator earlier, turns into the perfectly harmless $400/(10^8 + \sqrt{10^{16} - 800})$, which we can evaluate to high accuracy with an ordinary calculator. (The new quadratic formula has trouble with the big root.)

Another way: Note that $\sqrt{10^{16} - 800}$ is equal to $10^8\sqrt{1-\epsilon}$, where $\epsilon = 8.00 \times 10^{-14}$. We examine $\sqrt{1-\epsilon}$ more closely.

For any ϵ close to 0, $\sqrt{1-\epsilon}$ is close to $1 - \epsilon/2$. We can verify this informally by squaring $1 - \epsilon/2$, and noting that the result is $1 - \epsilon + \epsilon^2/4$. The number $\epsilon^2/4$ is utterly negligible in comparison with ϵ when the latter is close to 0.

Using this approximation to the square root, we find that the second root is about $(1/2)\left(10^8 - 10^8(1 - 4 \times 10^{-14})\right)$, that is, 2.00×10^{-6}.

Another way: Insofar as the small second root is concerned, the equation is really not a quadratic at all! The x^2 term only perturbs things a little bit, so for the smaller root we are really looking at the linear equation $-10^8 x + 200 = 0$, which has the solution $x = 2 \times 10^{-6}$. To see how much the x^2 term perturbs things, rewrite the equation as

$$x = \frac{200 + x^2}{10^8},$$

and substitute our first estimate 2×10^{-6} into the right-hand side to get an improved estimate. There is virtually no movement.

Comments. 1. We ran into trouble because when we compute the second root in the usual way, we are subtracting two large nearly equal numbers. Since the calculator only keeps internally a limited number of digits, usually no more than a dozen, the numbers 10^8 and $\sqrt{10^{16} - 400}$ are, from its point of view, equal, so when asked to subtract one from the other it reports that the answer is 0.

2. In the second way of calculating the small root, the expression that has a square root in the numerator is hard to work with, while the expression that has the square root in the denominator works just fine.

Many a high-school student has been penalized for leaving square roots in the denominator. That may have made sense once: paper and pencil division by an ugly number is more painful than multiplication by an ugly number. But simplified form for one purpose may be inappropriate for another.

IX-15. Find the smallest integer n such that 2^n has 1000 digits in its decimal representation.

Solution. We want the smallest n such that $2^n \geq 10^{999}$. Take logarithms to the base 10. Note that

$$\log(2^n) = n \log 2 \qquad \text{and} \qquad \log(10^{999}) = 999.$$

We want the smallest n such that $n \log 2 \geq 999$. The calculator says that $999/\log 2$ is approximately 3318.6062, so we can probably safely conclude that $3318 < 999/\log 2 < 3319$. It follows that 3319 is the smallest integer n that works.

Another way: The problem can be solved without explicit appeal to logarithms. We can calculate 2^n for various large n and experiment our way to the answer. But there is an *overflow* problem: most calculators refuse to calculate 2^n for $n > 232$. We will fool the calculator into cooperating.

Note that $2^{10} = 1024 = 1.024 \times 10^3$. So $2^{3000} = 1.024^{300} \times 10^{900}$. The calculator says that 1.024^{300} is about 1.23×10^{903}. We need to reach 10^{999}.

Since $2^{10} = 1.024 \times 10^3$, it follows that 2^{320} is about 2.136×10^{96}, and therefore 2^{3320} is about 2.63×10^{999}. If we divide by 2, we still have a 1000-digit number, but dividing by 2 again brings us below 10^{999}, so the smallest n is 3319.

Comment. Powers of 2 come up often in applications. The fact that 2^{10} is roughly 10^3, or equivalently that $\log 2$ is about 0.3, is useful for making ballpark estimates.

IX-16. A metal cube with initial volume $1000\,\text{cm}^3$ was uniformly electroplated. The volume of the cube increased by 10^{-8} cm^3. To 3 significant figures, how much did the side increase?

Solution. The new side is $\sqrt[3]{1000 + 10^{-8}}$. To find the change, take away 10. If we key this into an ordinary calculator, it says the answer is 0. So we take another approach.

Note that

$$\sqrt[3]{1000 + 10^{-8}} = 10 \sqrt[3]{1 + 10^{-11}}.$$

The cube root of $1 + 10^{-11}$ is a number very close to 1, say $1 + \epsilon$ for some small positive ϵ. If we cube $1 + \epsilon$, we get

$$1 + 10^{-11} = 1 + 3\epsilon + 3\epsilon^2 + \epsilon^3.$$

Since ϵ is very small, the terms $3\epsilon^2$ and ϵ^3 are negligibly small compared to 3ϵ. So to good accuracy we should have $10^{-11} = 3\epsilon$, and therefore $\epsilon = 3.33 \times 10^{-12}$. It follows that the change in side is about 3.33×10^{-11}.

We reached ϵ by an informal argument. But now we can verify in full detail that we have 3 significant figure accuracy. The ϵ that we computed is too small, since

$$3\epsilon^2 + \epsilon^3 < 4\epsilon^2 < 5 \times 10^{-23},$$

and $1 + 3\epsilon = 1 + 0.999 \times 10^{-11}$. And it is clear that 3.334×10^{-12} is too big.

Another way: Let $a = \sqrt[3]{1+\epsilon}$, where $\epsilon = 10^{-11}$. Then

$$a - 1 = \frac{a^3 - 1}{a^2 + a + 1} = \frac{\epsilon}{a^2 + a + 1}.$$

In particular, $a - 1 < \epsilon/3$. To get an inequality the other way, note that for example $(1 + 3.333332 \times 10^{-12})^3$ is less than $1 + 10^{-11}$.

IX-17. Let $n = 12345678987654321$. Find the first three digits after the decimal point in the decimal expansion of $\sqrt{n^2 + n} - n$.

Solution. We can't even key n into a calculator. To get some insight, ask the calculator to find $\sqrt{x^2 + x} - x$ for large—but not too large—values of x. If we try things like $x = 20000$, we are soon led to guess that the decimal expansion starts with 0.499. (But if we use $x = 10^{15}$, we will be seriously misled.)

Rationalize $\sqrt{n^2 + n} - n$ by multiplying "top" and "bottom" by $\sqrt{n^2 + n} + n$. We obtain

$$\sqrt{n^2 + n} - n = \frac{n}{\sqrt{n^2 + n} + n} = \frac{1}{\sqrt{1 + 1/n} + 1}.$$

The quantity $\sqrt{1 + 1/n}$ is imperceptibly larger than 1 when n is large, so the decimal expansion starts with 0.499.

Comment. Consider $\sqrt{y^2 + x}$, where y is positive and $|x|$ is much smaller than y^2. Rewrite $\sqrt{y^2 + x}$ as $y\sqrt{1 + x/y^2}$. Note that $\sqrt{1 + x/y^2}$ is very close to $1 + x/2y^2$, since $(1 + x/2y^2)^2 = 1 + x/y^2 + x^2/y^4$, and x^2/y^4 is negligible compared to $1 + x/y^2$. So $\sqrt{y^2 + x}$ is very close to $y + x/2y$.

Let $x = y = n$ and conclude that $\sqrt{n^2 + n}$ is close to $n + 1/2$. Even better, complete the square and rewrite $n^2 + n$ as $(n + 1/2)^2 - 1/4$. Then let $y = n + 1/2$, $x = -1/4$, and conclude that $\sqrt{n^2 + n}$ is very close to $n + 1/2 - 1/(8n + 4)$.

IX-18. Find $\log(2^{500} + 10^{150})$ correct to four decimal places.

Solution. Let's hope that one of 2^{500} or 10^{150} is negligible compared to the other—that would make things easy. Note that $\log(2^{500}) = 500 \log 2$, so $\log(2^{500})$ is about 150.5149978: our two large numbers have the same order of magnitude.

We conclude that 2^{500} is, with small relative error, approximately equal to

$$10^{0.5149978} \times 10^{150},$$

that is, about 3.27339×10^{150}. Now add 10^{150} and find the logarithm of the result. This is roughly $150+\log(4.27339)$, which, correct to four decimal places, is 150.6308.

IX-19. Find the three leftmost decimal digits of 2^{2000}.

Solution. Crudely direct computation of 2^{2000} with the x^y key doesn't work. Most calculators snicker, print -E-, and insist on being turned off.

We can get around the problem by using logarithms, say to the base 10. Let $N = 2^{2000}$. Then $\log N = 2000 \log 2$. The calculator says that $\log N$ is about 602.05999. Thus N is about $10^{602} \times 10^{0.05999}$, that is, $1.148130695 \times 10^{602}$. The three leftmost digits are 1, 1, and 4.

Another way: We can do the calculation with a mildly creative use of the x^y key, or even just the x^2 key. Note that $2^{10} = 1024 = 1.024 \times 10^3$. Now we can take quite high powers of 2^{10} without complaint from the calculator. For example, $2^{2000} = (2^{10})^{200} = 1.024^{200} \times 10^{600}$. The calculator says that 1.024^{200} is about 114.8130695. So the three leftmost digits are 1, 1, and 4.

IX-20. In 1614, John Napier published the first work on logarithms. His logarithm, which we will call $\operatorname{Nap} x$, was not quite the same as ours. It can be defined by

$$x = 10^7 \left(1 - \frac{1}{10^7}\right)^{\operatorname{Nap} x}.$$

(a) Solve the equation $\operatorname{Nap} x = 0$. (b) Solve the equation $\operatorname{Nap} x = 1$. (c) Show that if $st = uv$ then $\operatorname{Nap} s + \operatorname{Nap} t = \operatorname{Nap} u + \operatorname{Nap} v$.

Solution. (a) Put $\operatorname{Nap} x = 0$ in the defining formula for Nap; we get $x = 10^7$. (b) If we put $\operatorname{Nap} x = 1$, we get $x = 10^7(1 - 1/10^7) = 9999999$. (c) Temporarily, let $a = 1 - 1/10^7$. Then

$$st = 10^{14} a^{\operatorname{Nap} s + \operatorname{Nap} t} \qquad \text{and} \qquad uv = 10^{14} a^{\operatorname{Nap} u + \operatorname{Nap} v}.$$

Since $st = uv$, we obtain the desired result.

Comments. 1. Napier's definition was functionally equivalent to the one we gave, but quite different in appearance, and for very good reason: at the time, there was no notion of general exponentiation! Bürgi also discovered logarithms, at about the same time as Napier—before Napier, if you are Swiss, after, if you are British. Not much later, Briggs and others made the new tool more computationally useful by publishing tables of what are now called base 10 logarithms.

2. Part (c) is one version of the addition law for Napier's logarithms. We obtain another version by observing that $(xy)(1) = (x)(y)$, and therefore by part (c) $\operatorname{Nap} xy = \operatorname{Nap} x + \operatorname{Nap} y - \operatorname{Nap} 1$. To find Nap 1, we need to solve the equation $(1 - 1/10^7)^z = 1/10^7$. Ordinary logarithms will do the job, though we are close to the end of the comfort zone of the calculator. The answer is about 1.61181×10^8. This produces a rather ugly addition law—a good indication that Nap isn't quite the right logarithm.

3. The defining equation for Nap can be written

$$\frac{x}{10^7} = \left(\left(1 - \frac{1}{10^7}\right)^{10^7}\right)^{\frac{\operatorname{Nap} x}{10^7}}.$$

When n is large, $(1-1/n)^n$ is close to e^{-1}, where e is the base for natural logarithms. For $n = 10^7$, they agree to 7 decimal places. To very high accuracy, $\operatorname{Nap} x$ is about $-10^7 \ln(x/10^7)$.

4. The 10^7 that clutters things is there partly because in Napier's time decimal fractions were new to Western Europe. (They had been used for more than a thousand years in India.) Napier's tables were for $\operatorname{Nap} x$ where x is an integer.

Base 10 positional notation for integers had been used by merchants in Europe for hundreds of years, but well into the sixteenth century high precision computations, notably in astronomy, used the ancient Babylonian base 60 system!

IX-21. Find the number of digits in the decimal representation of P, where $P = 2^{6972592}\left(2^{6972593} - 1\right)$.

Solution. Any scientific calculator can handle this. Let $N = 2^{6972592+6972593}$. Then $\log_{10} N = 13945185 \log_{10} 2$, about 4197918.98. Since $P < N$, $\log_{10} P$ is less than 4197918.98. And since $10^{0.98}$ is about 9.54, and N/P is ludicrously close to 1, $\log_{10} P$ is greater than 4197918. Therefore P has 4197919 decimal digits.

Comment. The number P of the problem is at this time the largest *perfect number* known. For more information about perfect numbers, see VII-18.

IX-22. Let $N = 123456789$. Find the remainder when $(N-4)^9$ is divided by N.

Solution. We could calculate the ninth power and then divide, but that is neither quick nor pleasant nor instructive.

Instead, *imagine* multiplying out $(N-4)^9$—don't actually do it, that's messy and unnecessary. Is it obvious that the result has shape $kN - 4^9$, where k is an integer? Yes, if we use the *Binomial Theorem.* But we can convince ourselves in simpler ways. In general, for any m, $(am + b)(cm + d)$ has shape $km + bd$. So

$(N-4)(N-4)$ has shape $kN+(-4)^2$. But then $(N-4)^2(N-4)$ has shape $kN+(-4)^3$, and so on.

We have found that $(N-4)^9$ has shape $kN+(-4)^9$, which can be written as $(k-1)N+(N-2621444)$. Divide by N. The remainder is $N-262144$, that is, 123194645.

Comment. The old-fashioned phrase "has shape $kN+a$" is awkward, since k keeps changing. It has been largely replaced by the *congruence* notation. The calculation is shorter than it looks—most of the space was taken up with explanations. Once we have built up some theory the actual work is quick.

IX-23. My calculator says that $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$ is nearly equal to 1. Show that it is exactly equal to 1.

Solution. Maybe $2+\sqrt{5}$ is the cube of something nice, something of the shape $u+v\sqrt{5}$ where u and v are rational. Then $2-\sqrt{5}$ is the cube of $u-v\sqrt{5}$. If the sum is to be 1, then u must be $1/2$.

Is there a rational v that works? We need to have

$$\frac{1}{2}+v\sqrt{5}=\sqrt[3]{2+\sqrt{5}}.$$

The calculator suggests that $v=1/2$. It is now straightforward to verify by hand that indeed

$$\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)^3=2+\sqrt{5}.$$

Another way: Let $x=\sqrt[3]{2+\sqrt{5}}$, let $y=\sqrt[3]{2-\sqrt{5}}$, and let $x+y=w$. Then $w^3=x^3+y^3+3xy(x+y)$. Note that

$$x^3+y^3=4 \qquad \text{and} \qquad xy=\sqrt[3]{(2+\sqrt{5})(2-\sqrt{5})}=-1.$$

Thus $w^3+3w-4=0$. This has the obvious solution $w=1$. Since

$$w^3+3w-4=(w-1)(w^2+w+4),$$

and the equation $z^2+z+4=0$ has no real solution, w is indeed equal to 1.

Comment. The calculator's answer of 1 can be presumed correct to 9 decimal places, since the calculator does calculations internally to more places than it reveals, and there is no obvious source of large roundoff error. Thus in light of the calculation, it is reasonable to *conjecture* that the answer is *exactly* 1. In the first solution, the calculator suggested that v might be $1/2$, and this quickly led to a proof, so the solution is a (modest) example of a computer-aided proof. Recently, high precision calculations have given rise to a number of interesting mathematical conjectures, many of which have been proved correct.

IX-24. The year is 1600, and you need to find $(1.076)^{15}$, so you use a computer, that is, a human being who has mastered the art of multiplication and—sometimes—division. The computer charges a penny for each multiplication. There is an obvious way to find $(1.076)^{15}$ with 14 multiplications, but that costs 14 pennies. Can it be done more cheaply? How cheaply?

Solution. To calculate a^{15}, we can calculate in turn a^2, then a^4, then a^8, then a^{12}, a^{14}, a^{15} (the cost is six pennies). Somewhat more efficient is the sequence a^2, a^3, a^6, a^{12}, a^{15} or a^2, a^4, a^5, a^{10}, a^{15} (each costs five pennies).

IX-25. Triangle ABC has $AB = AC = 10^6 + 10^{-6}$ and $BC = 2 \times 10^6$. Find the area of $\triangle ABC$, correct to five significant digits.

Solution. Let M be the mid-point of BC. Using the Pythagorean Theorem, we compute the length of AM. Let $10^6 = u$ and $10^6 + 10^{-6} = v$. Then the length of AM is $\sqrt{v^2 - u^2}$. On most calculators, there is a problem if we try to key in v, directly or indirectly, for the calculator behaves as if $v = 10^6$, and reports that $v^2 - u^2 = 0$. But note that $v^2 - u^2 = (v + u)(v - u) = 2 + 10^{-12}$. The square root is very nearly $\sqrt{2}$. To find the area, multiply by 2×10^6 and divide by 2. The area is about 1.4142×10^6.

IX-26. The number

$$\frac{8765432}{(1234568)^3 - (1234570)(1234567)^2}.$$

lies between 2 and 3. Find the first digit after the decimal point.

Solution. If we compute in the natural way, most calculators do not give the correct answer, in fact do not give an answer between 2 and 3. The denominator is the difference between two nearly equal very large integers. Even though the calculator keeps *guard digits* beyond the ones it reveals, it can't handle the computation. A computer program like *Maple* that can calculate to "arbitrary" precision could handle the problem, but so can we.

Let $1234567 = x$. Then the denominator is $(x+1)^3 - (x+3)(x^2)$. But $(x+1)^3 = x^3 + 3x^2 + 3x + 1$, so the denominator is $3x + 1$, that is, 3703702. Divide. The answer, truncated to 3 decimal places, is 2.366.

IX-27. The number $(2 + \sqrt{7})^{11}$ can be expressed as $a + b\sqrt{7}$ where a and b are integers. Find b.

Solution. Note that

$$\text{if} \quad (p + q\sqrt{7})(r + s\sqrt{7}) = x + y\sqrt{7}, \quad \text{then} \quad (p - q\sqrt{7})(r - s\sqrt{7}) = x - y\sqrt{7}.$$

By applying this fact repeatedly we can see that $(2-\sqrt{7})^{11} = a - b\sqrt{7}$, and therefore

$$2b\sqrt{7} = (2 + \sqrt{7})^{11} - (2 - \sqrt{7})^{11}.$$

Compute. The calculator says that b is about 4111842.997, but we know that b is an integer. The calculation error is presumably not large, so $b = 4111843$.
Another way: We can simply multiply. Let $x = 2 + \sqrt{7}$. Then $x^2 = 11 + 4\sqrt{7}$. So $x^4 = 233 + 88\sqrt{7}$. Thus $x^8 = 108497 + 41008\sqrt{7}$. But $x^3 = 50 + 19\sqrt{7}$. Finally, $b = 108497 \cdot 19 + 41008 \cdot 50 = 4111843$.

Comment. There was nothing really special about the numbers chosen. The first solution exploits the symmetry between powers of $p + q\sqrt{d}$ and powers of $p - q\sqrt{d}$. That symmetry leads to interesting numerical facts. For instance, one probably would not immediately suspect that $(5+\sqrt{21})^{2001} + (5-\sqrt{21})^{2001}$ is an even integer, and that as a consequence $(5+\sqrt{21})^{2001}$ has a decimal expansion with an enormous number of zeros after the decimal point.

IX-28. Let $a = 57$ and $m = 2718$. Find efficiently the remainder when a^{520} is divided by m. Use the fact that if u and v are the remainders when x and y are divided by m, then the remainder when xy is divided by m is the same as the remainder when uv is divided by m.

Solution. First we find the remainder when a^2 is divided by m. This is easy: $a^2 = 3249$, so the remainder is 531. Now find the remainder when a^4, that is, $(a^2)(a^2)$, is divided by m. This is the same as the remainder when 531^2 is divided by m, and this is 2007. Now find the remainder when a^8, that is, $(a^4)(a^4)$, is divided by m. The result is 2691.

Continue. The remainder when a^8 is divided by m is 729. For a^{16}, it is 1431. For a^{32} to a^{512}, the remainders are 1107, 2349, 261, 171, and 2061. Finally, to find the remainder when a^{520} is divided by m, find the remainder when (2061)(729) is divided by m. The result is 2133.

Comment. The calculation took 10 steps, where a step consists of multiplying two numbers and finding the remainder when the result is divided by m. We could also have reached a^{520} in 519 steps, by repeatedly multiplying by a and calculating the remainder each time. Already for 520 the method we used is significantly more efficient than the naive method of repeatedly multiplying by a.

There are real applications, such as *public key cryptography,* in which if exponentiation were done in the naive way, encrypting/decrypting a simple message with a fast computer would take longer than the age of the universe, while if done efficiently it takes a small fraction of a second.

IX-29. Let $N = 98712888881234566666$. Find the remainder when N is divided by 31416.

Solution. We do an ordinary pencil and paper division, but use the calculator as much as possible. One difference is that instead of "bringing down" digits one at a time, to save time we bring them down five at time.

First find the remainder when 9871288888 is divided by 31416. How? Use the calculator to divide as usual. We get 314212.1495. Immediately subtract 314212, *using the calculator.* The displayed result is 0.1494777. The mysterious digits at the end are *guard digits*: internally, the calculator works to higher accuracy than it displays. We have just tricked the calculator into revealing its guard digits. Now multiply by 31416. We get 4695.969432. There has been some roundoff error, but we can conclude with reasonable confidence that the remainder at this stage is 4696.

Now "bring down" 12345, getting 469612345. Find the remainder on division by 31416, just as in the paragraph above. We get 5977. Bring down 66666, find the remainder. We get 14434.

Comments. 1. If we put in a few additional ideas, we get an efficient algorithm for calculating the quotient and remainder when N is divided by M, where N and M are allowed to be very large. For an excellent discussion, see Volume 2 of Knuth's *The Art of Computer Programming.*

2. The fact that calculators keep guard digits is one more reason to arrange calculations so that we seldom or never write down an intermediate answer and key it back in later. To rekey takes time, introduces the possibility of keying errors, and loses accuracy, since the guard digits are in effect thrown away. It is important to learn to use the memory feature of modern scientific calculators.

IX-30. A temperature was taken with a thermometer that reads in degrees Fahrenheit, and recorded to the nearest degree. The recorded number was later converted to degrees Celsius, using the formula $T_C = (5/9)(T_F - 32)$, where T_F is the Fahrenheit temperature, and T_C the Celsius temperature. The result was then rounded to the nearest degree Celsius. How large can the absolute difference be, in degrees Celsius, between the true temperature and the temperature computed through this process?

Solution. When the temperature was recorded, the rounding error was at most $1/2$ of a degree Fahrenheit, that is, $5/18$ of a degree Celsius. And in the rounding after the conversion, an error of no more than $1/2$ degree Celsius is introduced, so the final error is no more than $5/18 + 1/2$, that is, $7/9$.

That doesn't answer the question. The error can't be greater than $7/9$, but we have not shown that an error of $7/9$ can happen, and in fact it cannot.

For the error to be $7/9$, the true Fahrenheit temperature must be $x + 1/2$, where x is an integer. The recorded temperature is then $x + 1$, which is converted to $(5/9)(x + 1 - 32)$. To produce an an error of $1/2$ in the second rounding, we need $(5/9)(x - 31) = y + 1/2$, where y is an integer. This equation simplifies to $10x - 18y = 319$, which has no integer solution, for the left-hand side is always even but the right-hand side is odd.

Start again! Let x be the *recorded* temperature in degrees Fahrenheit, and let y be the nearest integer to $(5/9)(x - 32)$. Then $(5/9)(x - 32) = y + e$, where e is the rounding error, which may be negative.

We want $|e|$ to be as large as possible. The defining equation for e simplifies to $5x - 9y = 160 + 9e$. Thus $9e$ must be an integer. Since $|e| \leq 1/2$, it follows that

$|e|$ can't be greater than $4/9$. We now work backwards and construct an example where the second rounding error has absolute value equal to $4/9$.

Take $e = -4/9$. First we solve $5x - 9y = 156$ in integers: $x = 42$, $y = 6$ works. Let the true temperature in degrees Fahrenheit be 41.5; this is rounded up to 42. Conversion gives $5\frac{5}{9}$, which is rounded up to 6. The combined error of $13/18$ degrees Celsius is the largest possible error.

Comment. There is a subtle point in the construction of an example with maximum error. How do we round numbers whose decimal expansion ends in a 5, like 41.5? There are two common conventions to deal with this situation: round up, or round so that the last digit will be even. The example works with either of these conventions. If we try to make $e = 4/9$ rather than $-4/9$, and use the "round up" convention, then an error of absolute value exactly $13/18$ is not possible.

In a blood cell study that the author was once asked to interpret, the data had been subjected to a standard transformation. The results were rounded, and the original data were then discarded! It turned out that the rounding had in effect made the study useless.

Chapter 10

Maxima and Minima

Introduction

These problems require finding the maximum or minimum value of a function. About one-third are "word problems" in which the function isn't given explicitly.

In most cases, the problem can be solved by using the differential calculus. But even though this is usually not stated explicitly, the problems are meant to be done without calculus. Many senior secondary students haven't been introduced to the calculus. Even when they have, they often use it at a purely algorithmic level. Why use a mysterious piece of machinery when we can retain full control and *see* exactly what's going on?

A surprising number of the problems hinge on the simple fact that the maximum value of $\sin\theta$ is 1. Many more come down to observing that any square is non-negative. "Completing the square" is a useful tool in many areas of mathematics. It comes up in some form in half the problems, for instance X-1, X-10, and X-18. Even more important is interpreting the problem geometrically and visualizing correctly, as in, for example, X-11 and X-23. Many of the solutions take advantage of symmetries. Some of the problems, particularly from X-27 on, are quite difficult.

Problems and Solutions

X-1. Let $f(x) = 4 - (2x^2 + 2x - 1)^3$. Find the maximum value of $f(x)$ as x ranges over the reals.

Solution. Complete the square. For any x,

$$2x^2 + 2x - 1 = 2\left(x^2 + x - \frac{1}{2}\right) = 2\left(\left(x + \frac{1}{2}\right)^2 - \frac{3}{4}\right).$$

Since a square can't be negative, $2x^2+2x-1$ is never less than $-3/2$, and is equal to $-3/2$ when $x = -1/2$. The maximum value of $f(x)$ is therefore $4 - (-3/2)^3$, that is, $59/8$.

Comment. We started the completing the square process by dividing the quadratic by 2. T o avoid fractions for a while, *multiply* by 2 and observe that $4x^2+4x-2 = (2x-1)^2 - 3$.

Most textbook proofs of the quadratic formula "simplify" $ax^2 + bx + c = 0$ by dividing through by a, creating an immediate mess. It is better to *multiply* by $4a$ and then complete the square. We get

$$4a^2x^2 + 4abx + 4ac = 0 \qquad \text{if and only if} \qquad (2ax + b)^2 = b^2 - 4ac.$$

So $2ax = -b \pm \sqrt{b^2 - 4ac}$, and there is stuff in the denominator only at the end.

X-2. In Figure 10.1, $\triangle ABC$ is right-angled at C, and M is the midpoint of CB. Given that $\triangle ABC$ has area 1, how short can AM be?

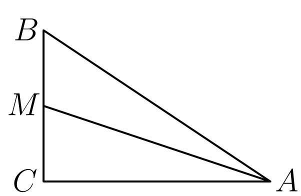

Figure 10.1: The Smallest Median

Solution. Let x be the length of CA and let $CB = 2y$. Since $\triangle ABC$ has area 1, we have $xy = 1$. By the Pythagorean Theorem, $(AM)^2 = x^2 + y^2$. We need to minimize $x^2 + y^2$ given that $xy = 1$. Note that

$$x^2 + y^2 = (x - y)^2 + 2xy = (x - y)^2 + 2.$$

The quantity on the right has minimum value 2, obtained by putting $x = y$. The shortest possible median AM has length $\sqrt{2}$.

Another way: We need to minimize $\sqrt{x^2 + y^2}$ given that x and y are positive and $xy = 1$. But $\sqrt{x^2 + y^2}$ is the distance from the point (x, y) to the origin. As (x, y) roams over the first-quadrant part of the hyperbola $xy = 1$, it comes closest to the origin when it reaches $(1, 1)$.

Comments. 1. If we substitute $1/x$ for y, the function to be minimized becomes

$$x^2 + \frac{1}{x^2},$$

and we can use the same argument, since $x^2 + 1/x^2 = (x - 1/x)^2 + 2$. Eliminating y is the standard first-year calculus approach. Keeping both x and y emphasizes the symmetry.

2. The height of $\triangle ABC$ was called $2y$ rather than y in order to make the expression for AM look (a bit) nicer. There is little harm in calling it y, but making things look nice *is* important.

3. The triangle AMB plays no role in the argument. Throw it away and complete $\triangle ACM$ to a rectangle. This rectangle has area 1. So the triangle problem is the same as the problem of finding the smallest diagonal among all rectangles of area 1 (it is $\sqrt{2}$, and the rectangle is a square). A closely related problem is finding the rectangle of *largest* area among all rectangles with given diagonal.

X-3. Use two thin sticks of length a and two of length b to make a kite-shaped figure that encloses the largest possible area and find that area. Please see Figure 10.2.

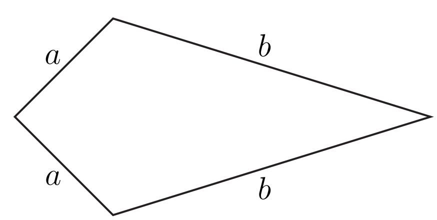

Figure 10.2: The Maximum Area of a Kite

Solution. Think of the kite as made up of two congruent triangles each of which has a side of length a and one of length b. We determine the angle between "a" and "b" that maximizes the areas of these triangles.

View each triangle as having base b. Then the larger the height, the larger the area of the triangle. The greatest height is reached when "a" is perpendicular to "b." Each triangle then has area $ab/2$, so the largest possible area is ab.

Comment. If two sides of a triangle have length a and b, and the angle between these sides is θ, then the triangle has area $(1/2)ab\sin\theta$. If we let θ vary, the area reaches a maximum when $\sin\theta = 1$. But it is better to see a hinge between "a" and "b," and "a" trying to reach as high as possible.

Any kite can be made by slicing a parallelogram with sides a and b along a diagonal, turning one of the halves upside down, and glueing along the diagonal. Among all parallelograms with sides a and b, the rectangle has the largest area.

X-4. Find the smallest possible value of

$$|x - 1^2| + |x - 2^2| + |x - 3^2| + \cdots + |x - 100^2|.$$

Solution. We look at a more general problem. Let $a_1 < a_2 < \cdots < a_n$; we want to minimize $|x - a_1| + |x - a_2| + \cdots + |x - a_n|$. Imagine travelling to the right on the x-axis, from a place x well to the left of a_1 to a place x to the right of a_n.

Since $|x - a_i|$ is the distance between x and a_i, $|x - a_1| + |x - a_2| + \cdots + |x - a_n|$ initially decreases as we travel rightward. Suppose that we have reached a point in the interval between a_k and a_{k+1} and take a tiny step of length ϵ rightward, still staying in the interval (a_k, a_{k+1}).

Our distance from each of a_1, a_2, $\dots$, a_k increases by ϵ and our distance from each of a_{k+1}, a_{k+2}, $\dots$, a_n decreases by ϵ. Thus the sum of the distances to *all* the a_i decreases as long as $k < n/2$, increases if $k > n/2$, and doesn't change if $k = n/2$.

In our problem, $n = 100$ and the smallest sum s is reached at any point b between 50^2 and 51^2, so

$$s = \left((b^2 - 1^2) + (b^2 - 2^2) + \cdots + (b^2 - 50^2)\right) + \left((51^2 - b^2) + (52^2 - b^2) + \cdots + (100^2 - b^2)\right).$$

Add up: all the b^2 terms cancel. By rearranging, we find that

$$s = (51^2 - 50^2) + (52^2 - 49^2) + \cdots + (100^2 - 1^2).$$

Factor each difference of squares. We conclude that $s = 101(1 + 3 + 5 + \cdots + 99)$. The sum of the arithmetic progression $1 + 3 + \cdots + 99$ is 2500, so $s = 252500$.

X-5. A teacher wants to put on a test an equation of the shape

$$(x - 1)(100 - x) = M,$$

and regrettably wants the roots to be integers. Find the biggest M that works.

Solution. Let $f(n) = (n - 1)(100 - n)$. We look for the biggest value of $f(n)$ as n ranges over the integers. For any n,

$$f(n + 1) - f(n) = (n)(99 - n) - (n - 1)(100 - n) = 100 - 2n.$$

So if $n < 50$ then $f(n+1) > f(n)$, if $n = 50$ then $f(n+1) = f(n)$, while if $n > 50$ then $f(n+1) < f(n)$. As n is incremented by 1's, $f(n)$ grows, briefly stays steady, and then falls. It is therefore biggest when $n = 50$ and when $n = 51$. Thus the biggest M that works is $49 \cdot 50$.

Another way: Complete the square. For any n,

$$f(n) = -n^2 + 101n - 100 = -\left(n - \frac{101}{2}\right)^2 + \left(\frac{101}{2}\right)^2 - 100.$$

Thus $f(n)$ is biggest when $|n - 101/2|$ is least. Since n is restricted to integers, the smallest possible value of $|n - 101/2|$ is $1/2$. It is reached when $n = 50$ and also when $n = 51$. Thus the biggest possible M is $f(50)$, that is, 2450.

Comments. 1. We prefer the first solution: it seems more direct, less mechanical. Users of calculus should note that the difference $f(n+1) - f(n)$, which we can rewrite as

$$\frac{f(n+1) - f(n)}{1}$$

bears a formal resemblance to the derivative. (It is used as a mathphobe's version of the derivative in some economics courses.) We examined where $f(n+1) - f(n)$ is positive and where it is negative. This is analogous to the standard calculus procedure of asking where the derivative is positive or negative.

2. In the second solution, the process of completing the square is a bit messy—too many fractions, too many minus signs. Note that

$$-4f(n) = 4n^2 - 404n + 400 = (2n - 101)^2 - 101^2 + 400.$$

So if instead of maximizing $f(n)$ we equivalently minimize $-4f(n)$, expressions look nicer and errors are less likely.

X-6. A circle $\mathcal{C}$ has center O, and P is a point in the interior of $\mathcal{C}$ other than O. Find all points X on $\mathcal{C}$ such that $\angle OXP$ is as big as possible. Please see Figure 10.3.

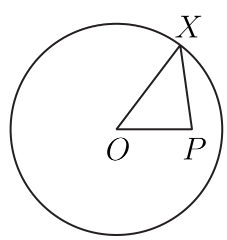

Figure 10.3: Maximizing $\angle OXP$

Solution. Let $\theta = \angle OXP$ and $\phi = \angle OPX$. Let the radius of $\mathcal{C}$ be r, and let a be the distance from O to P. By the Sine Law,

$$\frac{\sin\theta}{a} = \frac{\sin\phi}{r}.$$

Since θ must be acute, making θ big is equivalent to making $\sin\theta$ big. Since $\sin\theta = (a/r)\sin\phi$, we want to maximize $\sin\phi$.

A glance at Figure 10.3 shows that we can make ϕ a right angle, and therefore the biggest possible value of $\sin\phi$ is 1. Thus there are two points X that maximize

θ. They are the points where the chord that passes through P and is perpendicular to OP meets the circle.

X-7. Let $f(x) = x + \sqrt{1-x^2}$. Find the smallest and largest values reached by $f(x)$ as x ranges over the interval $-1 \le x \le 1$.

Solution. Note that $\sqrt{1-x^2}$ is never negative, so $f(x)$ can't be less than -1 on our interval. Since $f(x) = -1$ when $x = -1$, the smallest value reached by $f(x)$ is -1.

Now look for the largest value of $f(x)$. This largest value can't be reached at a negative x, so confine attention to $0 \le x \le 1$. Since $f(x)$ is positive on this interval, $f(x)$ is largest where $f^2(x)$ is largest. But

$$f^2(x) = (x + \sqrt{1-x^2})^2 = 1 + 2x\sqrt{1-x^2},$$

so it is enough to maximize $2x\sqrt{1-x^2}$, or equivalently to maximize $4x^2(1-x^2)$.

Let $u = x^2$. We want to maximize $4u - 4u^2$ on the interval $0 \le u \le 1$. Recall that $y = 4u - 4u^2$ is a downward-facing parabola with vertex at $u = 1/2$, or complete the square:

$$4u - 4u^2 = 1 - (2u-1)^2.$$

The right-hand side has maximum value 1 when $2u - 1 = 0$. Now we retrace our steps and conclude that the maximum value of $f^2(x)$ is 2, so the maximum value of $f(x)$ is $\sqrt{2}$.

Another way: For the maximum, we can confine attention to the interval $0 \le x \le 1$. Then $x = \cos\theta$ for some θ between 0 and $\pi/2$, and $\sqrt{1-x^2} = \sin\theta$. We want to maximize $\cos\theta + \sin\theta$. But

$$(\cos\theta + \sin\theta)^2 = \cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta = 1 + \sin 2\theta.$$

The largest value of $1 + \sin 2\theta$ is 2, reached when $\theta = \pi/4$, so the largest value of $f(x)$ is $\sqrt{2}$.

X-8. Find the largest value and the smallest value taken on by the function $\sqrt{16+x^4} - x^2$.

Solution. Rationalize the numerator:

$$\sqrt{16+x^4} - x^2 = \frac{(\sqrt{16+x^4} - x^2)(\sqrt{16+x^4} + x^2)}{\sqrt{16+x^4} + x^2} = \frac{16}{\sqrt{16+x^4} + x^2}.$$

As $|x|$ increases, $\sqrt{16+x^4} + x^2$ increases. So $\sqrt{16+x^4} + x^2$ is smallest when $x = 0$, and therefore $\sqrt{16+x^4} - x^2$ reaches its maximum value of 4 when $x = 0$.

As $|x|$ increases without bound, $\sqrt{16+x^4} + x^2$ also increases without bound, and therefore $\sqrt{16+x^4} - x^2$ decreases steadily toward 0. But it never reaches 0, and therefore there is no such thing as the smallest value *taken on* by the function.

X-9. Let $f(x) = x^2/(x^2 + x + 1)$. What values are taken on by $f(x)$?

Solution. If we use a graphing program to plot $y = x^2/(x^2 + x + 1)$, we can with some confidence read off the answer. But $f(x)$ is not too complicated, so it's worth exploring algebraic approaches.

If $x \neq 0$, we can divide "top" and "bottom" by x^2. Let $u = 1/x$. Then

$$f(x) = \frac{1}{1 + \frac{1}{x} + \frac{1}{x^2}} = \frac{1}{1 + u + u^2}.$$

Completing the square gives

$$1 + u + u^2 = \left(u - \frac{1}{2}\right)^2 + \frac{3}{4},$$

so the smallest value of $1 + u + u^2$ is $3/4$, reached when $u = 1/2$.

The only forbidden u is 0. As u ranges over the non-zero reals, $1+u+u^2$ takes on all values greater than or equal to $3/4$. (The value of $1+u+u^2$ at the forbidden $u = 0$ is also taken on at $u = -1$.)

We conclude that as x ranges over the non-zero reals, $f(x)$ ranges over the interval $0 < x \leq 4/3$. Since $f(0) = 0$, the function $f(x)$ ranges over the interval $0 \leq y \leq 4/3$ as x ranges over the reals.

Another way: Since $x^2 + x + 1$ is never 0, the equation $y = x^2/(x^2 + x + 1)$ is equivalent to $x^2 = y(x^2 + x + 1)$, that is, to $(y - 1)x^2 + yx + y = 0$.

Suppose that $y \neq 1$. By the quadratic formula,

$$x = \frac{-y \pm \sqrt{y^2 - 4(y-1)y}}{2(y-1)}.$$

The discriminant $4y - 3y^2$ is non-negative if and only if $0 \leq y \leq 4/3$. We conclude that if y is any number in this interval other than 1, then there is at least one x such that $y = x^2/(x^2 + x + 1)$. And $y = 1$ isn't exceptional, since $y = 1$ when $x = -1$.

X-10. The numbers x and y range over the reals, subject to the constraint $x^2 + 4y^2 - 8x - 16y - 4 = 0$. Find the smallest possible value of x.

Solution. Complete the squares:

$$x^2 - 8x = (x-4)^2 - 4^2 \qquad \text{and} \qquad 4y^2 - 8y = 4\left((y-2)^2 - 2^2\right).$$

Thus the constraint is equivalent to $(x - 4)^2 + 4(y - 2)^2 = 36$, and therefore (x, y) moves over a standard ellipse shifted 4 units to the right and 2 units up. The leftmost point on the standard ellipse $x^2 + 4y^2 = 36$ has $x = -6$, so the leftmost point on the shifted ellipse has $x = -2$.

Another way: If we are in a non-geometric mood—and we should hardly ever be—we can note that x reaches a mimimum precisely where $x - 4$ does. And from the equation $(x-4)^2 = 36 - 4(y-2)^2$ we can see that $(x-4)^2$ can't be larger than 36, and is equal to 36 only when $y = 2$. The smallest x is less than 4 and satisfies the equation $(x-4)^2 = 36$, so it is -2.

X-11. Find the maximum value taken on by $3x + 4y$ as (x, y) ranges over all ordered pairs such that $x^2 + y^2 \leq 1$.

Solution. We will be close to the answer as soon as we have a clear grasp of the geometry. The points (x, y) with $x^2 + y^2 \leq 1$ fill up the disk with center the origin O and radius 1.

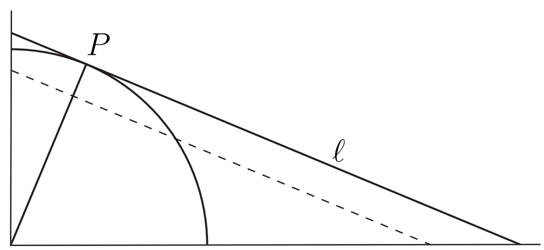

Figure 10.4: Maximizing $3x + 4y$ on the Unit Disk

For any fixed k, $3x + 4y = k$ is the equation of a line with slope $-3/4$. As k increases, the line moves upwards and to the right (see Figure 10.4). Thus the largest k such that the line $3x + 4y = k$ meets the disk is the k for which the line is tangent to $x^2 + y^2 = 1$ (and "above" it).

Let ℓ be this tangent line and P the point of tangency. Then the line OP is perpendicular to ℓ. Recall that if line ℓ has slope $m \neq 0$, then the lines perpendicular to ℓ have slope $-1/m$. The slope of ℓ is $-3/4$, so OP has slope $4/3$.

If the x-coordinate of P is a then the y-coordinate is $4a/3$. But P is on the circle, so $a^2 + 16a^2/9 = 1$ and therefore $a = 3/5$, and $P = (3/5, 4/5)$. But $3(3/5) + 4(4/5) = 5$, so the maximum of $3x + 4y$ as (x, y) ranges over the disk is 5.

Another way: Give $3x + 4y$ some convenient positive value, say 1. We *minimize* $x^2 + y^2$, that is,

$$x^2 + \frac{(1-3x)^2}{16}.$$

Simplify and complete the square. The minimum value of

$$\frac{\left(5x - \frac{3}{5}\right)^2 + \frac{16}{25}}{16} \qquad \text{is} \qquad \frac{1}{25}.$$

So the circle tangent to the line $3x + 4y = 1$ has radius $1/5$. Now scale up by a factor of 5: the positive k for which $3x + 4y = k$ is tangent to $x^2 + y^2 = 1$ is 5.

Another way: The general point on the unit circle has coordinates

$$x = \cos\theta, \qquad y = \sin\theta$$

for some θ between 0 and 2π. It follows that

$$3x + 4y = 5\left(\frac{3}{5}\cos\theta + \frac{4}{5}\sin\theta\right).$$

Let ϕ be the angle between 0 and $\pi/2$ whose sine is 3/5. Then $\cos\phi = 4/5$, and therefore

$$3x + 4y = 5(\sin\phi\cos\theta + \cos\phi\sin\theta) = 5\sin(\theta + \phi).$$

Choose θ so that $\theta+\phi = \pi/2$. Then $5\sin(\theta+\phi) = 5$, and no larger value is possible.

Another way: We present an algebraic solution that may look like magic. It uses an important identity that can be checked by expanding both sides:

$$(a^2 + b^2)(x^2 + y^2) = (ax \pm by)^2 + (ay \mp bx)^2.$$

Let $a = 3$ and $b = 4$. We want to make $ax + by$ as large as possible subject to the condition $x^2 + y^2 \leq 1$. From the above identity, we can see that to make $(ax + by)^2$ big, we should scale so as to make $x^2 + y^2 = 1$—that makes the left-hand side as big as possible—and then choose $(ay - bx)^2$ as small as possible. There are x and y such that $x^2 + y^2 = 1$ and $ax - by = 0$, so the maximum value of $(ax + by)^2$ is $a^2 + b^2$.

Comments. 1. The movement as k changes should be experienced kinesthetically. With a picture of the disk and a moving long ruler, we can see and feel the maximum k being reached just as we leave the disk behind, that is, at the point of tangency.

2. In the second solution, the quantity to be maximized and the constraint were in a sense interchanged, and the problem was turned into a minimization problem. This widely useful technique is sometimes called *dualizing* the problem.

3. The identity of the fourth solution is often called *Bachet's Identity,* even though Diophantus used the result more than 1350 years before Bachet. The identity is now sometimes called *Brahmagupta's Identity,* after the seventh-century mathematician who worked with the more general identity

$$(a^2 - Db^2)(x^2 - Dy^2) = (ax \pm Dby)^2 - D(ay \mp bx)^2.$$

Bachet's Identity can be used to express integers as sums of squares. Let c be a sum of two perfect squares, say $c = a^2 + b^2$, and let $z = x^2 + y^2$. The identity says that cz is a sum of two squares. For example, $401 = 1^2 + 20^2$ and $97 = 4^2 + 9^2$. Using Bachet's Identity, we can express $401 \cdot 97$ as the sum of two squares in two different ways.

4. Here is a seemingly more complicated but closely related problem: maximize $3x + 4y$ given that $x^2 + y^2 \leq 2x + 4y + 6$. By completing the squares we can rewrite the condition as $(x - 1)^2 + (y - 2)^2 \leq 1$. Let $u = x - 1$ and $v = y - 2$. Then $3x + 4y = 3u + 4v + 11$. The maximum value of $3u + 4v$ subject to the condition $u^2 + v^2 \leq 1$ is 5, so the required maximum value is 16.

X-12. Fourteen parallel blue lines are drawn across a standard 8.5×11 sheet of paper, dividing the sheet into 15 horizontal strips of equal height. Then 22 parallel red lines are drawn, dividing the sheet into 23 horizontal strips of equal height. Find the minimum distance between a blue line and a red line.

Solution. The problem is not really two-dimensional. Imagine an 11-inch line segment, with 14 blue dots dividing the segment into 15 intervals of equal length, and 22 red dots .... And the line segment might as well be $[0, 1]$: we can scale up at the end.

Then the blue dots are at $s/15$, where s ranges over the integers from 1 to 14, while the red dots are at $t/23$, where t ranges over the integers from 1 to 22. To find the minimum distance between a blue dot and a red dot, minimize

$$\left|\frac{s}{15} - \frac{t}{23}\right|, \qquad \text{that is,} \qquad \frac{|23s - 15t|}{15 \cdot 23}.$$

There are general methods for solving this kind of problem, but here the numbers are small and we can see that $|23s - 15t| = 1$ when $s = 2$ and $t = 3$. Since $|23s - 15t|$ can't be less than 1, the minimum distance between a blue line and a red line is $11/(15 \cdot 23)$ inches.

X-13. Find the smallest number m such that $x/(x^2 + 3) \leq m$ for all x.

Solution. We want the smallest m such that $y = m$ acts as a "ceiling" on the graph of $y = x/(x^2 + 3)$. This m is evidently positive, and the line $y = m$ is *tangent* to $y = x/(x^2 + 3)$. So the equation $m = x/(x^2 + 3)$ should have only one root, or more precisely two coinciding roots. Rewrite the equation as

$$x^2 - \frac{x}{m} + 3 = 0.$$

The product of the roots is 3 and their sum is $1/m$. So if the roots coincide then they are each $\sqrt{3}$, and $m = 1/(2\sqrt{3})$.
Another way: Let $x = \sqrt{3}\tan\theta$. (Any real x can be represented this way.) Using the identity $\tan^2\theta + 1 = \sec^2\theta$, we get

$$\frac{x}{x^2 + 3} = \frac{\sqrt{3}\tan\theta}{3\sec^2\theta} = \frac{\sin\theta\cos\theta}{\sqrt{3}} = \frac{\sin 2\theta}{2\sqrt{3}}.$$

The maximum value attained by $\sin 2\theta$ is 1, so $m = /(2\sqrt{3})$.

Comment. Look instead at $(x^2 + 3)/x$ and find its *minimum* as x ranges over the positive reals. For a solution of this version of the problem, see X-19.

X-14. Find the smallest value of $xy + yz + zx$ given that $x^2 + y^2 + z^2 = 1$.

Solution. Any square is non-negative, so $(x+y+z)^2 \geq 0$. Expand the square:

$$(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx.$$

Since $x^2+y^2+z^2=1$, we conclude that $1+2xy+2yz+2zx \geq 0$, and therefore $xy+yz+zx$ can never be less than $-1/2$.

To show that $xy+yz+zx$ can actually be $-1/2$, look back on the calculation and note that there is equality when $x+y+z=0$. So we need to show that there are numbers x, y, and z such that

$$x^2+y^2+z^2=1 \qquad \text{and} \qquad x+y+z=0.$$

This is evident from the geometry. The first equation is the equation of a sphere with center the origin and radius 1, while the second is the equation of a plane through the origin. The sphere and the plane meet at many places.

Alternately, we can find explicit x, y, and z that satisfy both equations. Let a, b, and c be three numbers, not all 0 but with sum 0, like 1, -1, and 0. Divide each by $\sqrt{a^2+b^2+c^2}$ and call the results x, y, and z.

X-15. Let a, b, and c be positive. Find the smallest value of $ax+by$ as (x, y) travels over the first-quadrant part of the curve $xy=c$.

Solution. Equivalently, minimize $(ax+by)^2$ under the constraint. Since

$$(ax+by)^2 = (ax-by)^2+4abxy = (ax-by)^2+4abc,$$

minimizing $(ax+by)^2$ is equivalent to minimizing $(ax-by)^2$.

The line $ax=by$ passes through the origin and has positive slope, so it meets the hyperbola $xy=c$. Thus $(ax-by)^2$ can be 0 on the hyperbola. The minimum value of $ax+by$ is therefore $2\sqrt{abc}$.

Another way: Let $u=ax$ and $v=by$. We want to minimize $u+v$ as (u, v) travels over the first-quadrant part of the hyperbola $uv=abc$. If we now use the preceding argument, the algebra is marginally simpler-looking.

The real reason for the change of variable is to enlist the aid of symmetry. Sketch the first-quadrant part of the hyperbola $uv=abc$, and draw the line $u+v=k$ for various k. The smallest k such that $u+v=k$ meets the hyperbola is the k for which the line is tangent to the hyperbola. By symmetry, tangency happens when $u=v=k/2$. But then $(k/2)^2=abc$, so the minimum is $2\sqrt{abc}$.

Comment. We could note that $y=c/x$ and minimize $ax+bc/x$. Since

$$ax+\frac{bc}{x} = \left(\sqrt{a}\sqrt{x}-\frac{\sqrt{bc}}{\sqrt{x}}\right)^2+2\sqrt{abc},$$

the minimum value is $2\sqrt{abc}$. This is a variant of the first solution, but it looks uglier and is therefore less useful.

X-16. Let $f(x) = (2x^2 - 8x + 125)/(2x^2 - 8x + 16)$. Find the biggest and smallest *integer* values taken on by $f(x)$ as x ranges over the reals.

Solution. Do the division:

$$f(x) = \frac{2x^2 - 8x + 125}{2x^2 - 8x + 16} = 1 + \frac{99}{2x^2 - 8x + 16}.$$

To make $99/(2x^2 - 8x + 16)$ big, we need to make y small, where

$$y = 2x^2 - 8x + 16 = 2\left((x-2)^2 + 4\right).$$

The minimum value of y is 8, attained when $x = 2$. But $99/8$ is not an integer.

As x moves away from 2, $99/y$ decreases away from 12.375. The biggest *integer* value of $99/y$ is 12, so the biggest integer value of $f(x)$ is 13.

Since $2x^2 - 8x + 16$ has minimum value 8 and increases without bound, there is an a such that $2a^2 - 8a + 16 = 99$, so $f(a) = 2$. Since $f(x)$ can't be an integer when $2x^2 - 8x + 16 > 99$, the smallest integer value taken on by $f(x)$ is 2.

Another way: Use a graphing program or a graphing calculator to plot the given function. The biggest and smallest *integer* values can be read exactly from the display. This strategy works for a wide variety of functions.

X-17. Let M be the largest value of $\sqrt{x^2 + y^2}$, and m the smallest value, given that $x^2 - 4x + y^2 - 2y + 4 = 0$. Find $M - m$.

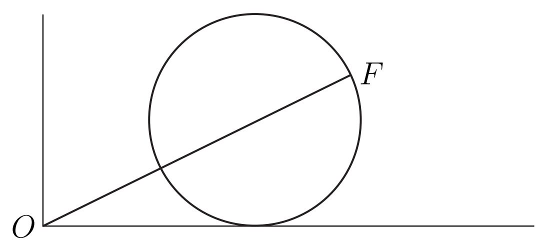

Figure 10.5: Points Far and Near

Solution. Interpret the problem geometrically. First complete the squares:

$$x^2 - 4x + y^2 - 2y + 4 = 0 \qquad \text{if and only if} \qquad (x-2)^2 + (y-1)^2 = 1.$$

So (x, y) travels around the circle of Figure 10.5. We want to maximize and minimize $\sqrt{x^2 + y^2}$, the distance of (x, y) from the origin O.

Let F be the point furthest from O. Line OF is perpendicular to the circle and therefore passes through the center of the circle, and also through the point on the circle nearest to O. It follows that $M - m$ is the diameter, namely 2.

Comment. It's not hard to find the coordinates of the near and far points. The line that joins the origin to the center of the circle has equation $y = x/2$. Substitute $2y$ for x in the equation of the circle and solve. But by using the geometry we didn't need to compute at all.

X-18. A curve has equation $x^4 + y^4 = x^2 + y^2$. Find the highest point on the curve.

Solution. Complete the squares and rewrite the equation as

$$\left(x^2 - \frac{1}{2}\right)^2 + \left(y^2 - \frac{1}{2}\right)^2 = \frac{1}{2}.$$

There are y greater than $1/\sqrt{2}$ that satisfy the equation. So to make y as big as possible, we make $y > 1/\sqrt{2}$ and $y^2 - 1/2$ as big as possible. To do that, set $x^2 - 1/2 = 0$. The biggest value of $y^2 - 1/2$ is thus $1/\sqrt{2}$, so the maximum value of y is $\sqrt{1/2 + \sqrt{1/2}}$.

X-19. (a) Show that if u and v are non-negative, then

$$\frac{u+v}{2} \geq \sqrt{uv},$$

with equality only when $u = v$.
(b) Let c be a positive constant. Use the inequality of part (a) to find the smallest value taken on by $x + c/x$ as x ranges over the positive reals.

Solution. (a) The desired result can be rewritten as

$$u - 2\sqrt{u}\sqrt{v} + v \geq 0, \qquad \text{or equivalently} \qquad (\sqrt{u} - \sqrt{v})^2 \geq 0.$$

But this last inequality is clear, for any square is non-negative. And equality can only happen when $\sqrt{u} - \sqrt{v} = 0$, that is, when $u = v$.
(b) Let $u = x$ and $v = c/x$. By the inequality of part (a), if x is positive then $\bigl(x + c/x\bigr)/2 \geq \sqrt{c}$, with equality only when $x = c/x$. So $x + c/x \geq 2\sqrt{c}$ for all positive x, with equality only when $x = \sqrt{c}$.

Comment. The inequality of part (a) is a special case of a useful result called the *Arithmetic Mean–Geometric Mean Inequality.* It asserts that if $u_1, u_2, \ldots, u_n$ are non-negative then

$$\frac{u_1 + u_2 + \cdots + u_n}{n} \geq (u_1 u_2 \cdots u_n)^{1/n},$$

with equality only when all the u_i are equal. Many of the "max/min" problems in first-year calculus books can be solved by using the above inequality instead of the derivative.

X-20. Find the points on the curve $x^4 + y^4 = 2x^2 + 2y^2$ which are furthest from the origin.

Solution. Note the symmetry between x and y: (a, b) is on the curve if and only if (b, a) is. So the curve is symmetrical about the line $y = x$. The curve is also symmetrical about $x + y = 0$ and about the coordinate axes.

We look for (x, y) on the curve such that $\sqrt{x^2 + y^2}$, or equivalently $x^2 + y^2$, is as large as possible. Let r be the largest radius such that the circle $x^2 + y^2 = r^2$ meets our curve. We want to find the points *where* it meets our curve.

It is almost clear that this happens when $y = \pm x$. When we substitute $\pm x$ for y in the equation of the curve, we get $x = \pm\sqrt{2}$, so there are four points furthest from the origin, $(\sqrt{2}, \pm\sqrt{2})$ and $(-\sqrt{2}, \pm\sqrt{2})$.

It is reasonable, particularly when there is no picture, to wonder whether the maximum really does occur when $y = \pm x$, so let's do the details. Complete the squares:

$$\left(x^2 - 1\right)^2 + \left(y^2 - 1\right)^2 = 2.$$

Let $u = x^2 - 1$ and $v = y^2 - 1$. We want to make $x^2 + y^2$ as large as possible, or equivalently to make $u + v$ as large as possible, subject to the condition $u^2 + v^2 = 2$.

Draw the circle $u^2 + v^2 = 2$, and a few lines with equations of shape $u + v = k$. As k increases, the line $u + v = k$ moves in a North–East direction. By using geometric reasoning close to that in X-11, we conclude that the maximum value of k consistent with $u^2 + v^2 = 2$ occurs when $u + v = k$ is tangent to $u^2 + v^2 = 2$, that is when $u = v = 1$.

X-21. Find all real a such that the roots of $x^2 + (2a - 1)x + 3a = 0$ are real and the sum of their squares is as small as possible.

Solution. The discriminant $4a^2 - 16a + 1$ is negative only when

$$\frac{4 - \sqrt{15}}{2} < a < \frac{4 + \sqrt{15}}{2},$$

so the roots are real unless a is in this interval.

Let p and q be the roots. Then $p + q = -2a + 1$ and $pq = 3a$, and therefore

$$p^2 + q^2 = (p + q)^2 - 2pq = 4a^2 - 10a + 1$$

(or compute the roots, then square and add).

Let $S(a) = 4a^2 - 10a + 1$. Complete the square:

$$S(a) = 4\left(a - \frac{5}{4}\right)^2 - \frac{21}{4}.$$

Thus $S(a)$ increases symmetrically as the distance of a from $5/4$ increases. The place closest to $5/4$ at which the roots are real is $(4 - \sqrt{15})/2$.

X-22. At a price of \$1.50 per cup, a convenience store sells 1000 cups of Burpee carbonated drink each week. For each 1 cent drop in price, it could sell 24 more cups per week. The ingredients for a cup of Burpee cost the store 2 cents, and the cup itself an additional 3 cents. At what price does the store maximize its weekly profit from Burpee sales?

Solution. We can experiment with a calculator. At the \$1.50 price, the profit is $1000 \cdot 145$ cents. If the price drops to \$1.49, the store sells 1024 cups, and the profit is $1024 \cdot 144$. At \$1.48, the profit is $1048 \cdot 143$, and so on. Note that for a while profit increases as price drops. After some tedium we can find the best price.

The process can be automated. Suppose the current price is $150 - x$ cents. So the store sells $1000 + 24x$ cups. Drop the price by 1 cent. The profit on these $1000 + 24x$ cups drops by $1000 + 24x$ cents. But 24 more cups are sold, at a profit of $(150 - x - 1) - 5$ cents each. So the net "increase" in profit is

$$24(144 - x) - (1000 + 24x), \qquad \text{or more simply} \qquad 2456 - 48x.$$

As long as $2456 > 48x$, it pays to drop the price by a penny. Since $2456/48$ is about 51.16, the optimal price is 98 cents.

Another way: Let $P(x)$ be the weekly profit if the price is x cents less than \$1.50. The profit per cup is $150 - 5 - x$, and $1000 + 24x$ cups are sold, so

$$P(x) = (1000 + 24x)(145 - x) = 145000 + 2480x - 24x^2.$$

To maximize $P(x)$, we maximize $2480 - 24x^2$.

The curve $y = 2480 - 24x^2$ is a downward-facing parabola that crosses the x-axis at $x = 0$ and at $x = 2480/24$. By symmetry, the highest point of the parabola is at $x = 1240/24$, that is, at $x \approx 51.67$. But price has to be an integer. The function $2480 - 24x^2$ is symmetrical about $x = 1240/24$, and decreases as x moves away from $1240/24$. So the best x is 52, the nearest integer to $1240/24$. A cup of Burpee should be priced at 98 cents.

X-23. Find the maximum value reached by xy as (x, y) travels around the ellipse $x^2/9 + y^2/4 = 1$.

Solution. Since xy is negative in the second and fourth quadrants, we needn't look there. Also, neither xy nor $x^2/9 + y^2/4$ changes if we change the sign of both x and y, so we can work in the first quadrant: the results transfer automatically to the third quadrant.

Imagine graphing the curves $xy = k$, where k is positive. We get a family of hyperbolas. As k increases, the hyperbolas recede from the origin. We want the largest k such that the hyperbola $xy = k$ meets the ellipse. For that largest k, the hyperbola and the ellipse are tangent to each other.

Substitute k/x for y in the equation of the ellipse and simplify. We get

$$4x^4 - 36x^2 + 9k^2 = 0.$$

If the discriminant $36^2 - 144k^2$ is positive, there are two positive values of x^2, and if the discriminant is negative there are no real values of x^2. The transition point occurs when the discriminant is 0. There is then a single intersection point, or more accurately two coinciding ones, that is, tangency. From $36^2 - 144k^2 = 0$ we conclude that $k = 3$.

Another way: Let $x = 3u$ and $y = 2v$. We want to maximize xy, that is, $6uv$, subject to the condition $u^2 + v^2 = 1$. Now as before look at hyperbolas $6uv = k$ as k increases. It is obvious from the symmetry that the largest value of $6uv$ is reached when $u = v = 1/\sqrt{2}$, so the maximum value is 3.

Another way: Since $x^2/9 + y^2/4 = 1$, we can let $x = 3\cos\theta$ and $y = 2\sin\theta$. Then $xy = 6\cos\theta\sin\theta = 3\sin 2\theta$. But $3\sin 2\theta$ reaches a maximum when $\sin 2\theta = 1$.

Comment. The second way is much nicer than the first! The ellipse didn't have enough symmetry for our taste, so by a change of variable we forced additional symmetry. The third way indirectly also changes the ellipse to a circle.

The second way is best, but there are many other approaches. We can maximize x^2y^2. Let $u = x^2/9$ and $v = y^2/4$. The ellipse turns into a line segment. Or else *minimize* $x^2/9 + y^2/4$ subject to say $xy = 1$, then rescale.

X-24. (a) Points P and Q are chosen on the curve $x^2 + 4y^2 = 1$ in such a way that the distance PQ is as large as possible. Find that distance. (b) Do the same calculation for the curve $x^4 + 16y^4 = 1$.

Solution. (a) The curve $x^2 + 4y^2 = 1$ is an ellipse that meets the x-axis at $(\pm 1, 0)$ and the y-axis at $(0, \pm 1/2)$. It is obvious that the two points farthest apart are $(1, 0)$ and $(-1, 0)$, so the maximum distance is 2. Is it really obvious? Only if ellipses look like their usual pictures. In fact, book representations of ellipses are often wrong. So to be certain we work a little more.

The problem is about *two* points P and Q. First we get rid of one of them. Let P and Q be two points on the ellipse, and let O be the origin. Without loss of generality we may assume that $OP \ge OQ$. Let P' be the reflection of P across the origin. Then $PP' \ge PQ$. So we need only find a point P that is as far from the origin as possible and then let $Q = P'$.

Let $P = (x, y)$. We want to maximize $x^2 + y^2$ given that $x^2 + 4y^2 = 1$. But

$$x^2 + y^2 = (x^2 + 4y^2) - 3y^2 = 1 - 3y^2,$$

and $1 - 3y^2$ reaches a maximum when $y = 0$. Thus the largest achievable distance is indeed 2.

(b) The curve $x^4 + 16y^4 = 1$ looks somewhat like an ellipse, and meets the axes exactly where $x^2 + 4y^2 = 1$ does. But the maximum distance is *not* 2. We can argue exactly as in (a) that it is enough to find a point P at maximum distance from the origin. We now maximize $x^2 + y^2$ given that $x^4 + 16y^4 = 1$.

Let $x^2 = u$ and $4y^2 = v$. We want to maximize $u + v/4$ given that $u^2 + v^2 = 1$. This is just Problem X-11 with different numbers, and any of the four solutions of that problem works. For example, we can argue that we need to find $k > 0$ such that the line $u + v/4 = k$ is tangent to the circle $u^2 + v^2 = 1$.

Let R be the point of tangency. The line $v = 4(k - u)$ has slope -4. The line joining the origin to R is perpendicular to the tangent line, so has slope $1/4$, and therefore equation $v = u/4$. This line meets $u^2 + v^2 = 1$ at $(4/\sqrt{17}, 1/\sqrt{17})$, so $k = \sqrt{17}/4$. Thus the largest distance between points on our curve is $\sqrt{17}/2$, a little more than 2.

Comment. Consider the curve $|x|^e + |2y|^e = 1$, where e is a positive constant. If $e = 2$ we get the ellipse of (a), and if $e = 4$ the curve of (b). It turns out that if $e \leq 2$, then the largest distance between points on the curve is 2, but if $e > 2$, then the largest distance is greater than 2. The exponent 2 is right on the boundary. So maybe the fact that 2 is the largest distance in the ellipse isn't so obvious after all!

X-25. Let $\triangle ABC$ have area 1. Divide it into three parts by choosing a point P between B and C and drawing lines through P parallel to AB and AC as in Figure 10.6. Show that the area of one of these parts is at least 4/9.

Solution. It is obvious that at least one part has area 1/3 or more, but getting to 4/9 takes some work. Let $BC = a$ and $BP = xa$. Then $PC = (1 - x)a$. Each of the two smaller triangles in Figure 10.6 is similar to $\triangle ABC$.

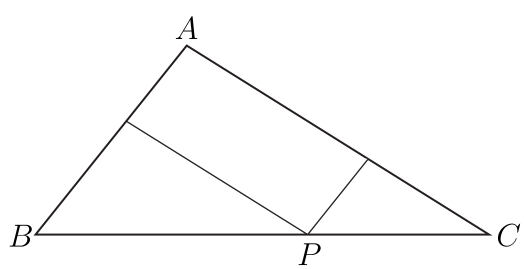

Figure 10.6: Dividing a Triangle into Three Parts

The triangle with base BP is obtained by scaling $\triangle ABC$ by the scaling factor x. But multiplying all lengths by x multiplies area by x^2, so the triangle with base BP has area x^2. Similarly, the triangle with base PC has area $(1 - x)^2$.

If $x^2 \geq 4/9$ or if $(1 - x)^2 \geq 4/9$, we are finished. So suppose that $x < 2/3$ and $1 - x < 2/3$, or equivalently $1/3 < x < 2/3$. The parallelogram in the picture has area $1 - \left(x^2 + (1 - x)^2\right)$, which simplifies to $2x - 2x^2$.

Look at the parabola $y = 2x - 2x^2$. Its vertex is at $x = 1/2$, so $2x - 2x^2$ decreases steadily as x moves away from 1/2. Note that $2x - 2x^2 = 4/9$ when $x = 1/3$ and when $x = 2/3$. It follows that $2x - 2x^2 > 4/9$ when $1/3 < x < 2/3$.

So if each small triangle has area less than 4/9, then the parallelogram must have area bigger than 4/9.

X-26. Let $A = (0, a)$. Find the shortest possible distance between A and Z as Z roams over the parabola $y = x^2/2$.

Solution. Let $Z = (x, y)$. The distance from A to Z is $\sqrt{x^2 + (y-a)^2}$. Choose Z so as to minimize this distance, or equivalently to minimize $x^2 + (y-a)^2$, the square of the distance. Since $x^2 = 2y$, we minimize $f(y)$, where $f(y) = 2y + (y-a)^2$. Complete the square:

$$f(y) = (y - (a-1))^2 + 2a - 1.$$

But $(y - (a-1))^2$ can't be negative. So $f(y)$ reaches a minimum when $y = a - 1$, and the minimum value is $2a - 1$. Thus the required minimum distance is $\sqrt{2a-1}$.

Or is it? Let $a = 0$. The minimum distance is 0, since $(0, 0)$ is on the parabola. The formula just obtained predicts that the minimum distance is $\sqrt{-1}$. And even a casual sketch (which is how a solution should *start*) shows that at least when a is negative the minimum distance is $|a|$. Something went awry.

Note that $y = x^2/2$, so $y \geq 0$. If $a - 1 \geq 0$, we can indeed find $y \geq 0$ such that $y = a - 1$, and the minimum distance is $\sqrt{2a-1}$. But if $a < 1$, the smallest value of $(y - (a-1))^2$ is obtained by setting $y = 0$, and therefore in that case the minimum distance is $|a|$.

X-27. The circles of Figure 10.7 have radius 3 and 4, the distance between their centers is 5, and they meet at A and B. A line drawn through A meets

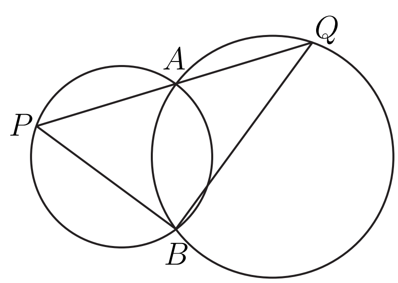

Figure 10.7: A Problem of Clavius

the smaller circle at P and the larger circle at Q, where A is between P and Q. Find the greatest possible length of PQ.

Solution. Look at $\angle APB$ as P travels from A to B on the outer arc of the smaller circle. Since $\angle APB$ is the angle subtended by chord AB, it is independent of P. Similarly, $\angle AQB$ is independent of Q. So even though the *size* of $\triangle PBQ$ varies, its *shape* doesn't.

Thus PQ is biggest when PB is biggest. That happens when PB is a diameter of the smaller circle. Since PB and QB are maximized simultaneously, QB is a diameter of the larger circle.

Let X be the center of the smaller circle and Y the center of the larger circle. If PB (and therefore QB) are diameters, then X bisects PB and Y bisects QB, so XY is half as long as PQ. But $XY = 5$, so $PQ = 10$. Note that as long as the circles meet, their radii are irrelevant: only the distance between their centers matters.

Comment. The sixteenth-century mathematician Clavius posed and solved this problem. Clavius is best known for his work on calendar reform that produced our current (Gregorian) calendar.

There are many variations on the same theme. For example, let two circles meet at A and B, and suppose that PB and QB are diameters of the circles. Show that PQ passes through A.

X-28. The Pyramid of Khafre originally had height 143.5 meters and a square base $ABCD$ of side 215.25 m. A long-ago ant travelled from A to C along the pyramid. If she took the shortest possible path, what height did she reach?

Solution. We solve the problem for a perfect pyramid of height h and base length b, partly because the ant did the Pyramid of Khufu the next day, but also because it is easier than with numbers.

Let O be the top of the pyramid. Imagine that the pyramid is just a cardboard hat. Slice out sides OCD and ODA, flatten what's left, and draw line AC. We end up with Figure 10.8. The shortest path from A to C is along line AC of the flattened pyramid, and P is the highest point reached.

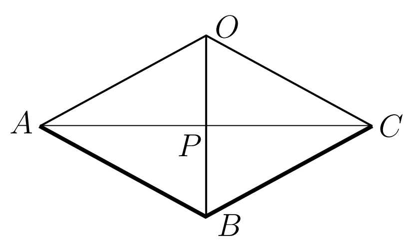

Figure 10.8: The Ant and the Pyramid

Drop a perpendicular from O to the center of the base of the pyramid and use the Pythagorean Theorem to find the slant height:

$$OA = OB = OC = \sqrt{h^2 + b^2/2}.$$

Let $\theta = \angle OBA$. By dropping a perpendicular from O to the midpoint of AB, we can see that $\cos\theta = (b/2)/\sqrt{h^2 + b^2/2}$. But since $\angle APB$ is a right angle, we have $PB = b\cos\theta$.

Let x be the height of P off the ground. By using similar triangles, we can see that $x/h = PB/OB$. Putting everything together, we conclude that

$$x = \frac{b^2 h}{2h^2 + b^2}.$$

For the values of h and b in the Pyramid of Khafre, x is about 76 meters.

Comment. In the same spirit, place the ant on the top edge of a conical drinking cup of depth h whose top is a circle of radius r. The ant wants to walk to the diametrically opposite point. Let $s = \sqrt{h^2 + r^2}$. The length of the shortest walk is $2s\sin(\pi r/2s)$.

X-29. Call a line *nice* if it crosses the positive x and y axes, and the x and y-intercepts, together with the origin, form a triangle of area 1. Find all numbers w such that $(1, w)$ lies on exactly one nice line.

Solution. We solve the problem in two ways, one for left-brain people, the other for the right-brained. The first solution is harder but maybe more interesting. Let $P = (1, w)$.

Suppose first that $w \leq 0$. Draw the line ℓ through P and the origin, then rotate ℓ clockwise around P. The area of the triangle formed by the origin and the intercepts increases steadily, so there is a unique position at which the area reaches 1, and therefore exactly one nice line through P.

Suppose now that $w > 0$. Take a line through P with negative slope close to 0, and rotate it clockwise about P until it is nearly vertical. Initially, the triangle formed by the origin and the intercepts has large area, and at the end it again has large area. Somewhere in between, the area reaches a minimum. If that minimum is *greater* than 1, there can't be a nice line through P. If the minimum is *less* than 1, then there is more than one nice line through P, for area must have been 1 twice. We conclude that there is a unique nice line if and only if the minimum area is 1.

We solve the minimization problem geometrically. Refer to Figure 10.9. Reflect $\triangle PAB$ in the line AP, and $\triangle PCD$ in the line PC. The two reflected triangles cover rectangle $OAPC$, and possibly more. So the area of $\triangle OBD$ is at least twice the area of the rectangle, with equality when $AB = OA$. The minimum area is 1 if and only if rectangle $OAPC$ has area $1/2$, so $w = 1/2$.

Another way: Let ℓ be the line $y = mx + b$. The y and x-intercepts of ℓ are b and $-b/m$. We want both of these to be positive, and we want the area of the triangle formed by the intercepts and the origin to be 1. That triangle has area $-b^2/2m$, and therefore ℓ has equation $(b^2/2)x - b + y = 0$. Thus ℓ passes through $(1, w)$ if and only if $b^2/2 - b + w = 0$.

There is single nice line if and only if exactly one positive b satisfies the preceding equation. The equation has roots $1 \pm \sqrt{1 - 2w}$. So there is exactly one positive solution (i) when $w = 1/2$ and (ii) when $w \leq 0$.

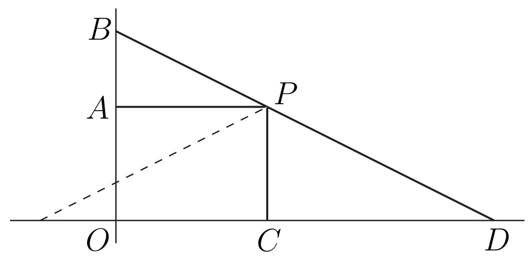

Figure 10.9: Nice Lines

Comment. A simpler problem with a similar theme is to ask for the triangle of smallest area whose sides are the positive axes and a line through, say, $(3, 2)$. This is a standard first-year calculus problem, but the reflection argument used above gives a purely geometric solution.

X-30. Jane started to cycle eastward at 5 meters per second along a country road. At the instant she started, her old dog Spot was 30 meters North and 40 meters East of Jane. After a quick computation, Spot chose a suitable line of travel, and running at full speed just managed to intercept Jane. Spot is smart but slow. If it were any slower there is no way it could have intercepted Jane. How long did it take for Spot to reach Jane?

Solution. Suppose that when the story begins Jane is at A and Spot is at B. Let C be the point where Spot intercepts Jane, and D the point on the road due South of B. Then $AD = 40$ and $BD = 30$. (Please see Figure 10.10.) Let $a = BC$ and $b = AC$.

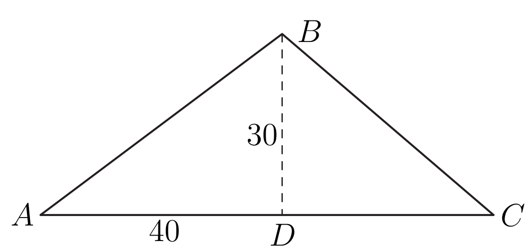

Figure 10.10: Run Spot Run

The key fact is that if Spot were any slower it couldn't have caught Jane. Let $\alpha = \angle BAC$ and let $\beta = \angle ABC$. Spot needs to choose β so that a/b is as small as possible. By the Sine Law,

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}, \qquad \text{and therefore} \qquad \frac{a}{b} = \frac{\sin\alpha}{\sin\beta}.$$

To minimize a/b, Spot needs to maximize $\sin\beta$. Thus $\beta = 90°$. By the Pythagorean Theorem, $AB = 50$, and therefore $\sin\alpha = 3/5$. It follows that $a/b = 3/5$, and therefore Spot's speed is 3. Since $a = 50\tan\alpha = 37.5$, Spot ran for 12.5 seconds.

Comment. This old standard first appeared in World War I. The setting was military, with one warship intercepting another. Once upon a time, North American children's first reading books featured Dick, Jane, and their dog Spot. This problem was worded in affectionate memory of the Dick and Jane series.

X-31. A triangle has area 1. Find its sides a, b, and c, given that $a \le b \le c$ and b is as small as possible.

Solution. Let the triangle have vertices A, B, and C, with as usual $BC = a$, $AC = b$, and $AB = c$. We show first that $\angle ACB$ must be $90°$ by showing it can be neither less than nor bigger than $90°$.

Suppose that $\angle ACB < 90°$. Imagine that legs CA and CB are hinged at C, and increase $\angle ACB$ to $90°$. We now have a triangle of area greater than 1 whose two smallest sides are a and b. By multiplying all sides by a suitable scale factor s, where $s < 1$, we get a triangle of area 1 whose two smallest sides are sa and sb. Since $sb < b$, this contradicts the fact that b is as small as possible.

Suppose now that $\angle ACB > 90°$. When we decrease $\angle ACB$ to $90°$, the new AB is still the longest side, and a and b are still the two smallest sides, but the area is now greater than 1. We reach a contradiction exactly as in the preceding paragraph.

Thus $\triangle ABC$ is right-angled. It has area $ab/2$, so $ab = 2$. Because $a \le b$, we have $b \ge \sqrt{2}$. But the right-angled isosceles triangle with legs $\sqrt{2}$ has area 1, so since b is as small as possible, $b = \sqrt{2}$. It follows that $a = \sqrt{2}$ and $c = 2$.

X-32. A 30-meter wide river flows East–West. Anna lives on the South side, 150 meters from the river. Bea lives on the North side, 60 meters from the river and 200 meters to the East of Anna's house. They build a footbridge across the river and perpendicular to it, in a way that minimizes the walking distance between their houses. Find that distance.

Solution. It is possible to solve the problem by using the calculus, but geometry is easier and more informative. Please see Figure 10.11. Let the North end of the bridge be at P, the South end at Q, let Anna's house be at A, and Bea's house at B. Finally, let C be the fourth corner of the parallelogram whose other three corners are B, P, and Q.

Assume that on land one can walk freely in any direction. Then the travel distance from A to B is $AQ + PB + 30$. To minimize this distance, we minimize $AQ + PB$. But note that $PB = QC$, so we must select Q so as to minimize $AQ + QC$. It is clear that Q should be on the line AC. In that case, AC is the hypotenuse of a right-angled triangle with legs 200 and $150 + 60$, so AC has length $\sqrt{(200)^2 + (210)^2}$, that is, 290. It follows that the minimal path has length 320.

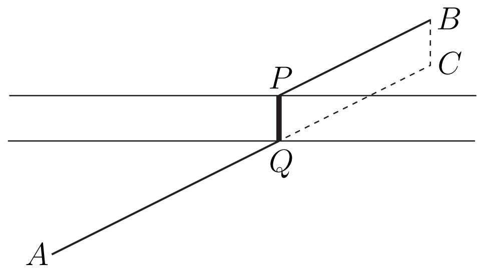

Figure 10.11: The Footbridge

Another way: Push the land North of the river southward, shrinking the river to zero, and draw the straight-line best path. Let this path meet the shrunken river at Q. Then push the land back up, making a bridge at Q. The final computation is the same as in the first solution.

X-33. Find all pairs (x, y) of real numbers such that there is a right-angled triangle with perimeter x and area y.

Solution. We can quickly *almost* find the relationship between x and y by using a simple but powerful idea called *dimensional analysis.*

Let k be the minimum perimeter among all triangles with area 1. Then for any $p \geq k$, there is a triangle with area 1 and perimeter p. Scale linear dimensions by the scale factor $\sqrt{y}$ to make the area y. Then perimeter scales to $k\sqrt{y}$, and therefore the condition we are looking for has the form $x \geq k\sqrt{y}$, or equivalently $y \leq x^2/k^2$.

The required pairs (x, y) are therefore all points to the right of the y-axis, above the x-axis, and on or below a certain parabola. The only remaining problem is to find k.

There is a right-angled triangle with area 1 and perimeter p if and only if there are positive reals u and v, the legs of the triangle, such that

$$uv = 2 \qquad \text{and} \qquad u + v + \sqrt{u^2 + v^2} = p.$$

To find k, note that it is the smallest p for which there are u and v satisfying the above equations, so it is the minimum value of $u + v + \sqrt{u^2 + v^2}$ as (u, v) ranges over the first-quadrant part of the hyperbola $uv = 2$.

To minimize p, it is enough to minimize either $u + v$ or if we prefer $u^2 + v^2$, since $u^2 + v^2 = (u + v)^2 - 2uv = (u + v)^2 - 4$. But

$$u^2 + v^2 = (u - v)^2 + 4uv = (u - v)^2 + 8,$$

so the minimum is reached when $u = v$. In that case, $u = \sqrt{2}$. The smallest possible perimeter k is therefore $2\sqrt{2} + 2$.

There are many other approaches. For example, let $w = \sqrt{u^2 + v^2}$, and let θ be the angle opposite one of the legs. Then the area is $(w^2 \sin\theta \cos\theta)/2$. But that area is 1, so $w^2 \sin 2\theta = 4$. To minimize w, make $\sin 2\theta$ as big as possible, namely 1.

Comment. The calculation goes through almost as easily if we use y as the area rather than setting it equal to 1. We chose the dimensional analysis approach to show how much information one can get almost for free.

X-34. Find the smallest value of $x^2 + y^2 + z^2$ given that $x + y + z = 1$.

Solution. In order to get some insight, look at the analogous problem in two variables: minimize $x^2 + y^2$, given that $x + y = 1$. Draw the line $x + y = 1$. We want to find a point (x, y) on this line such that $x^2 + y^2$, or equivalently $\sqrt{x^2 + y^2}$, is as small as possible. So we want to find the point on the line $x + y = 1$ which is closest to the origin. It is clear from the geometry that $x = y = 1/2$, and therefore the minimum value of $x^2 + y^2$ is $1/2$.

The idea extends to three variables. We can assign coordinates (x, y, z) to points in three-dimensional space. If we do, the equation $x + y + z = 1$ is the equation of a plane, and $\sqrt{x^2 + y^2 + z^2}$ is the distance from (x, y, z) to the origin. Once we are reasonably familiar with the geometry, it becomes intuitively clear that the point on the plane $x + y + z = 1$ which is nearest the origin is $(1/3, 1/3, 1/3)$, and therefore the minimum value of $x^2 + y^2 + z^2$ is $1/3$.

We now check that the intuition is corrrect. To bring out the symmetry, let

$$x = \frac{1}{3} + u, \qquad y = \frac{1}{3} + v, \qquad z = \frac{1}{3} + w.$$

We want to minimize

$$\left(\frac{1}{3} + u\right)^2 + \left(\frac{1}{3} + v\right)^2 + \left(\frac{1}{3} + w\right)^2$$

subject to the condition $u + v + w = 0$. Expand and simplify. The expression to be minimized becomes

$$\frac{1}{3} + u^2 + v^2 + w^2,$$

which is smallest when $u = v = w = 0$.

Comment. The same argument shows that the minimum value of

$$x_1^2 + x_2^2 + \cdots + x_n^2 \qquad \text{given} \qquad x_1 + x_2 + \cdots + x_n = 1$$

is reached when all the x_i are $1/n$.

X-35. An isosceles right triangle has legs of length 2. Find the length of the shortest line segment that cuts the triangle into two parts of equal area.

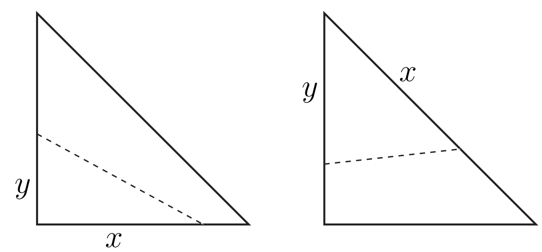

Figure 10.12: Cutting a Triangle in Half

Solution. Any cut divides the triangle into a triangle and a quadrilateral, or maybe two triangles. There are two cases: (i) the cut joins a point on one leg to a point on the other leg, as in the left triangle of Figure 10.12, or (ii) the cut joins a point on a leg to a point on the hypotenuse, as in the right triangle. We deal with both cases, but it is intuitively clear that the shortest cut is of type (ii). Let x and y be as in the pictures.

The original triangle has area 2, so in each case we must produce a triangle of area 1. In case (i), the area is $xy/2$, so we minimize $x^2 + y^2$, the square of the length of the cut, subject to $xy/2 = 1$. Note that

$$x^2 + y^2 = (x - y)^2 + 2xy = (x - y)^2 + 4,$$

so $\sqrt{x^2 + y^2}$ takes on its minimum value of 2 when $x = y$. (Case (i) is the same as Problem X-2.)

Now examine case (ii). The triangle has base y and height $x \sin 45°$, so it has area $xy/(2\sqrt{2})$. Thus $xy = 2\sqrt{2}$.

By the Cosine Law, the square of the length of the cut is

$$x^2 + y^2 - 2xy \cos 45°, \qquad \text{that is,} \qquad (x - y)^2 + 2xy - 2xy \cos 45°.$$

Since $xy = 2\sqrt{2}$, the cut is minimized by taking $x = y = \sqrt[4]{8}$. That makes the length of the cut $2\sqrt{\sqrt{2} - 1}$, much cheaper than the cheapest cut of case (i).

Comment. With minor modifications the method extends to any triangle. One could for example look at the 30–60–90 triangle.

Chapter 11

Quickies

Introduction

These problems are mostly of a more elementary nature than the problems in other chapters. But what really distinguishes them is that the solutions are short, requiring only one idea and not much computation.

Problems and Solutions

XI-1. In rectangle $ABCD$ of Figure 11.1, $AB = 12$, $BC = 5$, and $AP = 9$. Which of the shaded rectangles has the greater area?

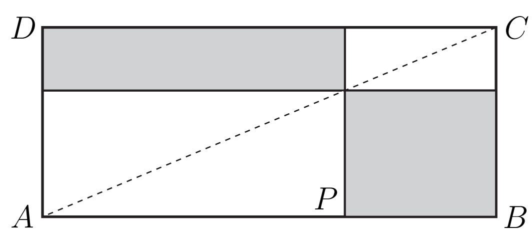

Figure 11.1: Comparing Rectangles

Solution. We could compute the areas by using the machinery of similar triangles. But a look at Figure 11.1 is enough. The shaded rectangle on the left is $\triangle ACD$ with two "white" triangles cut out. The shaded rectangle on the right is $\triangle CAB$ with two white triangles cut out. The parts cut out have the same area, and therefore so do the rectangles.

XI-2. Find a simple expression for a/b, where

$$a = 1 + 2 + 4 + \cdots + 2^{100} \qquad \text{and}$$
$$b = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{100}}.$$

Solution. It is possible to sum the two geometric series, then find the ratio of the sums and simplify. But it is better to look before using formulas. Note that $2^{100}b$ is equal to a (written backwards). Thus $a/b = 2^{100}$.

XI-3. Twenty people get seated at random on the twenty chairs around a circular banquet table. Find the probability that Ali and Ben are next to each other.

Solution. Paint a chair purple. Relative to the purple chair, there are 20 places for Ali to sit, and for each of these there are 19 places for Ben to sit, for a total of $20 \cdot 19$ equally likely ways. Now count the number of ways that Ali and Ben can be neighbours. Again, there are 20 places for Ali to sit, and for each of these there are 2 ways for Ben to sit beside her, a total of $20 \cdot 2$. So the required probability is $(20 \cdot 2)/(20 \cdot 19)$.

There is a simpler argument that doesn't require painting a chair. Ask Ali to sit down first. There remain 19 places for Ben to sit, and 2 of these are next to Ali, so the probability is $2/19$.

XI-4. One sheet of a newspaper section contains pages 9, 10, 27, and 28. How many sheets does the newspaper section have?

Solution. There are just as many pages after page 28 as there are before page 9, so there are $28 + 8$ pages in the section. Since there are 4 pages to a sheet, there are 9 sheets in the section.

XI-5. A 20 cm by 20 cm cake is 7 cm high. It has smooth icing on top and fruit glaze on the sides. Divide the cake among 5 people so that cake, icing, and glaze are all divided equally. Don't use a potato masher.

Solution. Put five equally spaced marks around the perimeter of the top of the cake, and slice with the slices meeting at the center of the top of the cake, as in Figure 11. It is clear that everyone gets equal amounts of fruit glaze. We show that everyone gets equal amounts of icing, and therefore of cake.

Any slice is either a triangle of base 16 and height 10, or, if it goes around a corner, can be decomposed into two triangles of height 10 whose bases add up to 16. Thus all slices have equal top area.

Comment. The same idea works for any regular polygon and any number of people. Any triangular cake can also be divided fairly into any number of pieces. Equally-spaced marks are placed around the perimeter, and these are joined to the center of the incircle of the triangle, that is, the point where the angle bisectors meet.

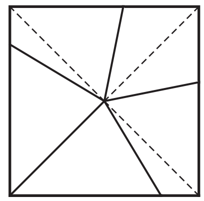

Figure 11.2: Dividing a Square Cake

XI-6. Local time in St. John's, Newfoundland is $4\frac{1}{2}$ hours ahead of time in Vancouver, British Columbia. A non-stop flight left Vancouver at 7:00 a.m. local time, and arrived at St. John's at 6:40 p.m. local time. After refueling for an hour, the plane headed back to Vancouver, arriving at 11:20 p.m. Because of tail winds, the trip to St. John's took an hour less than the trip back. What was the flight time to St. John's?

Solution. Since the pilot is going back to Vancouver, she might as well leave her watch on Vancouver time. Total elapsed time was 16:20 hours, so 15:20 hours were spent flying. Half of that is 7:40. The flight from Vancouver took an hour less than the flight back, so it took 7:10 hours.

XI-7. There will be either five or six people at a meeting. We need to pre-cut a pizza into slices, not necessarily all the same size, so that in either case the pizza can be shared equally. This can be done by cutting the pizza into 30 equal slices. Can we get by with fewer than 30 slices?

Solution. Cut the pizza into 6 equal slices, and divide one of these into 5 equal slices, for a total of 10. If there are five people, each gets a big slice and a small slice. If there are six, five get big slices, and one gets 5 small slices.

Comment. It is clear that we can't get by with fewer than 10 slices, and that if there will be n or $n+1$ people the minimum number of slices is $2n$. If there will be n or $n+d$ people where $d > 1$, the problem is more challenging.

XI-8. There are 100 purses, containing respectively 1, 2, 3, ..., 100 gold coins. Can these purses be distributed among five people so that everyone gets the same number of coins?

Solution. Yes, and there are many ways to do it. One way is first to divide the purses into groups of two: the purse that has 1 coin and the purse that has 100, the purse that has 2 and the purse that has 99, and so on, with finally the purse that has 50 and the purse that has 51. There are 50 groups of purses, and each group has 101 coins. Give 10 groups to each of the 5 people. Note that everyone also ends up with the same number of purses.

Comment. If there are instead 99 purses, add an imaginary empty purse, and group into "pairs" with sum 99. That gives 50 pairs, and the division can proceed.

Suppose that there are n purses, with 1, 2, 3, ..., n coins. For which n can the purses be divided into three groups so that each group has the same number of coins? What about four groups, five groups, six groups?

XI-9. The medians to the two equal sides of an isosceles triangle meet at right angles, and the third side has length 12. Find the area of the triangle.

Solution. Label points as in Figure 11.3. Four copies of $\triangle BOC$ can be arranged

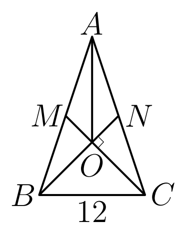

Figure 11.3: Perpendicular Medians

around O to make a square of side 12, so $\triangle BOC$ has area 36.

Triangles ACM and BCM have the same area (equal bases AM and BM and equal heights). For the same reason, $\triangle AOM$ and $\triangle BOM$ have the same area, and therefore by subtraction $\triangle AOC$ and $\triangle BOC$ have the same area. Thus $\triangle AOC$ has area 36. By symmetry, $\triangle AOB$ has area 36, so the whole triangle has area 108.

XI-10. Four cards lie on a table. On the top of each card Beth sees one printed symbol. These are E, T, 2, and 7. Beth knows that any card has a letter on one side and a number on the other. Alphonse claims that whenever a card has a vowel on one side, it has an even number on the other. Which cards must Beth turn over to make sure that all four cards satisfy Alphonse's rule?

Solution. The card that shows E must be turned over, for if the number on the other side isn't even, then the rule has been violated. The card that shows T needn't be touched, for the rule makes no claim about cards with consonants.

Should Beth turn over the card that shows 2? No, because a vowel on the other side is consistent with the rule, and a consonant is also consistent with the rule. But Beth must turn over the card that shows 7, because a vowel on the other side would violate the rule.

Comment. This problem is famous among cognitive psychologists; it is called the *Wason Test.* Many people get the wrong answer.

XI-11. In the United States and Canada, 180 billion soft drink cans are consumed each year. Two-thirds as many are thrown away as are recycled. How many cans are *not* recycled?

Solution. Let $3x$ be the number that are recycled. Then $2x$ are thrown away, and therefore $3x + 2x = 180$. (For convenience, the unit of measurement is a billion. That makes expressions shorter, and saves us the trouble of counting zeros.) So $x = 108$, and therefore 72 billion cans are not recycled. The number recycled was called $3x$ rather than x to avoid the unpleasantness of dealing with fractions.

Another way: "Guess" that 3 cans are recycled. So 2 are thrown away, for a total of 5. That's not quite 180 billion: scale up by multiplying everything by 36 billion.

XI-12. One second after firing a rifle at a bottle 224 meters away, Annie heard the sound of shattering glass. Given that sound travels at 336 meters per second, what was the average speed of the bullet?

Solution. The sound of the bottle breaking took 224/336 seconds, or more simply 2/3 of a second, to reach Annie. The bullet therefore took 1/3 of a second to reach the target, and travelled at an average speed of (224)/(1/3), that is, 672 meters per second.

Comment. Technically this isn't quite right. The solution assumed that the bullet travels in a straight line. The actual path is roughly parabolic, deviating, if there is no wind, by at most a half meter from the straight line path. The answer of 672 is correct for the average value of the component of velocity in the direction of the target. But the other components of velocity are much smaller than 672, so the average speed is 672 for all practical purposes.

XI-13. A 5×8 chocolate bar is made up of 1×1 squares. Find the *smallest* number of breaks that will break up this bar into its 40 individual squares. We are not allowed to put two pieces on top of each other and break them simultaneously.

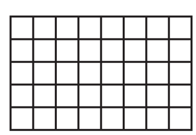

Figure 11.4: Breaking up a Chocolate Bar

Solution. Perhaps we should experiment with much smaller bars, starting with the 1×1 (this requires 0 breaks), and the 1×2 (1 break). It is obvious that a $1 \times n$ bar requires $n - 1$ steps. The 2×2 bar can be broken up in essentially one way, so we can't do any better than 3 steps.

The first interesting bar is the 2×3. The first break cuts it either into two 1×3's, and then we need 4 more breaks, for a total of 5 steps, or the first break cuts it into a 1×2 and a 2×2, again for a total of 5 steps. Try a few more. The work isn't hard, for we can recycle previous results—that's a hint that an induction argument may work. But we need to go through quite a few cases to get to the 8×5.

After a while we may suspect that no matter how we do the breaking, a $p \times q$ chocolate bar always takes $pq - 1$ steps, or equivalently that a bar with n squares takes $n - 1$ steps.

There is a simple proof (there are also ugly proofs). At each step the number of pieces increases by 1. We start with one piece and at the end we want n pieces. However we do it, that takes $n - 1$ steps, 39 in our case.

XI-14. A T-120 VHS videocassette tape can record exactly 2 hours in SP mode, 4 hours in LP mode, and 6 hours in EP mode. On a single tape, you have recorded three episodes of *Dawson's Creek,* the first one in SP, the next in LP, and the last in EP. Each episode lasts exactly one hour. How many minutes (in SP mode) of blank tape remain?

Solution. The first episode used 60 SP mode minutes, the second used 30, and the third 20, altogether 110 minutes, leaving 10 minutes of tape.

XI-15. In Mathematics 101, the term mark is calculated by averaging the test scores—all tests are out of 100, and test scores are integers. Just before the final test, Alicia realized that if she got 100 on that test, her term mark would be 91, but that even with a mark as low as 65, she would end up with a term mark of 86. What was her average going into the final test?

Solution. Let n be the total number of tests in the course, and S the sum of Alicia's grades on the first $n - 1$ tests. Then

$$S + 100 = 91n, \qquad \text{and} \qquad S + 65 = 86n.$$

Thus $n = 7$ and $S = 537$. The average on the first 6 tests is 89.5.

Another way: We don't need to use symbols. A swing of 35 marks on the final test, from 100 down to 65, would change Alicia's average by 5 marks. So the course has a total of 7 tests. It follows that right now her marks add up to $7 \cdot 91 - 100$. Divide by 6 to find her current average.

Or else observe that getting a mark of 65, that is, 21 marks under 86, would give her an average of 86. So right now she is 21 marks "over" an 86 average. Similarly, she is 9 marks "under" a 91 average. So her current average divides the interval from 86 to 91 in the ratio 21 : 9.

Comment. The solutions that don't use symbols require concentration, and errors of reasoning become more likely.

In the real world, the term mark would be probably be rounded to the nearest integer, with an average like 81.5 being rounded up. Under that interpretation, things become more interesting. It turns out that there could also be a total of 6 tests. If Alicia's first 5 marks add up to 448, then 100 on the last test gives a term average of 91.33, which rounds to 91, while 65 on the last test gives an average of 85.5, which rounds to 86. So the average going into the final test could also be 89.6.

XI-16. In Canada, the fuel efficiency of a car is defined as the number of liters of gasoline the car uses to travel 100 kilometers. The United States uses miles per gallon. A fuel additive claims to increase the number of miles per gallon by 22%. Find the corresponding percentage change in liters per 100 kilometers.

Solution. Oh no, conversions! But units don't matter in *percentage* change. If there is a 22% increase in the number of miles per gallon, there is also a 22% increase in the number of yards per tablespoon.

It is therefore enough to find the percentage change in the number of gallons per mile. If m is the usual number of miles per gallon, then m is increased by 22%, that is, multiplied by 1.22. So $1/m$, the number of gallons per mile, is divided by 1.22, that is, multiplied by about 0.81967. That represents a percentage decrease of about 18%.

XI-17. In Figure 11.5, OAB is a quarter-circle, $OPQR$ is an inscribed rectangle, $OP = 13$ and $PA = 5$. Calculate the length of PR.

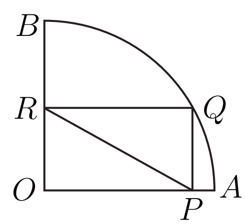

Figure 11.5: The Diagonal of an Inscribed Rectangle

Solution. Since OQ and PR are the two diagonals of a rectangle, $PR = OQ$. But OQ is equal to the radius of the quarter-circle, which is $13 + 5$.

XI-18. Two twins ran a 100 meter race. For a while they were neck and neck, but the one on steroids drew ahead and won by 3 meters. So the next time the faster twin started 3 meters behind the usual start line. What happened?

Solution. We need to make some assumptions, and these assumptions should be tested against reality. It is simplest to assume constant speed. Let the steroid twin's speed be v meters per second. The loser twin's speed is then $97v/100$. Thus the time for the steroid twin to run 103 meters is $103/v$. In that time, the loser twin runs $(103/v)(97v/100$, that is, 99.91 meters.

Unfortunately, the model we used is contrary to fact: sprinters ordinarily do not reach full speed until some 30 to 40 meters into the race. Here is a better argument. In the second race, when the steroid twin has run 100 meters, the loser twin has run only 97. So they are dead even at the 97 meter mark. But the steroid twin is going faster over the last three meters, and therefore wins.

Comment. The steroid twin's victory in the first race is due not to a faster start or a consistently higher speed—the racers were neck and neck for a while—but to a higher maximum speed. Experienced sprinters reach a maximum speed that they then maintain to the end. Horses are different. We see a horse surge ahead of another toward the end of a race and infer that it is speeding up. That's an optical illusion: both horses are slowing down, but one is slowing down faster.

XI-19. It takes 50 seconds to pump 20 dollars' worth of gas into a car. After 30 more seconds the amount pumped reaches 40 liters. How much does gas cost per liter?

Solution. So 40 liters "cost" 80 seconds of time, and therefore 1 liter costs 2 seconds. But each second costs 20/50 dollars. It follows that 1 liter costs 2(20/50) dollars, that is, 80 cents.

XI-20. We want to manufacture rectangular sheets of paper that can be cut into two pieces each similar to the original, that is, with the same length to width ratio. What shape should the sheets be?

Solution. Let the ratio of length to width in the original sheet be λ. When the sheet is cut into two pieces each similar to the original, the length must be cut in two. Since length to width ratio is unchanged, we conclude that $\lambda/1 = 1/(\lambda/2)$. It follows that $\lambda^2 = 2$, and therefore $\lambda = \sqrt{2}$.

Another way: Each daughter rectangle is similar to the original but has 1/2 the area. So in going from the original to a daughter, corresponding *linear* dimensions are multiplied by $1/\sqrt{2}$. If the original width was 1 unit, then the new length is 1, so the original length must have been $\sqrt{2}$.

Comment. British photographic enlarging paper has length to width ratio close to $\sqrt{2}$. For example, the A4 sheet measures 297 mm by 210 mm. The number 297/210 differs from $\sqrt{2}$ by less than 7.3×10^{-5} !

The same question can be asked about a box. It turns out that if the smallest side is 1 then the other two sides must be $2^{1/3}$ and $2^{2/3}$. This three-dimensional version unfortunately has no practical significance. A triangle can be divided into two pieces each similar to the original if and only if the triangle is right-angled.

XI-21. A number of the form $n(n+1)/2$, where n is a positive integer, is called a *triangular number*. The first five triangular numbers are 1, 3, 6, 10, and 15. How many triangular numbers are there from 50 to 5000?

Solution. We need to count the positive integers n such that

$$100 \le n(n+1) \le 10000.$$

These inequalities hold if and only if $10 \le n \le 99$, so there are 90.

XI-22. A number is called a *palindrome* if it reads the same forwards as backwards, like 7777 or 747. How many three-digit palindromes are there?

Solution. We can make a complete list and then count. Instead, *imagine* listing. The leftmost digit can be any one of 1, 2, ..., 9, a total of 9 possibilities. For *each* of these, there are 10 ways to choose the middle digit. And once the leftmost digit has been chosen, the rightmost one is determined. So there are 90 three-digit palindromes.

Comment. The problem of counting n-digit palindromes is almost as easy as the three-digit case.

First let n be even, say $n = 2k$. To make an n-digit palindrome, make a k-digit number, then append on the right the number written backwards. And there are $9 \times 10^{k-1}$ k-digit numbers.

Let n be odd, say $n = 2k+1$. To make an n-digit palindrome, take a $2k$-digit palindrome, and insert any digit in the middle. So there are 9×10^k palindromes. This isn't *quite* right. The reasoning works fine unless $k = 0$. How many 1-digit palindromes are there? If we count 0 as a 1-digit number, there are 10, not the 9 that the formula predicts. Maybe 0 shouldn't be called a 1-digit number.

XI-23. In Figure 11.6, PQ is a diameter of a circle of radius 1, RS is tangent to this circle, QR and PS are each perpendicular to RS, and $\angle SPQ$ is 45°. Find the sum of the lengths of PS and QR.

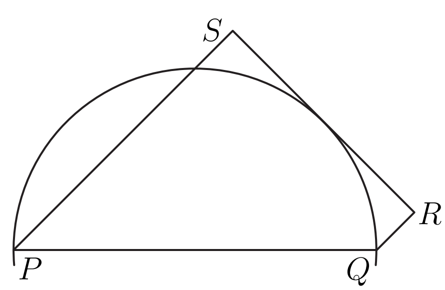

Figure 11.6: The Sum of PS and QR

Solution. There are many awkward ways to calculate. For example, extend SR until it meets the extension of PQ at say X. If O is the center of the circle, and T the point of tangency, then $\triangle OTX$ is an isosceles right-angled triangle. So $OX = \sqrt{2}$. It follows that $QX = \sqrt{2} - 1$, and therefore $QR = (\sqrt{2} - 1)/\sqrt{2}$. But $PX = 1 + \sqrt{2}$, and now we can find PS. Add. The result is 2. The simplicity of the answer means that there is probably

Another way: This solution yields a more general result, for it doesn't require knowing $\angle SPQ$. Let O be the center of the circle, and T the point of tangency. The radius OT is parallel to two sides of the trapezoid $PQRS$, and bisects PQ. It follows that the length of OT is halfway between the lengths of PS and QR, and therefore $PS + QR = 2$.

Another way: Rotate the figure $PQRS$ through $180°$ around the center of the circle. The rotated figure, together with $PQRS$, forms a rectangle two of whose opposite sides are tangent to the circle. Thus the perpendicular distance between these sides is equal to the diameter of the circle. But this perpendicular distance is just $PS + QR$, so $PS + QR = 2$.

XI-24. Find the smallest positive multiple of 12 whose decimal digits add up to 42.

Solution. If the decimal digits of a number add up to 42, then since 42 is a multiple of 3, the number is automatically a multiple of 3. We therefore look for the smallest positive multiple of 4 whose decimal digits add up to 42.

The number of digits should be as small as possible. Since a four-digit number can't have digit sum 42, look at five-digit numbers. The last digit must be even, so to get up to 42 the first digit must be at least 7. If it is 7, then the last digit must be 8, and the other digits must be 9's. But 79998 is not divisible by 4.

So the first digit is 8 or more. If it is 8, then since the last digit is 8 or less, the remaining digits must be an 8 and two 9's. But a number that ends in 98 is not divisible by 4, so the number is 89988.

XI-25. The *Lucas numbers* 1, 3, 4, 7, ... obey the rule that any number after the first two is the sum of the two previous ones. Find the remainder when the 48-th Lucas number is divided by 4.

Solution. The Lucas numbers grow quite fast, so it would be nice not to compute the first 48. Note that to calculate the remainder when $a + b$ is divided by m, it is enough to find the remainders when a and b are divided by m, add them, and if necessary subtract m to make the result less than m.

So instead of working with the actual Lucas numbers, we can work with remainders. The sequence of remainders is therefore 1, 3, 0, 3, 3, and so on. These numbers are pleasantly small, and we can get to the 48-th without much trouble. So we continue, getting 2, 1, 3, and so on.

But we can stop computing. The seventh and eighth entries are respectively the same as the first and second entries. Since the "next" remainder is always determined by the preceding two, there is cycling with period 6. Thus the 48-th remainder is the same as the sixth, namely 2.

Comment. The cycling helped a lot. Did we just get lucky? Look at the remainders when the Lucas numbers are divided by the positive integer m. There at most m possible remainders, and therefore at most m^2 pairs of consecutive remainders. Before we compute the $(m^2 + 2)$-th remainder, we must have seen two matching pairs of consecutive remainders. So the remainders ultimately are caught in an endless loop. It turns out that the loop begins with the first two remainders, that is, there is full cycling.

XI-26. A goat is tethered by a 15-foot rope to a post at the middle of a long side of a 10×5 shed. Over how large an area can she graze?

Solution. A quick sketch like Figure 11.7 shows that the goat can reach a half-circle of radius 15, together with two quarter-circles of radius 10 and two quarter-circles of radius 5. The combined area is 175π square feet.

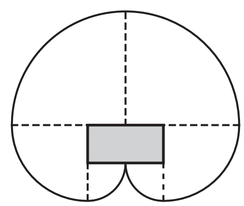

Figure 11.7: The Tethered Goat

XI-27. A bridge fanatic has five friends who come over every day to play. It takes four people to play a hand of bridge. The fanatic takes part in every hand, so two of the friends have to sit out each hand. This month the fanatic played 1500 hands. If each of her/his friends played an equal number of hands, how many hands did each friend play?

Solution. Imagine that after each hand, the fanatic gave a penny to each friend who played. Any hand involves three of the friends, so this month the fanatic spent 4500 pennies. These were equally distributed among five people, so each friend played 900 hands.

XI-28. A soccer ball is sewn together in a sweatshop from 32 pieces of leather, twelve of them regular pentagons and the rest regular hexagons. The pentagons and hexagons all have the same side length. A seam has to be sewn wherever two sides meet. How many seams are there?

Solution. The pentagons have a total of $5 \cdot 12$ edges, and the hexagons have $6 \cdot 20$ edges. Altogether, the polygons have 180 edges. Any seam takes care of two edges, so there are 90 seams.

XI-29. A block of cheese with a thin wax coating measures 20 cm by 16 cm by 16 cm. The entire block is cut into cubes of side 2 cm. What percentage of these cubes have no wax on them?

Solution. There are 640 cubes. The inner cubes form a $16 \times 12 \times 12$ block, so there are 288 of them. But $288/640 = 0.45$, so 45% of the cubes have no wax.

XI-30. How many times between noon and midnight are the hour hand and the minute hand of a clock at right angles to each other?

Solution. There is an easy solution: just stare at the clock for 12 hours. But a clock of the imagination moves faster.

Between noon and one o'clock, perpendicularity happens twice, shortly after 12:15, and a bit before 12:50. Between one o'clock and two, again it happens twice. Between two and three, it happens twice, the second time at exactly 3:00. Between three and four, it happens twice, the first time at exactly 3:00. Of course we won't count 3:00 twice. Continue. The complication that occurred at 3:00 occurs again at 9:00. So the hands are at right angles 22 times.

XI-31. The triangle ABC of Figure 11.8 has a right angle at B. Edge AB

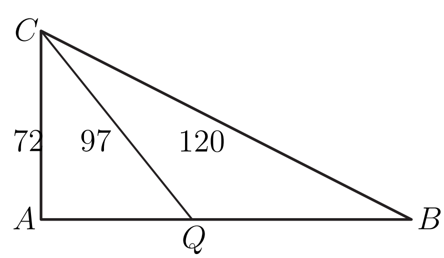

Figure 11.8: The Area of $\triangle AQC$

is 72 units long, and $AC = 120$. The point Q lies on line segment BC, and $AQ = 97$. Find the area of $\triangle AQC$.

Solution. By the Pythagorean Theorem, $(BC)^2 = 120^2 - 72^2$, so $BC = 96$. Again by the Pythagorean Theorem, $(BQ)^2 = 97^2 - 72^2$, so $BQ = 65$. Thus $QC = 31$, and the required area is 1116.

XI-32. The five-digit numbers made up of *different* digits chosen from 1, 2, 3, 4, 5, 6, 7 are listed in increasing order. So the first number is 12345 and the last is 76543. Find the last number in the first half of the list.

Solution. Just as many numbers start with 1, 2, or 3 as with 5, 6, or 7. So our number starts with 4. Half the numbers that start with 4 then continue with 1, 2, or 3, and the other half with 5, 6, or 7. Our number belongs to the first group, and is the largest in that group, so it is 43765.

Comment. We took full advantage of symmetry, and ended up doing no counting at all! It may be worthwhile to look at a slightly less symmetrical situation, such as finding the last number in the first quarter of the list. If instead we allow repetitions of digits, the problem is closely connected with the base 7 representation of a number.

XI-33. Alan entered a judo tournament in the 75 kg class. The rules say that a contestant is eliminated only after *losing* 2 matches. Spectators saw 40 matches in the 75 kg class, including 4 draws, before Alan won the championship. How many people competed in the 75 kg class?

Solution. There were 36 matches that resulted in a loss. Since it takes 2 losses to eliminate someone, 18 people were eliminated in these 36 matches. Everyone but Alan was eliminated, so 19 people competed in the 75 kg class.

XI-34. Three circular gold medallions have radius 6 mm, 6 mm, and 7 mm, and each is 2 mm thick. They are melted down to make a single 2 mm thick circular medallion. Find its radius.

Solution. The volume of a disk of given thickness is proportional to the area of the disk. Thus the volume is kr^2, where r is the radius and k is a constant. It so happens here that $k = 2\pi$, but we don't need to know that.

The combined volume of the three disks is therefore $k(6^2 + 6^2 + 7^2)$, that is, $121k$. If R is the radius of the new disk, then $kR^2 = 121k$, and therefore $R = 11$.

Comment. The shape of the medallions is irrelevant as long as (i) the thicknesses are constant and (ii) scaling aside, the shapes are the same.

XI-35. An Egyptian-style pyramid is 50 meters high. The distance from the top of the pyramid to any corner of the base is 100 meters. Find the area occupied by the base of the pyramid.

Solution. Let T be the top of the pyramid, and let M be the center of the base. Let A be one of the corners of the base. Then $\triangle TMA$ is right-angled, its hypotenuse is 100, and one of its sides is 50. Thus by the Pythagorean Theorem the square on MA has area 7500.

By drawing the diagonals of the base, we can cut the base into four right-angled isosceles triangles with legs MA. These triangles can be rearranged in pairs to make two squares of side MA. So the area of the base is 15000 square meters.

XI-36. The area of the region inside the bigger circle of Figure 11.9, but outside the smaller circle, is 144π square meters. The radius of the bigger circle is 4 meters more than the radius of the smaller circle. Find the sum

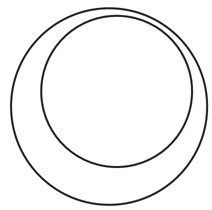

Figure 11.9: A Circle Inside a Circle

of the radii of the two circles.

Solution. Let R be the radius of the big circle, and r the radius of the small one. Then $\pi(R^2 - r^2) = 144\pi$, and therefore $R^2 - r^2 = 144$. Also, $R - r = 4$. Divide: $R + r = 144/4 = 36$.

XI-37. Find a triangle similar to the 8–15–17 triangle whose area in square meters is numerically equal to its perimeter in meters.

Solution. Since $8^2 + 15^2 = 17^2$, the converse of the Pythagorean Theorem shows that the triangle is right-angled. It follows that the triangle has area 60. Scale the 8–15–17 triangle by a scale factor λ. The perimeter of the new triangle is 40λ, and its area is $60\lambda^2$. Perimeter and area are equal if $\lambda = 40/60$.

Comment. This is a fairly silly problem, because the condition "perimeter is numerically equal to the area" depends on the units with which the sides and area are measured. So the question couldn't arise in a genuine physical setting.

XI-38. A regular hexagon $\mathcal{H}$ and an equilateral triangle $\mathcal{T}$ have the same perimeter. Find the ratio of their areas.

Solution. Join the center of $\mathcal{H}$ to the corners. That splits $\mathcal{H}$ into 6 equilateral triangles, which may be taken to have side of length 1 unit. The area of an equilateral triangle is proportional to the square of its side, so $\mathcal{H}$ has area $6k$ for some k. There is no need to know that $k = \sqrt{3}/4$. The equilateral triangle with the same perimeter as the hexagon has sides 2, so $\mathcal{T}$ has area $4k$. Thus the area of $\mathcal{H}$ is to the area of $\mathcal{T}$ as 6 is to 4.

The same idea can be carried out without formulas. Join the midpoints of the sides of $\mathcal{T}$. That splits $\mathcal{T}$ into 4 equilateral triangles of side 1. So the areas are in the proportion 6 : 4.

XI-39. In Figure 11.11, three of the rectangles have area 25, 30, and 35 as shown. Find the area of the fourth rectangle.

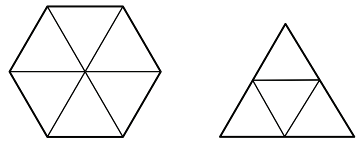

Figure 11.10: The Hexagon and the Triangle

25	30
35	??

Figure 11.11: The Area of the Fourth Rectangle

Solution. Two rectangles with the same height have areas that are proportional to their bases. So the bases of the two top rectangles are in the proportion 25 : 30. It follows that the area of the fourth rectangle is (35)(30/25).

XI-40. Suppose that any side of triangle $\mathcal{T}$ is greater than any side of triangle $\mathcal{T}'$. Is the area of $\mathcal{T}$ necessarily greater than the area of $\mathcal{T}'$? Find a proof or give a counterexample.

Solution. No, $\mathcal{T}'$ can have much bigger area than $\mathcal{T}$. For example, let the sides of $\mathcal{T}'$ all be 1. Let the sides of $\mathcal{T}$ be 4, $2+\epsilon$, and $2+\epsilon$, where ϵ is a positive number close to 0, like 10^{-8}. Then poor $\mathcal{T}$ is nearly flat and its area is only about 4×10^{-4}.

XI-41. In Figure 11.12, the outer square is 20×20. Find the area of the inner square.

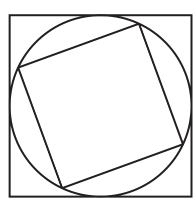

Figure 11.12: A Square Inside a Circle Inside a Square

Solution. The circle has diameter 20, and therefore the inner square has diagonal 20. Let s be the side of the inner square. By the Pythagorean Theorem, $s^2 + s^2 = 20^2 = 400$. So s^2, the area required, is 200. (It would be wasteful to take the square root and then square the result!)

Another way: Draw the two diagonals of the inner square, then rotate the inner square about its center until these diagonals are parallel to the sides of the outer square. Erase the circle. Note that the outer square is made up of 8 congruent isosceles right-angled triangles, while the inner square uses 4 of these triangles. So the area of the inner square is half the area of the outer square.

XI-42. The three altitudes of a triangle have length 1/5, 1/12, and 1/13. Find the sides of the triangle.

Solution. In any triangle, the product of base and height is twice the area. Thus the sides of the triangle are $5k$, $12k$, and $13k$, where k is twice the area. It remains to find k.

Note that $5^2 + 12^2 = 13^2$, so by the converse of the Pythagorean Theorem our triangle is right-angled. Thus twice the area of the triangle is $60k^2$. We conclude that $60k^2 = k$, and therefore $k = 1/60$.

XI-43. An isosceles triangle $\mathcal{T}$ has the property that the bisector of one of the angles divides $\mathcal{T}$ into two isosceles triangles. What can be said about the angles of $\mathcal{T}$?

Solution. Let $\mathcal{T}$ have vertices A, B, and C, with $AB = AC$. If the bisected angle is at A, then by symmetry $\mathcal{T}$ splits into two right triangles. If these are to be isosceles, then $\angle ABC = \angle ACB = 45°$, and $\angle BAC = 90°$.

Now suppose the bisected angle is at B. Let $\angle ABC = 2\theta$, and let the bisector of $\angle ABC$ meet AC at D. Since $\angle ACB = 2\theta$, it follows that $\angle BDC = 180° - 3\theta$. Since $4\theta < 180°$, in order for $\triangle BCD$ is to be isosceles we need $180° - 3\theta = 2\theta$, that is, $\theta = 36°$.

Finally, we check that this value of θ also makes $\triangle DAB$ isosceles. A bit of angle chasing shows that it does: $\angle BAD = \angle DBA = 36°$. The angles of $\mathcal{T}$ are $36°$, $72°$, and $72°$.

XI-44. Points A and B are 4 meters apart. Let $\mathcal{K}$ be the set of all points P in three-dimensional space such that $\triangle PAB$ has area 1 square meter. Describe what $\mathcal{K}$ looks like.

Solution. Let ℓ be the line through A and B, and let h be the perpendicular distance from P to ℓ. The area of $\triangle PAB$ is $4h/2$, so it is 1 if and only if $h = 1/2$. Thus $\mathcal{K}$ consists of all points P that are at perpendicular distance 1/2 from ℓ. These are the points on the infinite cylinder of radius 1/2 with central axis ℓ.

XI-45. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ lie in the same plane. The first has radius 11, the second has radius 7, and the circles meet at points 4 units apart. Let a_1 be the area of the region that lies inside $\mathcal{C}_1$ but outside $\mathcal{C}_2$, and let a_2 be the area of the region inside $\mathcal{C}_2$ but outside $\mathcal{C}_1$. Find $a_1 - a_2$.

Solution. It is possible, but not pleasant, to calculate the area of the region that lies inside *both* $\mathcal{C}_1$ and $\mathcal{C}_2$. But that's unnecessary. Let c be the area of this common region. Then $a_1 + c = 121\pi$, and $a_2 + c = 49\pi$, and therefore $a_1 - a_2 = (a_1 + c) - (a_2 + c) = 72\pi$.

XI-46. For any real number x, let $\lfloor x \rfloor$ be the largest integer which is less than or equal to x. For example, $\lfloor 2.7 \rfloor = 2$, and $\lfloor -2.7 \rfloor = -3$. Solve the equation $x\lfloor x \rfloor = 99$.

Solution. There is no positive solution x. For if $x \geq 10$, then $x\lfloor x \rfloor \geq 100$, while if $0 \leq x < 10$, then $x\lfloor x \rfloor < 90$.

Let's look for negative solutions. If $-9 \leq \lfloor x \rfloor < 0$, then $x\lfloor x \rfloor \leq 81$, while if $\lfloor x \rfloor \leq -11$, then $x\lfloor x \rfloor > 110$. So we look for a solution with $\lfloor x \rfloor = -10$. That forces $x = -9.9$, which works.

XI-47. In Figure 11.13, the big circle has radius 9, the small circle has radius 4, and the circles are tangent to each other. The big circle is tangent

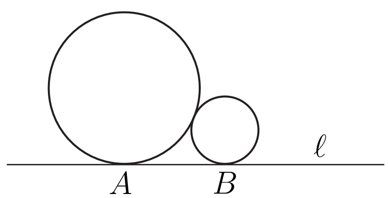

Figure 11.13: The Distance Between the Points of Tangency

to line ℓ at the point A, and the small circle is tangent to ℓ at B. Find the distance from A to B.

Solution. Let P be the center of the big circle and Q the center of the small circle. Let T be the point on the line segment PA which is at distance 4 from A. Then $TABQ$ is a rectangle and $PT = 5$. Since $PQ = 13$, it follows by the Pythagorean Theorem that TQ, and therefore AB, has length $\sqrt{13^2 - 5^2}$.

XI-48. Suppose that $x^2 + 2kx + k \geq 0$ for all x. What can we conclude about k?

Solution. Look at the parabola $y = x^2 + 2kx + k = (x + k)^2 - k^2 + k$. The minimum value of y is reached when $(x + k)^2$ is equal to 0, so the minimum value is $-k^2 + k$. It follows that $y \geq 0$ for all x precisely if $-k^2 + k \geq 0$, that is, if $0 \leq k \leq 1$.

XI-49. Solve the equation $\sqrt{x^2} + \sqrt{(x-1)^2} = 1$.

Solution. By generally accepted convention, $\sqrt{a}$ means the *non-negative* number whose square is a. In particular, the frequently seen assertion $\sqrt{4} = \pm 2$ is wrong, or at least ungrammatical. (There isn't universal agreement about this.)

In general $\sqrt{a^2} = a$ if $a \geq 0$, and $\sqrt{a^2} = -a$ if $a < 0$. Thus a shorter name for $\sqrt{a^2}$ is $|a|$, and $\sqrt{(x-a)^2}$ is the distance between x and a.

Our equation says that the sum of the distances of x from 0 and from 1 is equal to 1. That's true for all x such that $0 \leq x \leq 1$ and nowhere else.

XI-50. Find a polynomial $P(x)$ such that $P(x) = x^2$ when $x = 0, \pm 1, \pm 2$, but $P(3) = 17$.

Solution. There are infinitely many possible answers. A reasonable thing to try is $P(x) = x^2 + kx(x^2-1)(x^2-4)$, for the second term is 0 at $x = 0, \pm 1$, and ± 2, and therefore $P(x) = x^2$ at these five points. If we substitute 3 for x in the expression for $P(x)$ we get $17 = 9 + 120k$, so $k = 1/15$.

XI-51. The shadow of a spherical ball is four times as long as it is wide. What angle do the sun's rays make with the ground?

Solution. The sun is so far away that its rays are virtually parallel. If θ is the angle of elevation of the sun and d is the diameter of the ball, the geometry of the situation is captured by Figure 11.14. We conclude that $\sin\theta = 1/4$, and therefore

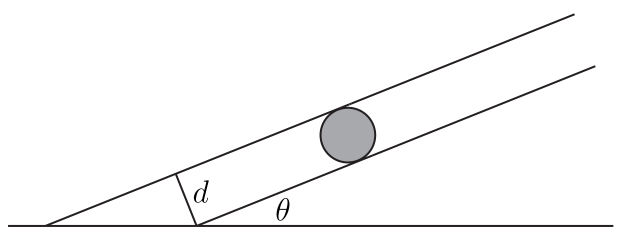

Figure 11.14: The Shadow of a Ball

θ is about 14.48 degrees.

XI-52. Find the angle between the hour and minute hands of an ordinary clock at 3:20.

Solution. The only problem here is the weirdness of the units. It would be better to have 10 hours per day, 10 days per week, 10 weeks per month, and 10 months per year. That could be done by appropriately adjusting the speed of rotation of the Earth and its distance from the Sun.

There are 12 hours in one revolution of the hour hand, and 60 minutes in an hour. Measure angles clockwise from 12:00. At 3:20, the minute hand makes an angle of (360)(20/60) with the upward vertical. The hour hand makes an angle of $(360)(3 + 20/60)/12$ with the upward vertical. The difference is 20°.

XI-53. Find, to high accuracy, the angle between 0 and 90 whose tangent is 0.5. Unfortunately, the calculator's $\tan^{-1}$ and *Solve* keys are broken. The remaining keys work.

Solution. Recall that $\cos^2 x + \sin^2 x = 1$ for any angle x. If we divide both sides of this identity by $\cos^2 x$, we obtain the identity $1 + \tan^2 x = \sec^2 x$. Thus if $\tan x = 0.5$, then $\sec^2 x = 1.25$. Since our angle has positive cosine, $\cos x = 1/\sqrt{1.25}$, and therefore $x = \cos^{-1}\left(1/\sqrt{1.25}\right)$. Now we can use the broken calculator. We get something like 26.56505118.

XI-54. Simplify

$$\frac{1+3+5+\cdots+97+99}{101+103+105+\cdots+197+199}.$$

Solution. This yields to standard formulas. The arithmetic series in the numerator has sum (50)(100)/2, and the arithmetic series in the denominator has sum (50)(300)/2. The ratio is 1/3.

Another way: Note that the denominator is equal to

$$(1+3+\cdots+97+99)+((1+3+\cdots+97+99)+(99+97+\cdots+3+1)).$$

So the denominator consists of three copies of the numerator.

Comment. There is of course nothing special about 99. Galileo mentioned the general result in 1615. Why should Galileo care? This little problem is connected with an important development in the history of Physics.

Galileo had, through a combination of experiment and theory, noticed that when a ball rolls down an inclined plane, and if the distance it travels in the first second is called 1 unit, then in the next second it travels 3 units, in the second after that it travels 5 units, and so on. Nowadays this is expressed by saying that the distance travelled in the first t seconds is kt^2 for some constant k. That Galileo's version and ours are essentially the same is connected with the fact that 1, $1+3$, $1+3+5$, $1+3+5+7$, and so on are the consecutive perfect squares.

XI-55. The sequence a_1, a_2, a_3, ... has the property that any term after the second is equal to the sum of the two preceding terms. Suppose that the fourth term is 4 and the seventh term is 7. Find the tenth term.

Solution. Let $a_5 = x$. Then $a_6 = 4 + x$, and therefore $a_7 = x + (4 + x)$. It follows that $2x + 4 = 7$. We conclude that $x = 1.5$ and therefore $a_6 = 5.5$. Now calculate: $a_8 = 12.5$, $a_9 = 19.5$, and $a_{10} = 32$.

XI-56. Let p and q be consecutive odd primes. Show that there are integers a, b, and c, all greater than 1, such that $p+q = abc$. For example, $31+37 = 2 \cdot 2 \cdot 17$.

Solution. Note that $p + q$ is even and $(p + q)/2 \neq 1$. Since $(p + q)/2$ lies between the consecutive primes p and q, it follows that $(p + q)/2$ is composite. So there are integers b and c, both greater than 1, such that $(p + q)/2 = bc$, and therefore $p + q = 2bc$.

XI-57. Express 79 as a sum of perfect fourth powers, using as few fourth powers as possible.

Solution. There aren't many possibilities to consider. We can only use 1's and 16's. If we use four 16's, we need to use fifteen 1's, for a total of 19 fourth powers. If we use three or fewer 16's, the total number of fourth powers is far higher. So the smallest possible number is 19.

Comment. In his *Meditationes Algebraicae* (1770), Edward Waring asserted that every positive integer is the sum of at most 4 positive squares, at most 9 positive cubes, at most 19 positive fourth powers, at most 37 positive fifth powers, "and so on." The result for squares had just been proved by Lagrange. The assertions about cubes and fourth powers were only proved a few decades ago. It turns out that 79 is the only positive integer that requires 19 fourth powers. The complex of problems of this type is called *Waring's Problem.* There are still many unresolved questions connected with Waring's Problem.

XI-58. A group of 19 mathematics students ordered pizza worth \$90.00. They split the cost as evenly as possible, but since 19 doesn't divide 9000 evenly, not all the students paid the same price. How many paid the lower price?

Solution. Divide 9000 by 19. The quotient is 473 (unimportant) and the remainder is 13. Collect 473 cents from everyone: another 13 cents are needed. So 13 students each paid an extra cent, and 6 students paid the lower price.

XI-59. There are 6! different six-digit numbers that use each digit from 1 to 6 exactly once. How many of them are divisible by 6?

Solution. If n is one of our six-digit numbers, then the *sum* of the digits of n is 21. Since 3 divides 21, it follows that 3 divides *all* of our numbers.

Thus 6 divides n precisely if n is even. But n is even if and only if its last digit is even. Since three of our digits are odd and three are even, by symmetry half the numbers have an even last digit. So 360 of our numbers are divisible by 6.

XI-60. Maybe the NFL will decide that the kick for "the point after" is too dull, and that only two methods of scoring are allowed: touchdown, for 7 points, and field goal, for 3 points. What scores would then be mathematically impossible for a team to get?

Solution. It isn't possible to score -17 points, or π points. But let's confine attention to non-negative integers! Experiment for a while. Scores of 1, 2, 4, 5, 8, and 11 are clearly impossible. Calculate a bit longer, looking without success for the next impossible number. We prove that 11 is the last impossible number.

Any number $n > 14$ differs from one of 12, 13, or 14 by a multiple of 3. Since 12, 13, and 14 can be reached, any $n > 14$ can be reached from one of these by adding a bunch of field goals.

Comment. Let a and b be positive integers. What integers can be expressed in the form $ax + by$, where x and y range over the non-negative integers?

If for example a and b are both even, then we can't find integers x and y such that $ax + by$ is odd. To get rid of uninteresting examples like this one, assume that a and b have *no* common divisor greater than 1.

It turns out that $ab - a - b$ is the largest integer which is *not* of the form $ax + by$, with x and y non-negative integers. In the football example, $a = 3$, $b = 7$, and indeed the largest impossible score is $3 \cdot 7 - 3 - 7$. The following question can lead to useful numerical exploration: Florida Fried Fat pieces come in boxes of 9 and buckets of 19. What is the largest number of pieces that *can't* be reached exactly by using a combination of boxes and/or buckets?

XI-61. Find the smallest positive integer n such that there are no 9's in the decimal representation of $(999)n$.

Solution. We should experiment first with multiples of 99. With a calculator that can be done pretty fast. Without one it can be done faster. The smallest positive n for which $99n$ has no 9's is 12. Now look at multiples of 999.

Note that $999n = 1000n - n$. The decimal representation of $1000n$ ends with at least three zeros. And $1000 - n$ has at least one 9 in it for a long while, certainly for $n = 1$ to 100, and even longer, until $1000 - n = 888$. So there is at least one 9 in the decimal expansion whenever $n < 112$. There are no 9's in the decimal expansion of $999 \cdot 112$, so $n = 112$.

XI-62. A kitchen has two electronic timers. One of them, after being turned on, beeps every five minutes. The second timer beeps every eleven minutes after it is started. How can these timers be used to time a three-minute egg?

Solution. Start both timers. At the second beep of the eleven-minute timer (that is, at the 22 minute mark) put the egg in the boiling water. At the fifth beep of the five-minute timer, that is, at the 25 minute mark, take the egg out.

Comment. Suppose we have an a-minute timer and a b-minute timer, where a and b are integers. It can be shown that if a and b have no common divisor greater than 1, then any positive integer number of minutes can be timed. So there was almost nothing special about 5 and 11, and we can freely invent new problems.

XI-63. Find all four-digit perfect squares N whose decimal expansion has the form $xxyy$, where x and y are digits.

Solution. Even if we don't do any preliminary thinking, there are only 90 candidates, and we can eliminate many of them quickly without tedious calculation. For example, a perfect square can only end in 0, 1, 4, 5, 6, or 9. But as usual we try to get some structural insight.

Note that $N = 100(11x) + 11y$, so 11 divides N exactly. Since N is a perfect square, 11^2 must divide N, and therefore $100x + y$ must be a multiple of 11. But $100x + y = 99x + (x + y)$, so 11 must divide $x + y$. Since x and y are *digits,* we conclude that $x + y = 11$. Now there are very few cases to deal with. But we can simplify further.

If we replace y by $11 - x$ in the equation $N = 11(100x + y)$, we obtain $N = 11^2(9x + 1)$. So $9x + 1$ has to be a perfect square. Run x quickly from 2 to 9; only $x = 7$ works. Thus $y = 4$ and $N = 7744$.