Logic-Based Merging : the infinite case

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Abstract

We extend the results of Konieczny and Pino Pérez (*J. Log. and Comp.* 2002) concerning merging operators in a finite logical framework to the infinite case (countably many propositional variables). The main result is the representation theorem. Some postulates had to be restated in a new form, equivalent only in the finite case, but more appropriate to deal with the infinite case. The construction of merging operators starting from distances between valuations is also generalized. We introduce a new kind of operators built upon the so called Cantor distance.

Keywords: Merging operators, Belief revision, Integrity constraints, syncretic assignment

Introduction

The general problem of merging information is to extract a coherent common information from several sources of information. The most natural thing to do, which consists in taking the union of pieces of information does not work in general because two (or more) sources of information can be contradictory. In this case, their union will be necessarily contradictory.

A lot of methods have been introduced to merge information in a logical framework (Cholvy & Hunter 1997; Cholvy 1998; Baral *et al.* 1992; Lin & Mendelzon 1999; Revesz 1997; Konieczny 1999). Different sets of logical properties that have to be satisfied by belief merging operators, have been proposed (Revesz 1997; Liberatore & Schaerf 1998; Lin & Mendelzon 1999; Konieczny & Pino-Pérez 1998; 1999). These works offer a classification of (logical) merging operators. In fact, they give some interesting results concerning a semantic characterization of their operators. These results are known as Representation Theorems. Their basic framework is the finite Propositional Logic. Since then, nothing has been done to extend these results to infinite logical framework. This is what we do in this work.

The motivations to study the case in which we have countably many propositional variables -the infinite case- are given now. First of all, in the logic framework used to represent a piece of information, the propositional variables play an essential role: they represent the factual information. For instance, facts like *the device is broken, inflation is stopped,* etc., are traditionally represented by propositional variables. However, to assume that the number of propositional variables is finite supposes modeling a situation completely closed in which there is no room for new facts. In some cases it is interesting to have the possibility of introducing new facts. For instance, when merging information about the causes of bad performance of a microchip, it can be useful to introduce the fact that the intensity of the current is always changing, a new variable not considered up to now. Another situation in which it is very common to introduce new variables, concerns the arguments about the acceptance or rejection of an article for a conference, a journal, etc. In many cases, one source -a reviewer- presents a new argument that can be decisive in the final appreciation.

Thus, in order to introduce the fact that one source of information can carry out infinite information, we have to allow infinite propositional variables. This does not mean that a source has to give always an infinite list of facts or more complex pieces of information even if sometimes it is necessary to have those kinds of complex representations. We know that some kind of infinite information can be encoded with finite information. For instance, the theory generated by a finite set of formulas is a truly infinite object which is completely encoded by a finite number of finite objects. The approach used in this work will be to represent the piece of information of an agent (a source) by a logical theory (not always finitely represented), *i.e.* a set of propositional formulas closed by logical deduction. Of course, the set of propositional variables will be infinite.

The kind of representation of information used here -theories- is not new in the area of dynamical reasoning. For instance, in Belief Revision ((Alchourrón, Gärdenfors, & Makinson 1985; Gärdenfors 1988)), the epistemic states are represented by logical theories. This kind of representation is quite natural and rich enough. Nevertheless, this is not the richest representation of epistemic states as can be seen in the works of Konieczny, Konieczny and Pino Pérez and Benferhat et al. ((Konieczny 1999; Konieczny & Pino-Pérez 2000; 2002b; Benferhat *et al.* 2000)). However, in order to deal with the infinite case, the representation of information by a logical theory will give us sufficient insight to see the problems about postulates and to find a general technique for the representation theorems.

The representation theorems stated here are, roughly speaking, of the following kind (for more precision see Theorem 3):

> *A merging operator is represented by an assignment which maps each group of sources of information to a total pre-order over valuations. The result of merging the group under the constraints will be the theory of the minimal models of the constraints (minimal with respect to the pre-order associated to the group).*

The difficult part in this kind of theorem is, on one hand, to find the definition of the assignment and, on the other hand, to prove that the definition works. We introduce here a general technique to define this kind of assignments (see Definition 4). We will prove that the definition can be stated in different equivalent ways, each of them useful to prove some properties. All this is inspired by a similar technique used to prove representation theorems for nonmonotonic relations introduced in (Pino-Pérez & Uzcategui 2000).

Merging operators and representation theorems

In this section we define merging operators adapting the postulates in (Konieczny & Pino-Pérez 2002a). We define the notion of the syncretic assignment needed to state the representation theorems. This is also a modification of a similar notion defined in (Konieczny & Pino-Pérez 2002a) in the finite case. We end the section stating the representation theorems.

Our logical framework is the infinite propositional calculus. The set of valuations will be denoted by $\mathcal{V}$. When A is a subset of $\mathcal{V}$, we will denote $Th(A)$, the theory of A, the set $\{\varphi \in \mathcal{L} : \forall M \in A, M \models \varphi\}$. Let $\mathcal{T}$ be the collection of all the theories. Let $\mathcal{T}^*$ the set of consistent theories. Let $\mathcal{B} = \mathcal{M}(\mathcal{T}^*)$ where $\mathcal{M}(\mathcal{T}^*)$ is the collection of finite multi-sets with elements in $\mathcal{T}^*$. The elements of $\mathcal{B}$ they will be denoted with uppercase Greek letters $\Phi, \Psi, \ldots$ (possibly with subscripts); they are called *knowledge sets*, while the elements of $\mathcal{T}$ are called *knowledge bases*. We use the notation K, H, R, S, T (eventually with subscripts) for the theories. We denote $\cup\Phi = \{\alpha : \exists K \in \Phi \text{ such that } \alpha \in K\}$, and $\bigwedge\!\!\!\bigwedge \Phi = \mathrm{C}_n(\cup\Phi)$, where C_n is the operator giving the set of classical logic consequences of a set of formulas. The union of multi-sets is denoted by the symbol $\sqcup$. For convenience, we write $K \sqcup T$ instead of $\{K\} \sqcup \{T\}$, and K^n for the multi-set $\{K, K \ldots K\}$ where K appears n times.

The candidates to be merging operators are applications of the following form

$$\Delta : \mathcal{B} \times \mathcal{T} \longrightarrow \mathcal{T}$$

$\Delta(\Phi, R)$ will be denoted by $\Delta_R(\Phi)$. R represents the integrity constraints. Such applications will be called *operators*. Such an operator Δ is said to be a *merging operator* if the following postulates hold:

(IC0) $\Delta_R(\Phi) \supset R$

(IC1) $R \not\vdash \bot \quad \Rightarrow \quad \Delta_R(\Phi) \not\vdash \bot$

(IC2) $R \cup (\bigwedge\!\!\!\bigwedge \Phi) \not\vdash \bot \quad \Rightarrow \quad \Delta_R(\Phi) = \mathrm{C}_n(R \cup (\bigwedge\!\!\!\bigwedge \Phi))$

(IC3) $[R \subset K \wedge R \subset K' \wedge \Delta_R(K \sqcup K') \cup K \not\vdash \bot] \Rightarrow \Delta_R(K \sqcup K') \cup K' \not\vdash \bot$

(IC4) $\mathrm{C}_n(\Delta_R(\Phi_1) \cup \Delta_R(\Phi_2)) \supset \Delta_R(\Phi_1 \sqcup \Phi_2)$

(IC5) $\Delta_R(\Phi_1) \cup \Delta_R(\Phi_2) \not\vdash \bot \quad \Rightarrow \quad \Delta_R(\Phi_1 \sqcup \Phi_2) \supset \mathrm{C}_n(\Delta_R(\Phi_1) \cup \Delta_R(\Phi_2))$

(IC6) $\mathrm{C}_n(\Delta_{R_1}(\Phi) \cup R_2) \supset \Delta_{\mathrm{C}_n(R_1 \cup R_2)}(\Phi)$

(IC7) $\Delta_{R_1}(\Phi) \cup R_2 \not\vdash \bot \quad \Rightarrow \quad \Delta_{\mathrm{C}_n(R_1 \cup R_2)}(\Phi) \supset \mathrm{C}_n(\Delta_{R_1}(\Phi) \cup R_2)$

For an interpretation of these postulates see (Konieczny & Pino-Pérez 2002a). But note that in the works (Konieczny & Pino-Pérez 1999; 2002a) concerning the finite case, the postulate (IC3) refers to the independence of the syntax. In that case they used formulas to represent a piece of information, so it is required that the operators were invariant under logical equivalence between formulas and an equivalence between the knowledge sets. In our framework, dealing with theories, those equivalences become identities and therefore the corresponding postulate is tautological. Thus, there is a displacement in the enumeration of the postulates with respect to the work mentioned above.

Next, we are interested in studying mappings from $\mathcal{B}$ into the collection of total pre-orders over valuations (*i.e.* reflexive and transitive relations which are total). Some of these mappings will be enough to represent our merging operators. In order to define the *good* assignments we need to introduce explicitly a property which is trivially satisfied in the finite case.

Definition 1 *A total pre-order $\leq_\Phi$ over $\mathcal{V}$ is said to be* **smooth** *if for all $K \in \mathcal{T}$ and for all $M \models K$ that is not minimal in* mod (K)*, there exists $N \models K$ such that N is minimal in* mod (K) *and $N <_\Phi M$.*

Notice that, for a total pre-order $\leq_\Phi$ the smoothness condition can be expressed by the following

$$\forall K \in \mathcal{T}^*, \min(mod(K), \leq_\Phi) \neq \varnothing.$$

Definition 2 *A* **syncretic assignment** *is a mapping ($\Phi \mapsto \leq_\Phi$) that assigns to each knowledge set $\Phi \in \mathcal{B}$ a total pre-order $\leq_\Phi$ over the set of valuations $\mathcal{V}$ verifying the following conditions:*

1) If $M \models \bigwedge\!\!\!\bigwedge \Phi, N \models \bigwedge\!\!\!\bigwedge \Phi$, then $M \simeq_\Phi N$

2) If $M \models \bigwedge\!\!\!\bigwedge \Phi, N \not\models \bigwedge\!\!\!\bigwedge \Phi$, then $M <_\Phi N$

3) $\forall M \models K\ \exists N \models T$ such that $N \leq_{K \sqcup T} M$

4) If $M \leq_{\Phi_1} N$ and $M \leq_{\Phi_2} N$, then $M \leq_{\Phi_1 \sqcup \Phi_2} N$

5) If $M <_{\Phi_1} N$ and $M \leq_{\Phi_2} N$, then $M <_{\Phi_1 \sqcup \Phi_2} N$

6) $\leq_\Phi$ is smooth

The link between these assignments and merging operators is showed in the next (representation) theorem:

Theorem 3 *An application $\Delta : \mathcal{B} \times \mathcal{T} \longrightarrow \mathcal{T}$ is a merging operator if and only if there exists a syncretic assignment $\Phi \mapsto \leq_\Phi$ such that*

$$\Delta_K(\Phi) = Th(min(mod(K), \leq_\Phi)) \qquad (1)$$

Given a syncretic assignment, it is quite straightforward to see that the operator defined by the previous equality verifies the postulates of merging operator. The converse is not so straightforward. The crucial point is the definition of the total pre-order $\leq_\Phi$ associated to Φ when we have a merging operator Δ. The next definition, inspired by Pino Pérez and Uzcátegui's work about representation of nonmonotonic relations (Pino-Pérez & Uzcategui 2000), tells us how to proceed.

Definition 4 *Let Δ be a mapping of $\mathcal{B} \times \mathcal{T}$ into $\mathcal{T}$. For any $\Phi \in \mathcal{B}$ we define a relation $\leq_\Phi$ over $\mathcal{V}$ by putting*

$$M \leq_\Phi N \overset{def}{\iff} \forall K, T \in \mathcal{T}[M \models \Delta_K(\Phi) \wedge N \models \Delta_T(\Phi) \rightarrow M \models \Delta_{K \cap T}(\Phi)]$$

Most of the work will consist in showing that this relation is a total pre-order when Δ is a merging operator and that in fact this pre-order represents the operator, *i.e.* the equation 1 holds.

There are another classes of natural operators that can be characterized in terms of some family of assignments. First define the following postulate:

(IC5') $\Delta_R(\Phi_1) \cup \Delta_R(\Phi_2) \not\vdash \bot \Rightarrow \Delta_R(\Phi_1 \sqcup \Phi_2) \supset \Delta_R(\Phi_1) \cap \Delta_R(\Phi_2)$

A operator Δ is said to be **quasi-merging operator** if the postulates (IC0)-(IC4),(IC5′),(IC6),(IC7) hold.

The counterpart in terms of assignments is as follows. An assignment $\Phi \mapsto \leq_\Phi$, where $\leq_\Phi$ is a total pre-order over $\mathcal{V}$, is said to be a **quasi-syncretic assignment** if it satisfies the conditions $1-4$ and 6 given in Definition 2 plus the following:

5') If $M <_{\Phi_1} N$ and $M <_{\Phi_2} N$, then $M <_{\Phi_1 \sqcup \Phi_2} N$

Now, we can state our second representation theorem:

Theorem 5 *An operator Δ is a quasi-merging operator if and only if there exists a quasi-syncretic assignment $\Phi \mapsto \leq_\Phi$ such that*

$$\Delta_K(\Phi) = Th(min(mod(K), \leq_\Phi))$$

Two classes of merging operators: Majority and arbitration

We define in this section two classes of merging operators. The first one is the class of *majority* merging operators. The operators in this class are supposed to give account of some majority behavior in extracting information from several sources: if many sources give us a piece of information, this piece of information will persist in the result of merging.

The second class introduced is the class of *arbitration* merging operators. The operators in this class are supposed to give account of some consensual behavior in extracting information from several sources: a piece of information that is unlikely for one source of information will have a tendency to be rejected in the result of merging. Conversely, a piece of information that is likely for all the sources will have a tendency to remain in the result after merging.

We define also two classes of assignments corresponding exactly to these classes of operators. For each class we state the corresponding representation theorem.

A merging operator is said to be a *majority* merging operator iff the following postulate holds

(Maj) $\forall \Phi_1, \forall \Phi_2 \exists n$ such that $\forall K \in \mathcal{T}$, $\Delta_K(\Phi_1 \sqcup {\Phi_2}^n) \supset \Delta_K(\Phi_2)$.

Note that this new version of (Maj) is stronger than the correspondent postulate in (Konieczny & Pino-Pérez 2002a). Indeed in the finite case they are equivalent. We conjecture that in the infinite case the ancient version of (Maj) does not imply the new one.

A syncretic assignment is **majority** syncretic assignment iff the following condition holds:

7) If $M <_{\Phi_2} N$, then $\exists n, M <_{\Phi_1 \sqcup {\Phi_2}^n} N$

Theorem 6 *An operator Δ is a majority merging operator iff there exists a majority syncretic assignment such that*

$$\Delta_K(\Phi) = Th(min(mod(K), \leq_\Phi)).$$

In order to establish the postulate characterizing the merging operators having a consensual behavior we need to introduce some concepts and notation. Let A and B be two sets, we denote $A \triangle B$ their *symmetrical difference*, *i.e.* $A \triangle B = (A \setminus B) \cup (B \setminus A)$ where $A \setminus B = \{x \in A : x \notin B\}$. Let K and T be two theories, their symmetrical difference, denoted $K \diamond T$ is defined as follows:

$$K \diamond T = Th\left(mod(K) \triangle mod(T)\right).$$

A merging operator is said to be an *arbitration merging operator* if the following postulate holds:

(Arb) If $K_1 \not\subseteq K_2$, $K_2 \not\subseteq K_1$, $\Delta_{K_1}(H_1) = \Delta_{K_2}(H_2)$ and $\Delta_{K_1 \diamond K_2}(H_1 \sqcup H_2) = K_1 \diamond K_2$ then $\Delta_{K_1 \cap K_2}(H_1 \sqcup H_2) = \Delta_{K_1}(H_1)$

A syncretic assignment $\Phi \mapsto \leq_\Phi$ is said to be an *arbitration syncretic assignment* if the following condition holds

8) If $M <_{H_1} N, M <_{H_2} P, N \simeq_{H_1 \sqcup H_2} P$ then $M <_{H_1 \sqcup H_2} N$

Theorem 7 *An operator Δ is an* arbitration merging operator *if and only if there exists an* arbitration syncretic assignment $\Phi \mapsto \leq_\Phi$ *such that*

$$\Delta_K(\Phi) = Th(min(mod(K), \leq_\Phi)).$$

General construction of merging operators

In this section we consider three general methods to build merging operators from (pseudo) distances between valuations (see definition below). In fact we give methods to construct syncretic assignments and then we use the representation theorems to obtain the operators.

Let $\mathbb{R}^+$ be the set $\{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}^* = \mathbb{R}^+ \cup \{\infty\}$ with the usual order $\leq$ over elements of $\mathbb{R}$ and putting $x \leq \infty$, for any $x \in \mathbb{R}^+$. Remember that for any set $A \subseteq \mathbb{R}^+$ which is non empty, $\inf(A)$ denotes the greatest lower bound of A. We can extend the inf to non-empty subsets of $\mathbb{R}^*$ by putting

$$\inf(B) = \begin{cases} \inf(B \cap \mathbb{R}^+) & \text{if } B \cap \mathbb{R}^+ \neq \emptyset \\ \infty & \text{otherwise} \end{cases}$$

Definition 8 *A function $d : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^*$ is said to be a* pseudo-distance *if the following conditions hold:*

1. $d(M, N) = d(N, M)$.
2. $d(M, N) = 0$ *iff* $M = N$.
3. *If K and T are consistent theories, there are $M \models T$ and $N \models K$such that $d(M, N) = \inf\{d(Q, P) : Q \models T, P \models K\}$.*

We can extend d to a function $\bar{d} : \mathcal{V} \times \mathcal{T} \to \mathbb{R}^*$ in the following way:

$$\bar{d}(M, K) = \inf_{N \models K} d(M, N)$$

In turn we extend the function $\bar{d}$ to a function $\bar{\bar{d}} : \mathcal{T} \times \mathcal{T} \to \mathbb{R}^*$ as follows

$$\bar{\bar{d}}(K, S) = \inf_{N \models K} \bar{d}(N, S)$$

By an abuse of notation we will write d instead of $\bar{d}$ and instead of $\bar{\bar{d}}$. It is easy to see that

$$d(K, S) = \inf_{N \models K, M \models S} d(M, N) = \inf_{M \models S} d(M, K)$$

Notice that, by the condition 3 of the definition of pseudo-distance, the inf above is indeed a minimum.

Σ and Max operators

Let d be a pseudo-distance as defined above; we define $d_\Sigma : \mathcal{V} \times \mathcal{B} \to \mathbb{R}^*$ and $d_{max} : \mathcal{V} \times \mathcal{B} \to \mathbb{R}^*$ as follows:

$$d_\Sigma(M, \Phi) = \sum_{K \in \Phi} d(M, K)$$
$$d_{max}(M, \Phi) = \max\{d(M, K) : K \in \Phi\}$$

Now for each Φ we define two relations $\leq_\Phi^\Sigma$ and $\leq_\Phi^{max}$ over $\mathcal{V}$ as follows:

$$M \leq_\Phi^\Sigma N \quad \text{iff} \quad d_\Sigma(M, \Phi) \leq d_\Sigma(N, \Phi)$$
$$M \leq_\Phi^{max} N \quad \text{iff} \quad d_{max}(M, \Phi) \leq d_{max}(N, \Phi)$$

Proposition 9 *(i)* $\Phi \mapsto \leq_\Phi^\Sigma$ *is a majority syncretic assignment.*
(ii) $\Phi \mapsto \leq_\Phi^{max}$ *is a quasi-syncretic assignment.*

As a corollary of the previous Proposition and Theorems 5 and 6 we have the following result

Corollary 10 *(i) The operator* $\Delta^\Sigma : \mathcal{B} \times \mathcal{T} \to \mathcal{T}$ *defined by*

$$\Delta_R^\Sigma(\Phi) = Th(\min(mod(R), \leq_\Phi^\Sigma))$$

is a majority merging operator.
(ii) The operator $\Delta^{max} : \mathcal{B} \times \mathcal{T} \to \mathcal{T}$ *defined by*

$$\Delta_R^{max}(\Phi) = Th(\min(mod(R), \leq_\Phi^{max}))$$

is a quasi-merging operator.

There is an interesting property dealing with the iterative behavior of an operator that is satisfied by Δ^Σ and Δ^{max} when d is indeed a distance, *i.e.* when d satisfies the triangle inequality. To be more precise let us define first the so called *iteration property.*

Let R and T knowledge bases, Φ a knowledge set and Δ an operator, we define the sequence $(\Delta_R^n(\Phi, T))_{n \geq 1}$ in the following way:

1) $\Delta_R^1(\Phi, T) = \Delta_R(\Phi \sqcup T)$, and
2) $\Delta_R^{n+1}(\Phi, T) = \Delta_R(\Delta_R^n(\Phi, T) \sqcup T)$

The following property is called the *iteration property*

(**ICit**) If $T \supset R$ then $\exists n\ \Delta_R^n(\Phi, T) \supset T$

Theorem 11 *If the pseudo-distance* $d : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^*$ *satisfies the triangle inequality,* i.e. $d(M, N) \leq d(M, P) + d(P, N)$, *then* Δ^Σ *and* Δ^{max} *satisfy* (**ICit**).

Gmax operators

Starting from a pseudo-distance d we are going to build an arbitration syncretic assignment which induces, via the representation theorem 7, an arbitration merging operator that is actually a refinement of the operator Δ^{max}.

Definition 12 *Let* $\Phi = \{K_1, K_2, \ldots K_n\}$ *be a knowledge set. For any valuation* M *we put* $(d_1^M, d_2^M \ldots d_n^M)$ *where* $d_i^M = d(M, K_i)$, *for* $i = 1, \ldots, n$. *Let* L_M^Φ *be the list* $(d_1^M, d_2^M \ldots d_n^M)$ *ordered decreasingly. Let* $\leq_{lex}$ *The lexicographical order between lists. Finally we define the following relation:*

$$M \leq_\Phi^{Gmax} N \quad \textit{iff} \quad L_M^\Phi \leq_{lex} L_N^\Phi$$

We denote d_{Gmax} the function mapping a pair (M, Φ) to the list L_M^Φ, and we call this the distance Gmax between the valuation M and the knowledge set Φ.

Theorem 13 Δ^{Gmax} *defined by*

$$\Delta_K^{Gmax}(\Phi) = Th(\min(mod(K), \leq_\Phi^{Gmax}))$$

is an arbitration merging operator. Moreover, if the pseudo-distance d *, satisfies the triagular inequality, then* Δ^{Gmax} *satisfies* (**ICit**).

Concrete merging operators

This section is devoted to define concrete operators using the techniques explained in the previous section. Thus, first, we define some distances from which we define our operators.

Remember that in the finite case the Dalal distance (Dalal 1988) between a valuation and a theory is defined using the Hamming distance between valuations, *i.e.* the distance between M and N, is the number of propositional variables in which they differ. For instance, the Hamming distance between $(1,1,1,0,0)$ and $(1,0,1,1,0)$ is equal to 2 because they differ exactly in the second and in the fourth variables.

In the infinite case (when the number of propositional variable is infinite) we define the *generalized Dalal distance* below.

First we adopt the following notation: given a valuation M, we will write $M(i)$ instead of $M(p_i)$ the value of M in the variable p_i.

Now, the generalized Dalal distance $d_1 : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^*$ is defined by putting

$$d_1(M,N) = \sum_{i=1}^{\infty} |M(i) - N(i)|$$

This function verifies the condition 1 and 2 of pseudo-distance given at the beginning of Section in page 5. Moreover as the rank of this function is $\mathbb{N} \cup \{\infty\}$ we have that for any pair of theories T, K the set $\{d(M,P); M \models T, P \models K\}$ has a minimum, so the condition 3 is verified. Also, it is clear that d satisfies the triangle inequality.

Now we define the *discrete distance*, $d_2 : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^*$ by putting

$$d_2(M,N) = \begin{cases} 0 & \text{if}, M = N \\ 1 & \text{if}, M \neq N. \end{cases}$$

The verification that d_2 satisfies the conditions of pseudo-distance given in 5 is straightforward. The condition 3 is due to the fact that the rank of d_2 is the set $\{0,1\}$. Indeed, d_2 is a distance, *i.e.* it satisfies the triangle inequality.

It is quite interesting to notice that starting from this discrete distance we have

$$\leq_{\Phi}^{GMax} = \leq_{\Phi}^{\Sigma}$$

This is because L_{M}^{Φ} is a decreasing sequence of $1's$ and $0's$, since $d(M,K)$ is 1 or 0 for any valuation M and any knowledge base K; indeed it is equal to 1 if $M \models K$ and it is equal to 0 if $M \not\models K$. Thus, it is clear that $L_{\mathrm{M}}^{\Phi} <_{lex} L_{\mathrm{N}}^{\Phi}$ if and only if the number of $1's$ in L_{M}^{Φ} is less than the number of $1's$ in L_{N}^{Φ}, and this is equivalent to $\sum_{K \in \Phi} d(M,K) < \sum_{K \in \Phi} d(N,K)$.

Putting together this observation with Corollary 10, we have the following result

Corollary 14 *There are merging operators that are both arbitration and majority.*

The third distance we consider is the so called *Cantor distance*, $d_3 : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^*$ defined in the following way:

$$d_3(M,N) = \sum_{i=1}^{\infty} \frac{|M(i) - N(i)|}{2^i}$$

Notice that this distance gives a hierarchy over the propositional variables: the first variable is the most important and the importance decreases as the subscript of the variable increases.

First of all, let us remark this well known fact: the set of valuations with the distance d_3 (see below) is in fact isometric to Cantor's space; this is the reason to call d_3 Cantor's distance.

The conditions 1 and 2 of pseudo-distance are clearly satisfied. Also, it is straightforward to verify the triangle inequality for d_3. In order to verify the condition 3, let us notice that d_3 is a continuous function mapping the product of Cantor space by itself in $[0,1]$ with the topology inherited of $\mathbb{R}$. But the Cantor space is compact because it is an infinite product of the space $\{0,1\}$ with the discrete topology which is compact, and by the Tychonoff theorem the product of compact spaces is compact. Since $mod(T)$ is a closed set for any T, $mod(T)$ is compact. Therefore $mod(K) \times mod(T)$ is compact and so, the continuous function d_3 takes a minimum value in that set, that is to say the condition 3 holds.

Examples and observations

In order to distinguish the operators after the distance used to build them we make explicit mention of it. Thus $\Delta^{\Sigma(d_i)}$, $\Delta^{max(d_j)}$ and $\Delta^{Gmax(d_k)}$ are the operators Σ, Max and $Gmax$ built from d_i, d_j and d_k respectively where $i,j,k \in \{1,2,3\}$.

The following observation tells us that if in the knowledge set two sources totally disagree, then the Σ operator and the Max operator built from Dalal distance choose exactly the whole knowledge base representing the integrity constraints. More precisely:

Observation 15 *Let $\Phi = \{K_1, K_2, \ldots, K_n\}$ be a knowledge set such that there are K_i, K_j with $d(K_i, K_j) = \infty$. Then for any R*

$$\Delta_R^{\Sigma(d)} = R = \Delta_R^{max(d)}$$

To check this observation notice that since d_1 satisfies the triangle inequality, for any valuation M we have $d(M, K_i) = \infty$ or $d(M, K_j) = \infty$. From this, it follows that $d_{\Sigma}(M, \Phi) = \infty$ and $d_{max}(M, \Phi) = \infty$. This is the end of the verification.

Nevertheless, under the same assumptions, if there is K_i in Φ such that $d(R, K_i) < \infty$ then the operator $\Delta^{Gmax(d)}$ might give nontrivial results.

Let us illustrate the behavior of these operators ar work with the following example.

Example 16 *Let $\Phi = \{K_1, K_2, K_3, K_4\}$ be a knowledge set, where*

$$K_1 = \mathrm{C}_n(\{p_2, \neg p_3, \neg p_4\} \cup \{p_i : i \geq 6\})$$
$$K_2 = \mathrm{C}_n(\{p_1, \neg p_2, \neg p_3\} \cup \{p_i : i \geq 6\})$$
$$K_3 = \mathrm{C}_n(\{\neg p_1, p_3, p_4, p_5\} \cup \{p_i : i \geq 6\})$$
$$K_4 = \mathrm{C}_n(\{p_1, \neg p_2, p_3, \neg p_4, \neg p_5\} \cup \{p_i : i \geq 6\})$$

That is to say

$$mod(K_1) = (*, 1, 0, 0, *, \bar{1})$$
$$mod(K_2) = (1, 0, 0, *, *, \bar{1})$$
$$mod(K_3) = (0, *, 1, 1, 1, \bar{1})$$
$$mod(K_4) = (1, 0, 1, 0, 0, \bar{1})$$

where $\bar{1}$ denotes the sequence equal to 1 and $*$ denotes any value in $\{0, 1\}$ (analogously $\bar{0}$ and $\bar{*}$ denotes the infinite sequence equal to 0 and any sequence of 0's and 1's respectively) and by abuse we identify $(*, 1, 0, 0, *, \bar{1})$ with the set of models of this form, etc.

Let $R = \mathrm{C}_n(\{p_3 \wedge p_4 \wedge \neg p_5\})$ so $mod(R) = (*, *, 1, 1, 0, \bar{*})$ We proceed to find the minimum values for $d_\Sigma(M, \Phi)$, $d_{max}(M, \Phi)$ and $d_{Gmax}(M, \Phi)$ when $M \models R$ and d is one of d_i for $i = 1, 2, 3$. First notice that any model N of K_i verifies $N(i) = 1$ for all $i \geq 6$. So, the models M of R realizing the minimum values have to verify $M(i) = 1$ for all $i \geq 6$. There are four possible cases:

Case 1 $M(1) = M(2) = 1$, i.e. $M = (1, 1, 1, 1, 0, \bar{1})$.

Case 2 $M(1) = 0, M(2) = 1$, i.e. $M = (0, 1, 1, 1, 0, \bar{1})$.

Case 3 $M(1) = M(2) = 0$, i.e. $M = (0, 0, 1, 1, 0, \bar{1})$.

Case 4 $M(1) = 1, M(2) = 0$, i.e. $M = (1, 0, 1, 1, 0, \bar{1})$.

The Table 1 is very useful to calculate the Dalal distance. In the boxes of the table we find a 1 in the positions in which the models realizing the minimun differ.

In Case 1 we have that the minimum values are $d_1(M, K_1) = 2, d_1(M, K_2) = 2, d_1(M, K_3) = 2, d_1(M, K_4) = 1$ therefore $d_\Sigma(M, \Phi) = 8$, $d_{max}(M, \Phi) = 2$ and $d_{Gmax}(M, \Phi) = (2, 2, 2, 2)$.

In Case 2 we have that the minimum values are $d_1(M, K_1) = 2, d_1(M, K_2) = 3, d_1(M, K_3) = 1, d_1(M, K_4) = 2$ therefore $d_\Sigma(M, \Phi) = 8$, $d_{max}(M, \Phi) = 3$ and $d_{Gmax}(M, \Phi) = (3, 2, 2, 1)$.

In Case 3 we have that the minimum values are $d_1(M, K_1) = 3, d_1(M, K_2) = 2, d_1(M, K_3) = 1, d_1(M, K_4) = 2$ therefore $d_\Sigma(M, \Phi) = 8$, $d_{max}(M, \Phi) = 3$ and $d_{Gmax}(M, \Phi) = (3, 2, 2, 1)$.

In Case 4 we have that the minimum values are $d_1(M, K_1) = 3, d_1(M, K_2) = 1, d_1(M, K_3) = 2, d_1(M, K_4) = 1$ therefore $d_\Sigma(M, \Phi) = 7$, $d_{max}(M, \Phi) = 3$ and $d_{Gmax}(M, \Phi) = (3, 2, 1, 1)$.

From these observations follows that

$mod(\Delta_R^{\Sigma(d_1)}(\Phi)) = \{(1, 0, 1, 1, 0, \bar{1})\}$ and $mod(\Delta_R^{max(d_1)}(\Phi)) = mod(\Delta_R^{Gmax(d_1)}(\Phi)) = \{(1, 1, 1, 1, 0, \bar{1})\}$

Now we treat the discrete distance. Notice that the model M' of R defined by $M' = (0, 1, 1, 1, 0, 1, 1, 1, 1, \ldots, 1, \ldots)$ is a model of K_3, but there are no models of R which are models of K_1, K_2 or K_4.

Thus, for any model M of R different to M' we have $d_2(M, K_1) = d_2(M, K_2) = d_2(M, K_3) = d_2(M, K_4) = 1$. Moreover $d_2(M', K_1) = d_2(M', K_2) = d_2(M', K_4) = 1$ and $d(M', K_3) = 0$ From these observations we obtain
$\Delta_R^{max(d_2)}(\Phi) = R$, $mod(\Delta_R^{\Sigma(d_2)}(\Phi)) = mod(\Delta_R^{Gmax(d_2)}(\Phi)) = \{M'\}$

Finally we deal with Cantor distance. Using the previous table we proceed to find the minimum values.

Case 1 $M(2) = M(3) = 1$, i.e. $M = (1, 1, 1, 1, 0, \bar{1})$. *In this case $d_3(M, K_1) = \frac{1}{2^3} + \frac{1}{2^4}$, $d_3(M, K_2) = \frac{1}{2^2} + \frac{1}{2^3}$ $d_3(M, K_3) = \frac{1}{2} + \frac{1}{2^5}$ and $d_3(M, K_4) = \frac{1}{2^2} + \frac{1}{2^4}$. Therefore $d_\Sigma(M, \Phi) = \frac{45}{2^5}$, $d_{max}(M, \Phi) = \frac{17}{2^5}$, $d_{Gmax}(M, \Phi) = \left(\frac{17}{2^5}, \frac{3}{2^3}, \frac{5}{2^4}, \frac{3}{2^4}\right)$*

Case 2 $M(2) = 1, M(3) = 0$, i.e. $M = (0, 1, 1, 1, 0, \bar{1})$. *In this case $d_3(M, K_1) = \frac{1}{2^3} + \frac{1}{2^4}$, $d_3(M, K_2) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$, $d_3(M, K_3) = \frac{1}{2^5}$, $d_3(M, K_4) = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^4}$. Therefore $d_\Sigma(M, \Phi) = \frac{61}{2^5}$, $d_{max}(M, \Phi) = \frac{7}{2^3}$, $d_{Gmax}(M, \Phi) = \left(\frac{7}{2^3}, \frac{13}{2^4}, \frac{3}{2^4}, \frac{1}{2^5}\right)$.*

Case 3 $M(2) = M(3) = 0$, i.e. $M = (0, 0, 1, 1, 0, \bar{1})$. *In this case $d_3(M, K_1) = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$, $d_3(M, K_2) = \frac{1}{2} + \frac{1}{2^3}$, $d_3(M, K_3) = \frac{1}{2^5}$, $d_3(M, K_4) = \frac{1}{2} + \frac{1}{2^4}$. Therefore*

$mod(K_i)$ \ $mod(R)$	$(1,1,1,1,0,\bar{1})$	$(0,1,1,1,0,\bar{1})$	$(0,0,1,1,0,\bar{1})$	$(1,0,1,1,0,\bar{1})$
$(*,1,0,0,*,\bar{1})$	$0,0,1,1,0,\bar{0}$	$0,0,1,1,0,\bar{0}$	$0,1,1,1,0,\bar{0}$	$0,1,1,1,0,\bar{0}$
$(1,0,0,*,*,\bar{1})$	$0,1,1,0,0,\bar{0}$	$1,1,1,0,0,\bar{0}$	$1,0,1,0,0,\bar{0}$	$0,0,1,0,0,\bar{0}$
$(0,*,1,1,1,\bar{1})$	$1,0,0,0,1,\bar{0}$	$0,0,0,0,1,\bar{0}$	$0,0,0,0,1,\bar{0}$	$1,0,0,0,1,\bar{0}$
$(1,0,1,0,0,\bar{1})$	$0,1,0,1,0,\bar{0}$	$1,1,0,1,0,\bar{0}$	$1,0,0,1,0,\bar{0}$	$0,0,0,1,0,\bar{0}$

Table 1: Calculating Dalal distances.

$d_\Sigma(M,\Phi) = \frac{53}{2^5}$,
$d_{max}(M,\Phi) = \frac{5}{2^3}$,
$d_{Gmax}(M,\Phi) = \left(\frac{5}{2^3}, \frac{9}{2^4}, \frac{7}{2^4}, \frac{1}{2^5}\right)$

Case 4 $M(2) = 0, M(3) = 1$, i.e. $M = (1,0,1,1,0,\bar{1})$. *In this case* $d_3(M,K_1) = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}$ $d_3(M,K_2) = \frac{1}{2^3}$, $d_3(M,K_3) = \frac{1}{2} + \frac{1}{2^5}$ *and* $d_3(M,K_4) = \frac{1}{2^4}$ *Therefore*

$d_\Sigma(M,\Phi) = \frac{37}{2^5}$,
$d_{max}(M,\Phi) = \frac{17}{2^5}$,
$d_{Gmax}(M,\Phi) = \left(\frac{17}{2^5}, \frac{7}{2^4}, \frac{1}{2^3}, \frac{1}{2^4}\right)$

Finally from the previous observations we obtain:

$$mod(\Delta_R^{\Sigma(d_3)}(\Phi)) = \{(1,0,1,1,0,\bar{1})\}$$

$$mod(\Delta_R^{max(d_3)}(\Phi)) = \{(1,1,1,1,0,\bar{1}),(1,0,1,1,0,\bar{1})\}$$

$$mod(\Delta_R^{Gmax(d_3)}(\Phi)) = \{(1,1,1,1,0,\bar{1})\}$$

Final remarks and questions

We have defined merging operators in the infinite framework. This is important because in many situations we do not know in advance how many variables will be involved. So, we have to dispose of mechanisms to treat these situations.

An interesting result of studying the infinite framework for merging operators is to fix some postulates (e.g. the majority postulate). The infinite case tells us that the distances used to define operators have to be very particular: they have to satisfy the condition 3, *i.e.* a realization condition which is satisfied, for instance, by continuous functions over compact spaces. Another thing very interesting is the smoothness property which guarantees the consistency of operators defined by smooth assignments.

An interesting question is to study the behaviour of operators when we restraint the codomain and the nature of information. For instance what happen if instead of $\mathcal{T}$ (the set of all theories) we take the set of theories finitely generated? Do the representation theorems hold?

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