Paraconsistent Default Reasoning*

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Abstract

In this paper, a novel technique called bi-default theory is proposed for handling inconsistent knowledge simultaneously in the context of default logic without leading to triviality of the extension. To this end, the positive and negative transformations of propositional formulas are defined such that the semantic link between a literal and its negation is split. It is proven that the bi-default theory preserves many nice properties of Reiter's original theory and guarantees the existence of consistent modified bi-extensions. Thus, the bi-default logic is a generalization of default logic in the presence of inconsistency. Furthermore, a method is provided as an alternative approach for making the reasoning ability of paraconsistent logic as powerful as classical one.

Introduction

The reasoning systems based on classical logic suppose to reasoning with consistent information; otherwise, a single contradiction may destroy the vast amount of meaningful information. Even if the pursuit of consistency, nonmonotonic reasoning has also the problem when faced with inconsistency. Default logic (Reiter 1980) is a widely investigated formalism of nonmonotonic reasoning. In the context of default logic, it is well-known that once the set of axioms of a default theory is inconsistent, the default extension will collapse into triviality immediately. Theoretically, nonmonotonic logic in general and default logic in particular may lead to inconsistency (Hanks and McDermott 1987). On the other hand, it is advisable to introduce paraconsistency to conquer the trivial problem of reasoning in the presence of inconsistency. Some formalizations of paraconsistent and nonmonotonic reasoning have been proposed, in which a common technique is by appeal to multiple-valued logics, in particular a four-valued logic ((Belnap 1977a; 1977b; Priest 1991; Kifer and Lozinskii 1992; Lin 1995; Arieli and Avron 1998), among others). However, it will take much effort to use a multiple-valued logic directly as the underlying logic of the default theory.

In this paper, we investigate the issue of simultaneously handling inconsistent information and consistently revising beliefs in the context of default logic. A novel technique called bi-default theory is developed to reason with inconsistent knowledge which allows the set of axioms of a default theory to be inconsistent . Comparing with Reiter's original formalism, the bi-default theory does not lead to triviality of the extension. Technically, we transform a default theory T into a pair $T^{B} = (T^{+}, T^{-})$. Though T may be inconsistent, both T^{+} and T^{-} are consistent. Consequently, the truth value of a formula φ comes from two parts: one is the positive part $\overline{\varphi}^{+}$; the other is the negative part $\neg\overline{\varphi}^{-}$. Indeed, we have a classical two-valued semantics for the formula of the default theory in the viewpoint of the four-valued setting. Thus, the bi-default logic is both paraconsistent and nonmonotonic. In other words, the bi-default logic can be regarded as a formalization of commonsense reasoning with inconsistent and incomplete knowledge.

A similar technique appeared in (Arieli and Denecker 2003), where the authors showed how multiple-valued theories can be shifted back to two-valued classical theories through a polynomial transformation. Their transformation of a formula φ is really the same as the positive part $\overline{\varphi}^{+}$ in our setting and based on a mapping from four-valued valuation to two-valued one. We use a new transformation for getting the negative part $\neg\overline{\varphi}^{-}$ as well. In fact, the original inspiration comes from Ginsberg's *bilattices* (Ginsberg 1988) that naturally generalize Belnap's $FOUR$ (Belnap 1977a; 1977b), where a pair of truth values, representing the degree of belief for or against an assertion, composes a whole judgment of the assertion.

In (Besnard and Schaub 1998), the signed systems were introduced by transforming an inconsistent theory into a consistent one, in the same way as the positive transformation in our setting. While the semantic link between an atom and its denial was restored by appeal to default logic which at last resulted in a family of paraconsistent consequence relations. Roughly speaking, the signed systems don't aim at dealing with inconsistent default theories especially, since the defaults in signed systems are used to reestablish the context between renamed atoms and the atoms from the original theory. Nevertheless, it may be readily verified that the signed systems have the same results as that of the bi-default theory when the latter is applied to improve the reasoning ability of four-valued logics. In this sense, the bi-default theory would be regarded as an alternative formalization of signed systems.

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In (Pequeno and Buchsbaum 1991), a formalization of inconsistent default reasoning was proposed based on a particular paraconsistent logic LEI. The main difference between that approach and the bi-default theory is that the latter's underlying logic is still classical two-valued logic and thus enjoys nice properties of classical logic naturally.

Another advantage of the technique behind the bi-default theory is that it improves the reasoning ability of Belnap's four-valued logic. As well known, Belnap's four-valued logic is strictly weaker than classical logic even in the case of consistent theories. For instance, the disjunctive syllogism: φ, $\neg\varphi \vee \phi$ implying ϕ, does not hold in the four-valued logic. To resolve this weakness, Priest (Priest 1989; 1991) first proposed the solution by introducing nonmonotonicity into a paraconsistent logic. In our setting, the disjunctive syllogism works well in the consistent premises but is effectively blocked in the case of inconsistent theories without appealing to nonmonotonicity. This method gives a novel syntactic approach for reasoning from inconsistent theories as well.

The rest of this paper is organized as follows. Firstly, we briefly review Reiter's default logic. Secondly, we defines two forms of transforming a propositional formula φ to its counterparts $\overline{\varphi}^+$ and $\overline{\varphi}^-$. Thirdly, we introduce the bi-default theory. Finally, we make conclusion in the concluding section.

Default Logic

Through out of this paper, let $\mathcal{L}$ be a propositional language. A theory is a set of formulas in $\mathcal{L}$. We write Th for the consequence operator and $\vdash$ for the provability relation, respectively.

In Reiter's default logic, a *default* is an expression of the form

$$\frac{\alpha : \beta_1, \ldots, \beta_k}{\gamma}$$

where $\alpha, \beta_1, \ldots, \beta_k$ and γ are formulas in $\mathcal{L}$. α is said the *prerequisite*, $\beta_1, \ldots, \beta_k$ the *justifications* and γ the *consequent* of a default. A *default theory* is defined as a pair $T = (W, D)$, where W is a set of formulas and D is a set of defaults. A default is said *normal* if it is of the form $\frac{\alpha:\gamma}{\gamma}$, *prerequisite-free* if it is of the form $\frac{:\beta_1,\ldots,\beta_k}{\gamma}$ and *prerequisite-free normal* if it is of the form $\frac{:\gamma}{\gamma}$. $T = (W, D)$ is said a *normal default theory* (resp. *prerequisite-free normal default theory*) if every default $d \in D$ is normal (resp. prerequisite-free normal).

A set E of formulas in $\mathcal{L}$ is an *extension* of $T = (W, D)$ if it is a fixed point of the operator Γ, i.e. $E = \Gamma(E)$, where the operator Γ is defined as follows: Given a set of formulas S, $\Gamma(S)$ is the smallest set of formulas such that
(D1) $\Gamma(S) = Th(\Gamma(S))$
(D2) $W \subseteq \Gamma(S)$
(D3) If $(\alpha : \beta_1, \ldots, \beta_k/\gamma) \in D$, $\alpha \in \Gamma(S)$ and $\neg\beta_1 \notin S, \ldots, \neg\beta_k \notin S$, then $\gamma \in \Gamma(S)$.

A default theory may have none, one or multiple extensions in general. By $ext(W, D)$ we denote the family of all extensions of a default theory $T = (W, D)$. The set of *generating defaults for E w.r.t. T*, written $GD(E, T)$, is defined by $GD(E, T) = \{(\frac{\alpha:\beta_1,\ldots,\beta_k}{\gamma}) \in D \mid \alpha \in E \ and \ \neg\beta_1 \notin E, \ldots, \neg\beta_k \notin E\}$. $CONS(GD(E, T))$ denotes the set of consequents of the defaults from $GD(E, T)$.

Proposition 1 (Reiter 1980) A default theory $T = (W, D)$ has an inconsistent extension iff W is inconsistent.

Proposition 2 (Reiter 1980) If E is an extension of a default theory $T = (W, D)$, then

$$E = Th(W \cup CONS(GD(E, T))).$$

Let $T = (W, D)$ be a default theory. D^w denotes the set $\{\frac{:\varphi}{\varphi} \mid \varphi \in W\}$, i.e. the set of prerequisite-free normal default form of the axioms of the default theory T.

Marek, Treur and Truszczyński in (Marek, Treur, and Truszczyński 1997) described the family of extensions of an arbitrary prerequisite-free normal default theory.

Proposition 3 (Marek, Treur, and Truszczyński 1997) Let $W, \Psi \subseteq \mathcal{L}$. Let $D = \{\frac{:\varphi}{\varphi} \mid \varphi \in \Psi\}$. If W is inconsistent then $ext(W, D) = \{\mathcal{L}\}$. Otherwise, $ext(W, D)$ is exactly the family of all theories of the form $Th(W \cup \Phi)$, where Φ is a maximal subset of Ψ such that $W \cup \Phi$ is consistent.

According to Proposition 3, $T = (W, D)$ and $T = (W, D^w \cup D)$ have the same extensions. Without loss of generality, we can assume all default theories have the form $T = (W, D^w \cup D)$, and abbreviate it to $T = (W, D)$.

Transformations

We firstly give a brief review of the transforming technique proposed by Arieli in (Arieli and Denecker 2003). Let φ be a formula in $\mathcal{L}$. Define the scope of a negation operator $\neg$ in the formula $\neg\varphi$ as the set of all occurrences of propositional symbols in φ. An occurrence of atomic formula p in φ is *positive*, if it appears in the scope of an even number of negation operators in φ; otherwise, it is *negative*. For example, let $\varphi = \neg(p \vee \neg q) \vee \neg q$, then the first occurrence of q in φ is positive, and the second occurrence of q in φ is negative. Note that Arieli's transformation needs all formulas to be written in their logically equivalent negation normal form. In (Besnard and Schaub 1998), Besnard and Schaub gave a more general definition by the notion of polarity.

Arieli's transformation is defined as follows: Let φ be a formula in $\mathcal{L}$. Substitute every positive occurrence in φ of an atomic formula p by a new symbol p^+, and every negative occurrence in φ of an atomic formula p by $\neg p^-$, then the resulting formula is denoted by $\overline{\varphi}$. The language obtained from $\mathcal{L}$ by Arieli's transformation is denoted by $\overline{\mathcal{L}}$.

We use the similar notations of Arieli's transformation and define two transformations as follows.

Definition 1 Let φ be a formula in $\mathcal{L}$. The *positive transformation* (*p-trans*, for short) is to substitute every positive occurrence in φ of an atomic formula p by a new symbol p^+, and every negative occurrence in φ of an atomic formula p by $\neg p^-$. The resulting formula is denoted by $\overline{\varphi}^+$. The *negative transformation* (*n-trans*, for short) is to substitute every positive occurrence in φ of an atomic formula p by $\neg p^-$, and every negative occurrence in φ of an atomic formula p by p^+. The resulting formula is denoted by $\overline{\varphi}^-$.

The language obtained from $\mathcal{L}$ by the transformations defined in Definition 1 is still denoted by $\overline{\mathcal{L}}$.

Example 1 Let $\varphi = \neg(p \vee \neg q) \vee \neg q$. Then

$$\overline{\varphi}^+ = \neg(\neg p^- \vee \neg q^+) \vee \neg\neg q^- = (p^- \wedge q^+) \vee q^-$$

and

$$\overline{\varphi}^- = \neg(p^+ \vee \neg\neg q^-) \vee \neg q^+ = (\neg p^+ \wedge \neg q^-) \vee \neg q^+.$$

Given a propositional theory Δ, Δ^+ represents the set $\{\overline{\varphi}^+ \mid \varphi \in \Delta\}$, and Δ^- the set $\{\overline{\varphi}^- \mid \varphi \in \Delta\}$. $\Delta^\pm$ denotes $\Delta^+ \cup \Delta^-$.

It is clear that p-trans makes Δ^+ be classically equivalent to a formula in which negation does not occur, and n-trans makes Δ^- be classically equivalent to a formula in which there is a single occurrence of negation in front of each atomic formula p^+ (or p^-). Therefore, if we view p^+ and p^- as two independent atomic formulas, both Δ^+ and Δ^- will be always consistent.

Here are some properties of p/n-trans.

Proposition 4 $\neg\overline{\varphi}^+ = \overline{\neg\varphi}^-$ and $\neg\overline{\varphi}^- = \overline{\neg\varphi}^+$.

Theorem 5 Let Δ be a propositional theory. Δ is consistent iff $\Delta^\pm$ is consistent.

Theorem 6 Let Δ be a consistent propositional theory and φ is a formula in $\mathcal{L}$. If $\Delta^\pm \vdash \overline{\varphi}^+$ or $\Delta^\pm \vdash \overline{\varphi}^-$, then $\Delta \vdash \varphi$.

Theorem 7 Let Δ be a consistent propositional theory and φ is a formula that is not a tautology in $\mathcal{L}$. If $\Delta \vdash \varphi$, then $\Delta^\pm \vdash \overline{\varphi}^+$ and $\Delta^\pm \vdash \overline{\varphi}^-$.

Regarding p^+ and p^- as two independent atomic formulas, the reasoning ability of a single Δ^+ (or Δ^-) transformed from a given propositional theory Δ is very weak. For instance, let $\Delta = \{p, \neg p \vee q\}$, then $\Delta^+ = \{p^+, p^- \vee q^+\}$ and $\Delta^- = \{\neg p^-, \neg p^+ \vee \neg q^-\}$, and hence $\Delta^\pm = \Delta^+ \cup \Delta^- = \{p^+, p^- \vee q^+, \neg p^-, \neg p^+ \vee \neg q^-\}$. It is clear that the disjunctive syllogism works on $\Delta^\pm$ but not on Δ^+ and Δ^- separately. The same issue will be further discussed in the next section as the application of a special family of the bi-default theories.

Bi-default Theory

In this section, the so-called bi-default theory is defined by the application of the p/n-trans in a default theory, which can be well interpreted by a four-valued semantics. We will prove that the bi-default theory has nice properties in several respects.

Definition 2 Let d be a default of the form $\frac{\alpha:\beta_1,\ldots,\beta_k}{\gamma}$, then $\frac{\overline{\alpha}^+:\overline{\beta}_1^+,\ldots,\overline{\beta}_k^+}{\overline{\gamma}^+}$ is the p-trans result of d, denoted by $\overline{d}^+$, and $\frac{\overline{\alpha}^-:\overline{\beta}_1^-,\ldots,\overline{\beta}_k^-}{\overline{\gamma}^-}$ is the n-trans result of d, denoted by $\overline{d}^-$. $\overline{d}^+$ and $\overline{d}^-$ are called *bi-defaults*. D^+ represents the set $\{\overline{d}^+ \mid d \in D\}$, and D^- the set $\{\overline{d}^- \mid d \in D\}$.

Definition 3 A *bi-default theory* w.r.t. the default theory $T = (W, D)$ is a pair $T^B = (T^+, T^-)$, where $T^+ = (W^+, D^+)$ and $T^- = (W^-, D^-)$.

Definition 4 Let $T^B = (T^+, T^-)$ be a bi-default theory over a propositional language $\overline{\mathcal{L}}$. For any pair of sets of formulas S^+, $S^- \subseteq \overline{\mathcal{L}}$, let $\Gamma(S^+, S^-)$ be the pair of smallest sets of propositional formulas S'^+, S'^- from $\overline{\mathcal{L}}$ such that
(D1') $S'^+ = Th(S'^+)$ and $S'^- = Th(S'^-)$
(D2') $W^+ \subseteq S'^+$ and $W^- \subseteq S'^-$
(D3') If $(\overline{\alpha}^+ : \overline{\beta}_1^+, \ldots, \overline{\beta}_k^+ / \overline{\gamma}^+) \in D^+$, $\overline{\alpha}^+ \in S'^+$ and $\neg\overline{\beta}_1^+ \notin S^-$,…,$\neg\overline{\beta}_k^+ \notin S^-$, then $\overline{\gamma}^+ \in S'^+$ and $\overline{\gamma}^+ \in S'^-$; If $(\overline{\alpha}^- : \overline{\beta}_1^-, \ldots, \overline{\beta}_k^- / \overline{\gamma}^-) \in D^-$, $\overline{\alpha}^- \in S'^-$ and $\neg\overline{\beta}_1^- \notin S^+$,…,$\neg\overline{\beta}_k^- \notin S^+$, then $\overline{\gamma}^- \in S'^-$ and $\overline{\gamma}^- \in S'^+$.
A pair of sets of propositional formulas $E^B = (E^+, E^-)$, where $E^+, E^- \subseteq \overline{\mathcal{L}}$, is a *bi-extension* of T^B iff $(E^+, E^-) = \Gamma(E^+, E^-)$, i.e. iff (E^+, E^-) is a fixed point of the operator Γ.

By Proposition 4, $\neg\overline{\beta_i}^+ = \overline{\neg\beta_i}^-$ and $\neg\overline{\beta_i}^- = \overline{\neg\beta_i}^+$ $(1 \le i \le k)$, so in (D3') of Definition 4, $\neg\overline{\beta_i}^+$ (resp. $\neg\overline{\beta_i}^-$) is compared with S^- (resp. S^+) for consistency checking. As we have pointed out in the last Section, $\overline{\gamma}^+$ and $\overline{\gamma}^-$ are added to both S'^+ and S'^- in order to strengthen the reasoning ability of a single transform S'^+ (or S'^-). This also explains why we presuppose the set of defaults has the form $D^w \cup D$. By this assumption, when applying the bi-defaults, consistent formulas of W^+ and W^- will be mixed up, but the inconsistent ones will be kept splitting. To illustrate it, considering a simple default theory $T = (W, D)$ where $W = \{p\}$ and $D = \emptyset$ and thus $D^w \cup D = \{\frac{:p}{p}\}$, one may check that $E^B = (E^+, E^-)$, where $E^+ = E^- = Th(\{p^+, \neg p^-\})$, is a bi-extension of T^B. The bi-default $\overline{d}^+ = \frac{:p^+}{p^+}$ is an applicable bi-default since W^- doesn't include $\neg p^+$, then p^+ is added into both E^+ and E^-, the same is $\overline{d}^-$. But if $W = \{p, \neg p\}$, $\overline{d}^+$ is not an applicable bi-default again since W^- contains $\neg p^+$, and so $E^B = (E^+, E^-)$, where $E^+ = Th(\{p^+, p^-\})$ and $E^- = Th(\{\neg p^-, \neg p^+\})$, is a bi-extension of T^B. On the other hand, if $\{p^+, p^-\} \subseteq W^+$, then W must be inconsistent since at this time we have $\{p, \neg p\} \subseteq W$. By these considerations, Definition 4 is reasonable and gives us a hint on how to define a paraconsistent semantics for the bi-default logic as we shall see later.

The next example further explains how the bi-default theory works.

Example 2 Let $T^B = (T^+, T^-)$ be a bi-default theory w.r.t. the default theory $T = (W, D)$, where $W = \{z, \neg z, r \wedge q\}$ and

$$D = \{\frac{:z}{z}, \frac{:\neg z}{\neg z}, \frac{:r \wedge q}{r \wedge q}, \frac{r:\neg p}{\neg p}, \frac{q:p}{p}\}.$$

One interpretation of this theory reads r as "republican", q as "quaker", p as "pacifist" and z is any inconsistent information. It is easy to see that $T^+ = (W^+, D^+)$ and $T^- = (W^-, D^-)$, where

$$\begin{aligned} W^+ &= \{z^+, z^-, r^+ \wedge q^+\}, \\ W^- &= \{\neg z^-, \neg z^+, \neg r^- \wedge \neg q^-\} \end{aligned}$$

and

$$D^+ = \{\frac{: z^+}{z^+}, \frac{: z^-}{z^-}, \frac{: r^+ \wedge q^+}{r^+ \wedge q^+}, \frac{r^+ : p^-}{p^-}, \frac{q^+ : p^+}{p^+}\}$$
$$D^- = \{\frac{: \neg z^-}{\neg z^-}, \frac{: \neg z^+}{\neg z^+}, \frac{: \neg r^- \wedge \neg q^-}{\neg r^- \wedge \neg q^-}, \frac{\neg r^- : \neg p^+}{\neg p^+}, \frac{\neg q^- : \neg p^-}{\neg p^-}\}.$$

Since W is inconsistent, according to Reiter's default theory, T has only one extension $\mathcal{L}$. It is a trivial theory. But according to the bi-default theory, T^B has four bi-extensions which are given by $E_i^B = (E_i^+, E_i^-)$ (i=1,2,3,4), where

$$\begin{aligned} E_1^+ &= Th(W^+ \cup \{\neg r^- \wedge \neg q^-, p^-, \neg p^+\}), \\ E_2^+ &= Th(W^+ \cup \{\neg r^- \wedge \neg q^-, p^+, \neg p^-\}), \\ E_3^+ &= Th(W^+ \cup \{\neg r^- \wedge \neg q^-, p^-, p^+\}), \\ E_4^+ &= Th(W^+ \cup \{\neg r^- \wedge \neg q^-, \neg p^+, \neg p^-\}) \end{aligned}$$

and

$$\begin{aligned} E_1^- &= Th(W^- \cup \{r^+ \wedge q^+, p^-, \neg p^+\}), \\ E_2^- &= Th(W^- \cup \{r^+ \wedge q^+, p^+, \neg p^-\}), \\ E_3^- &= Th(W^- \cup \{r^+ \wedge q^+, p^-, p^+\}), \\ E_4^- &= Th(W^- \cup \{r^+ \wedge q^+, \neg p^+, \neg p^-\}). \end{aligned}$$

Note that both E_i^+ and E_i^- (i=1,2,3,4) are consistent over the language $\overline{\mathcal{L}}$.

Belnap's structure $FOUR$(Belnap 1977a; 1977b) contains four truth values: the classical truth values t and f, the inconsistent truth value $\top$ and the incomplete truth value $\bot$. By means of the bi-default theory, any formula φ in the language $\mathcal{L}$ could be given a four-valued interpretation w.r.t. a bi-extension E^B.

Definition 5 Given a default theory $T = (W, D)$, $E^B = (E^+, E^-)$ is a bi-extension of T^B w.r.t. T, the mapping v_{E^B} associates a propositional formula φ with a truth value from $FOUR$ as follows:

$$v_{E^B}(\varphi) = \begin{cases} t & \text{if } \overline{\varphi}^+ \in E^+, \overline{\varphi}^- \in E^- \\ & \text{and } \overline{\neg\varphi}^+ \notin E^+, \overline{\neg\varphi}^- \notin E^-; \\ \top & \text{if } \{\overline{\varphi}^+, \overline{\neg\varphi}^+\} \subseteq E^+ \\ & \text{or } \{\overline{\varphi}^-, \overline{\neg\varphi}^-\} \subseteq E^-; \\ f & \text{if } \overline{\neg\varphi}^+ \in E^+, \overline{\neg\varphi}^- \in E^- \\ & \text{and } \overline{\varphi}^+ \notin E^+, \overline{\varphi}^- \notin E^-; \\ \bot & otherwise. \end{cases}$$

On the one hand, it should be noted that the mapping v_{E^B} is really a single-value function; on the other hand, Definition 5 requires the existence of bi-extensions, but as we will see later, in general case, bi-default extensions may not exist. Hence a syntactically restricted form of the bi-default theory and a definition of modified bi-extension will be developed to overcome this problem.

Example 2 (*continued*)
$v_{E_1^B}(z) = \top, v_{E_1^B}(\neg z) = \top, v_{E_1^B}(r \wedge q) = t, v_{E_1^B}(p) = f$ and $v_{E_1^B}(\neg p) = t$.
$v_{E_2^B}(z) = \top, v_{E_2^B}(\neg z) = \top, v_{E_2^B}(r \wedge q) = t, v_{E_2^B}(p) = t$ and $v_{E_2^B}(\neg p) = f$.
$v_{E_3^B}(z) = \top, v_{E_3^B}(\neg z) = \top, v_{E_3^B}(r \wedge q) = t, v_{E_3^B}(p) = \top$ and $v_{E_3^B}(\neg p) = \top$.
$v_{E_4^B}(z) = \top, v_{E_4^B}(\neg z) = \top, v_{E_4^B}(r \wedge q) = t, v_{E_4^B}(p) = \top$ and $v_{E_4^B}(\neg p) = \top$.

Intuitively, without the consideration of $\{z, \neg z\}$, E_1^B (resp. E_2^B) is the corresponding bi-extension of Reiter's original extension of the default theory T which includes p (resp. $\neg p$); E_3^B and E_4^B are new bi-extensions which mean that both p and $\neg p$ hold in the same extension of T , therefore they are the corresponding bi-extensions of Reiter's inconsistent but non-trivial extensions (although they don't really exist in Reiter's default theory).

Here are some properties of the bi-default theory. In fact, many results of Reiter's default logic could be reproduced in the setting of the bi-default logic, for instance, the next theorem provides a recursive characterization of the bi-extensions.

Theorem 8 If $T^B = (T^+, T^-)$ is a bi-default theory w.r.t. to the default theory $T = (W, D)$, then a pair of sets of propositional formulas $E^B = (E^+, E^-)$ is a bi-extension of T^B iff $E^+ = \bigcup_{i=0}^{\infty} E_i^+$ and $E^- = \bigcup_{i=0}^{\infty} E_i^-$, where

$$E_0^+ = W^+, \ E_0^- = W^-$$

and for $i \geq 0$

$$E_{i+1}^+ = Th(E_i^+) \cup \Gamma_i^{\pm}, \ E_{i+1}^- = Th(E_i^-) \cup \Gamma_i^{\pm}$$

where

$$\begin{aligned} \Gamma_i^{\pm} = & \{\overline{\gamma}^+ \mid (\overline{\alpha}^+ : \overline{\beta}_1^+, \ldots, \overline{\beta}_k^+ / \overline{\gamma}^+) \in D^+, where \\ & E_i^+ \vdash \overline{\alpha}^+ \ and \ \neg\overline{\beta}_1^+ \notin E^-, \ldots, \neg\overline{\beta}_k^+ \notin E^-\} \\ \cup & \{\overline{\gamma}^- \mid (\overline{\alpha}^- : \overline{\beta}_1^-, \ldots, \overline{\beta}_k^- / \overline{\gamma}^-) \in D^-, where \\ & E_i^- \vdash \overline{\alpha}^- \ and \ \neg\overline{\beta}_1^- \notin E^+, \ldots, \neg\overline{\beta}_k^- \notin E^+\}. \end{aligned}$$

Definition 6 Let T^B be a bi-default theory and suppose that E^B is a bi-extension of T^B. The set of *generating bi-defaults for* E^B *w.r.t.* T^B, written $GD(E^B, T^B)$, is defined by

$$\begin{aligned} GD(E^B, T^B) = & \{(\overline{\alpha}^+ : \overline{\beta}_1^+, \ldots, \overline{\beta}_k^+ / \overline{\gamma}^+) \in D^+ \mid \overline{\alpha}^+ \in \\ & E^+ \ and \ \neg\overline{\beta}_1^+ \notin E^-, \ldots, \neg\overline{\beta}_k^+ \notin E^-\} \\ \cup & \{(\overline{\alpha}^- : \overline{\beta}_1^-, \ldots, \overline{\beta}_k^- / \overline{\gamma}^-) \in D^- \mid \overline{\alpha}^- \in \\ & E^- \ and \ \neg\overline{\beta}_1^- \notin E^+, \ldots, \neg\overline{\beta}_k^- \notin E^+\}. \end{aligned}$$

Theorem 9 If $E^B = (E^+, E^-)$ is a bi-extension of a bi-default theory T^B w.r.t. $T = (W, D)$, then

$$E^+ = Th(W^+ \cup CONS(GD(E^B, T^B)))$$

and

$$E^- = Th(W^- \cup CONS(GD(E^B, T^B))).$$

Corollary 10 Given a bi-default theory T^B w.r.t. $T = (W, D)$, if $E^B = (E^+, E^-)$ is a bi-extension of T^B, then both E^+ and E^- are consistent.

Definition 7 Given bi-extensions $E^B = (E^+, E^-)$ and $F^B = (F^+, F^-)$,

$$E^B = F^B \ iff \ E^+ = F^+ \ and \ E^- = F^-$$
$$E^B \subseteq F^B \ iff \ E^+ \subseteq F^+ \ and \ E^- \subseteq F^-.$$

The next theorem is the maximality of the bi-extensions.

Theorem 11 If $E^B = (E^+, E^-)$ and $F^B = (F^+, F^-)$ are bi-extensions of a bi-default theory T^B and $E^B \subseteq F^B$, then $E^B = F^B$.

Similar to the default theory, a bi-default theory may have none, one or multiple bi-extensions. Example 2 is an illustration for multiple bi-extensions. T^B w.r.t. $T = (\emptyset, \{\frac{:p}{q}, \frac{:p}{\neg q}\})$ has no bi-extension. T^B w.r.t. $T = (\{p\}, \emptyset)$ has only one bi-extension. But for a normal bi-default theory, the bi-extension can be proven to exist.

Definition 8 Let T^B be a bi-default theory w.r.t. the default theory $T = (W, D)$. If T is a normal default theory, then T^B is called a *normal bi-default theory*.

Theorem 12 Every normal bi-default theory has a consistent bi-extension.

A normal bi-default theory also satisfies orthogonality and semi-monotonicity.

Theorem 13 If a normal bi-default theory T^B has two bi-extensions $E^B = (E^+, E^-)$ and $F^B = (F^+, F^-)$, then either $E^+ \cup F^+$ or $E^- \cup F^-$ is inconsistent.

Theorem 14 Let D, D' be two sets of normal defaults such that $D \subseteq D'$. If $E^B = (E^+, E^-)$ is a bi-extension of T^B w.r.t. $T = (W, D)$, then there exists a bi-extension E'^B for T'^B w.r.t. $T' = (W, D')$ such that $E^B \subseteq E'^B$.

The next two theorems reveal the relation between the default extension of a default theory and the bi-extension of a bi-default theory. That is, for any default theory $T = (W, D)$, under some restrictions, every default extension of the default theory corresponds to a bi-default extension of the corresponding bi-default theory and vice versa. In other words, the bi-default logic is a generalization of Reiter's default logic in the presence of inconsistency.

Theorem 15 Let $T = (W, D)$ be a default theory such that W is consistent and for every default $(\alpha : \beta_1, \dots, \beta_k/\gamma)$ from D, $\alpha, \neg\beta_1, \dots, \neg\beta_k$ and γ are not tautologies. If $E = Th(W \cup CONS(GD(E, T)))$ is an extension of T, then $E^B = (\Lambda, \Lambda)$ is a bi-extension of the bi-default theory T^B w.r.t. T, where $\Lambda = Th(W^{\pm} \cup CONS^{\pm}(GD(E, T)))$.

Theorem 16 Let $T = (W, D)$ be a default theory such that W is consistent and for every default $(\alpha : \beta_1, \dots, \beta_k/\gamma)$ from D, $\alpha, \neg\beta_1, \dots, \neg\beta_k$ and γ are not tautologies. If (Λ, Λ) is a bi-extension of the bi-default theory T^B w.r.t. T and for every $\varphi \in \mathcal{L}$, $\overline{\varphi}^+ \in \Lambda$ iff $\overline{\varphi}^- \in \Lambda$, then there exists $F \subseteq \mathcal{L}$ such that $Th(F^{\pm}) = \Lambda$ and $E = Th(F)$ is an extension of T.

Without these restrictions, however, there are circumstances in which the bi-default theory T^B w.r.t. the default theory T has bi-extensions but T may have no default extension. For example, one may check that the bi-default theory T^B w.r.t. the default theory $T = (\emptyset, \{\frac{:\neg p}{p}\})$ has two bi-extensions $E_1^B = (Th(\{p^+\}), Th(\{p^+\}))$ and $E_2^B = (Th(\{\neg p^-\}), Th(\{\neg p^-\}))$, but T has no extension. We point out this coincides with the fact that the law of excluded middle is not valid in Belnap's four-valued logic.

As we have mentioned above, in general case, a bi-default theory may have no bi-extension. A full account of inconsistency handling in default logic should guarantee the existence of bi-extensions as well. To this end, the syntactically restricted form of a bi-default theory named normal bi-default theory has been developed previously. Here we will modify the definition of bi-extension according to Łukaszewicz's method (Łukaszewicz 1988), by which the modified bi-extension of a bi-default theory is guaranteed to exist. For the sake of simplicity, we use one-justification default theories to give the formal definition.

Definition 9 Let $T^B = (T^+, T^-)$ be a bi-default theory over a propositional language $\overline{\mathcal{L}}$. For any pair of sets of formulas $S^+, S^- \subseteq \overline{\mathcal{L}}$ and any set of formulas $U^{\pm} \subseteq \overline{\mathcal{L}}$, let $\Gamma[(S^+, S^-), U^{\pm}]$ be the smallest sets of propositional formulas S'^+, S'^- and $U'^{\pm}$ from $\overline{\mathcal{L}}$ such that
(D1") $S'^+ = Th(S'^+)$ and $S'^- = Th(S'^-)$
(D2") $W^+ \subseteq S'^+$ and $W^- \subseteq S'^-$
(D3") If $(\overline{\alpha}^+ : \overline{\beta}^+/\overline{\gamma}^+) \in D^+$, $\overline{\alpha}^+ \in S'^+$ and for all $\eta \in U^{\pm} \cup \{\overline{\beta}^+, \overline{\beta}^-\}$, $S^- \cup \{\overline{\gamma}^+\} \not\vdash \neg\eta$, then $\overline{\gamma}^+ \in S'^+$, $\overline{\gamma}^+ \in S'^-$ and $\overline{\beta}^+, \overline{\beta}^- \in U'^{\pm}$; If $(\overline{\alpha}^- : \overline{\beta}^-/\overline{\gamma}^-) \in D^-$, $\overline{\alpha}^- \in S'^-$ and for all $\eta \in U^{\pm} \cup \{\overline{\beta}^+, \overline{\beta}^-\}$, $S^+ \cup \{\overline{\gamma}^-\} \not\vdash \neg\eta$, then $\overline{\gamma}^- \in S'^-$, $\overline{\gamma}^- \in S'^+$ and $\overline{\beta}^+, \overline{\beta}^- \in U'^{\pm}$.
A pair of sets of propositional formulas $E^B = (E^+, E^-)$, where $E^+, E^- \subseteq \overline{\mathcal{L}}$, is a *modified bi-extension* of T^B for a set of propositional formulas $J^{\pm} \subseteq \overline{\mathcal{L}}$ iff $[(E^+, E^-), J^{\pm}] = \Gamma[(E^+, E^-), J^{\pm}]$.

This variant of bi-default logic guarantees two fundamental properties: the existence of modified bi-extensions and semi-monotonicity, showed as follows.

Theorem 17 Every bi-default theory has a consistent modified bi-extension.

Theorem 18 Let D, D' be two sets of defaults such that $D \subseteq D'$. If $E^{^B} = (E^+, E^-)$ is a modified bi-extension of $T^{^B}$ w.r.t. $T = (W, D)$, then there exists a modified bi-extension ${E'}^{^B}$ for ${T'}^{^B}$ w.r.t. $T' = (W, D')$ such that $E^{^B} \subseteq {E'}^{^B}$.

In fact, similar to Reiter's original formalism, we firmly believe that many other variants could evolve from the bi-default logic as well, which exhibits its flexibility.

Finally, a special family of bi-default theories $T^{^B}$ w.r.t. $T = (W, D^w)$ is very attractive. In Proposition 3, assuming W is consistent and $\Psi = W$, we immediately get that the default theory $T = (W, D)$ has an unique extension $E = Th(W)$. In view of this, in Belnap's four-valued logic, given a theory W, we may consider its corresponding bi-default theory $T^{^B}$ w.r.t. $T = (W, D^w)$ and reason under $T^{^B}$. By Theorem 15, if W is consistent, under the four-valued semantics, we shall get most conclusions excluding tautologies which could be derived from the classical propositional theory W. As to inconsistent theory, the bi-default logic still gives as many conclusions as possible.

Denoted by $\models^4$ the four-valued consequence relation, and define $W \models^{^B} \varphi$ iff there exists a mapping v_{E^B} defined in Definition 5 such that $v_{E^B}(\varphi) \in \{t, \top\}$.

Proposition 19 $\models^{^B}$ is nonmonotonic and paraconsistent.

Theorem 20 Let W be a propositional theory. If $W \models^4 \varphi$ then $W \models^{^B} \varphi$.

Theorem 21 If W is a consistent propositional theory and φ is not a tautology, then $W \vdash \varphi$ iff $W \models^{^B} \varphi$.

By Theorem 20 and Theorem 21, it is clear that the reasoning ability of $\models^{^B}$ is far stronger than that of four-valued consequence relation. The following example illustrates this statement.

Example 3 Let $W = \{p, \neg p \vee q\}$. One may easily check that bi-default theory $T^{^B}$ w.r.t. $T = (W, D^w)$ has an unique bi-extension $E^{^B} = (E^+, E^-)$, where

$$\begin{aligned} E^+ &= Th(\{p^+, \neg p^-, p^- \vee q^+, \neg p^+ \vee \neg q^-\}) \\ E^- &= Th(\{p^+, \neg p^-, p^- \vee q^+, \neg p^+ \vee \neg q^-\}), \end{aligned}$$

and so we have $W \models^{^B} p$, $W \models^{^B} \neg p \vee q$ and $W \models^{^B} q$.

When adding $\neg p$ into W, one may readily check that

$$\begin{aligned} E^+ &= Th(\{p^+, p^-, p^- \vee q^+, \neg p^+ \vee \neg q^-\}) \\ E^- &= Th(\{\neg p^-, \neg p^+, p^- \vee q^+, \neg p^+ \vee \neg q^-\}). \end{aligned}$$

And so, we have $W \models^{^B} p$, $W \models^{^B} \neg p$, $W \models^{^B} \neg p \vee q$ but $W \not\models^{^B} q$.

As well-known, Belnap's four-valued logic is strictly weaker than classical logic even in the case of consistent premises. Interestingly, by the above example we have seen that, the bi-default theory can be regarded as a novel technique on how to strengthen the reasoning ability of Belnap's four-valued logic. Moreover, due to the syntactic approach of the bi-default theory, it can be viewed as an alternative approach to making paraconsistent reasoning as powerful as classical one.

Conclusion

Our main contribution in this paper is to provide default logic with the ability for handling inconsistency and nonmonotonicity simultaneously. Thus, the bi-default theory has potential applications in the practice of commonsense reasoning in presence of inconsistency and incompleteness.

The bi-default theory can be well interpreted by a four-valued semantics. We firmly believe that most results of default logic in the literature could be reproduced in the setting of the bi-default logic, because the bi-default logic is a generalization of Reiter's default logic under the four-valued semantics. A byproduct is that the bi-default theory can be applied to strengthen the reasoning ability of Belnap's four-valued logic, which provides an alternative approach for making paraconsistent reasoning as powerful as classical one in the premise of consistency.

The results of this paper is limited on propositional level, we will extend it to first-order case and make more comprehensive investigation on the bi-default theory in the future work.

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