Wang, Xiaoming

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Florida State University

Stochastic and Probabilistic Methods in Ocean-Atmosphere Dynamics, Victoria, July 2008

Main reference

The general area of geophysical fluid mechanics is truly interdisciplinary, diseas from statistical physics area now bring applied in novel ways to inhomogeneous complex systems such as atmospheres and oceans. In the book, the basic diseas of geophysics, probability theory, information theory, nonlinear dynamics, and equilibrium statistical nucleanies are innoduced and applied to large-time softwork approaches of equilibrium statistical mechanics for geophysical flows are systematically compared and constrast of from the viewpoint of molean applied multimetization and constrast of from the viewpoint of molean applied multimetizations. Novel applications of information theory are utilized to quartify aspects Novel applications of information theory are utilized to quartify aspects freedom. The book is the first to adopt this approach and it comains may recent ideas and results. Its audience eranges from graduate students to researchers in both applied mathematics and the geophysical sciences, qualitative models, and numerical aintufations, which combine in the merging area of computational actions. Majda and Wang

Nonlinear Dynamics and Statistical Theories for Basic Geophysical Flows

Nonlinear Dynamics and Statistical Theories for Basic Geophysical Flows

Andrew J. Majda and Xiaoming Wang

resigned by Zoe Navlor



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Complete Statistical Mechanics and Emergence of Large Scale Coherent Struct

CAMBRIDGE

• Approximate PDE by ODE system.

 $\frac{d\vec{X}}{dt}=\vec{F}(\vec{X}),\ \vec{X}\in\mathcal{R}^N,\ \vec{F}=(F_1,\cdots,F_N),\quad N\gg 1,\quad \vec{X}|_{t=0}=\vec{X}_0.$

S(t): solution semi-group.

• Liouville property

$$abla_{\vec{X}}\vec{F} = \sum_{j=1}^{N} \frac{\partial F_j}{\partial X_j} = 0.$$

Example: Hamiltonian system.

• Liouville equation and its solution

$$\frac{\partial p}{\partial t} + \vec{F} \cdot \nabla_{\vec{X}} p = 0$$
$$p(\vec{X}, t) = p_0 \left(S^{-1}(t)(\vec{X}) \right)$$

Conservation

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Conservation

- Conserved quantities $E_l(\vec{X}(t)) = E_l(\vec{X}_0), \quad 1 \le l \le L$
- Statistical version of the conserved quantities $\overline{E}_{l} = \langle E_{l} \rangle_{p} \equiv \int_{\mathcal{R}^{N}} E_{l}(\vec{X}) p(\vec{X}) d\vec{X}, \quad 1 \leq l \leq L.$
- Conservation of the statistical form $\langle E_I \rangle_{p(t)} = \langle E_I \rangle_{p_0}$, for all *t*.
- Maximum entropy principle

$$\mathcal{S}(p^*) = \max_{p \in \mathcal{C}} \mathcal{S}(p), \quad \mathcal{S}(p) = -\int_{\mathcal{R}^N} p(\vec{X}) \ln p(\vec{X}) \, d\vec{X},$$

$$\mathcal{C} = \left\{ p(\vec{X}) \ge 0, \int_{\mathcal{R}^N} p(\vec{X}) \, d\vec{X} = 1, \langle E_l \rangle_p = \overline{E}_l, 1 \le l \le L \right\}$$

• Most probable pdf (Gibbs measure)

$$p^*(\vec{X}) = \mathcal{G}_{\vec{\theta}}(\vec{X}) = \mathcal{Z}^{-1} \exp(-\sum_{l=1}^L \theta_l E_l(\vec{X})), \quad \mathcal{Z} = \int_{\mathcal{R}^N} \exp(-\sum_{l=1}^L \theta_l E_l(\vec{X}))$$

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Josiah Willard Gibbs

Figure: Josiah Willard Gibbs, 1839-1903



Wang, Xiaoming wxm@math.fsu.edu Complete Statistical Mechanics and Emergence of Large Scale Coherent Structure

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Remarks on the maximum entropy principle

• Entropy is conserved in the deterministic case.



Fokker-Planck equation (Kolmogorov, Smoluchowski)

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Remarks on the maximum entropy principle

• Entropy is conserved in the deterministic case.

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Noise enhances mixing and statistical uniqueness.

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Remarks on the maximum relative entropy principle

• Relative entropy is conserved in the deterministic case.

$$\frac{d\vec{X}}{dt} = \vec{F}(\vec{X}) + \epsilon \frac{d\vec{W}}{dt}$$

• Fokker-Planck equation (Kolmogorov, Smoluchowski)

$$\frac{\partial \boldsymbol{p}}{\partial t} + \vec{\boldsymbol{F}} \cdot \nabla_{\vec{X}} \boldsymbol{p} - \frac{\epsilon^2}{2} \Delta_{\vec{X}} \boldsymbol{p} = \boldsymbol{0}$$

• Equation for the density of relative entropy $Q = -p \ln \frac{p}{q}$

$$\frac{\partial Q}{\partial t} + \nabla_{\vec{X}} \cdot (\vec{F}Q) - \frac{\epsilon^2}{2} \Delta_{\vec{X}}Q = \frac{\epsilon^2}{2p} \frac{p^3}{q^2} |\nabla_{\vec{P}}^{\vec{Q}}|^2$$

• Monotonicity of relative entropy $\mathcal{S}(p,q) = -\int p \ln \frac{p}{q}$

$$\frac{d}{dt}\mathcal{S}(p(t),q(t))\geq 0.$$

Noise enhances mixing and statistical uniqueness.

Adriaan Fokker, Max Planck, Andrey Kolmogorov, Marian Smoluchowski

Figure:
Fokker,
1887-
1972





Figure: Smoluchowski, 1872-1917







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Barotropic quasi-geostrophic equation with topography

Barotropic quasi-geostrophic equation

$$\frac{\partial \boldsymbol{q}}{\partial t} + \nabla^{\perp} \psi \cdot \nabla \boldsymbol{q} = \boldsymbol{0}, \quad \boldsymbol{q} = \Delta \psi + \boldsymbol{h}$$

q: potential vorticity, ψ : stream-function, *h*: bottom topography, $\Omega = (0, 2\pi) \times (0, 2\pi)$, per. b.c.

• Conserved quantities: kinetic energy E and total enstrophy \mathcal{E}

$$E = -\frac{1}{2|\Omega|} \int_{\Omega} \psi \Delta \psi \, d\mathbf{x},$$
$$\mathcal{E} = -\frac{1}{2|\Omega|} \int_{\Omega} q^2 \, d\mathbf{x}$$

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Finite dimensional (Galerkin) truncation (approximate PDE by ODE)

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Jule Charney, Robert Kraichnan

Figure: Jule Charney, 1917-1981

chamey.gif (GIF Image, 896x688 pixels)

http://www-das.uwyo.edu/~geerts/cwx/notes/chap10/charney.gi



Figure: Robert Kraichnan, 1928-2008



Truncated QG

• Fourier basis: $B_{\Lambda} = \left\{ \exp\left(i\vec{k}\cdot\vec{x}\right) \mid 1 \leq |\vec{k}|^2 \leq \Lambda \right\}.$

$$\begin{split} \psi_{\Lambda} &\equiv \sum_{1 \leq |\vec{k}|^2 \leq \Lambda} \hat{\psi}_{\vec{k}}(t) e^{i \vec{x} \cdot \vec{k}} = -\sum_{1 \leq |\vec{k}|^2 \leq \Lambda} \frac{1}{|\vec{k}|^2} \hat{\omega}_{\vec{k}}(t) e^{i \vec{x} \cdot \vec{k}}, \\ h_{\Lambda} &\equiv \sum_{1 \leq |\vec{k}|^2 \leq \Lambda} \hat{h}_{\vec{k}}(t) e^{i \vec{x} \cdot \vec{k}}, \\ \omega_{\Lambda} &\equiv \sum_{1 \leq |\vec{k}|^2 \leq \Lambda} \hat{\omega}_{\vec{k}}(t) e^{i \vec{x} \cdot \vec{k}} = \sum_{1 \leq |\vec{k}|^2 \leq \Lambda} (-|\vec{k}|^2 \hat{\psi}_{\vec{k}}(t)) e^{i \vec{x} \cdot \vec{k}}, \end{split}$$

$$\frac{\partial \omega_{\Lambda}}{\partial t} + P_{\Lambda} \left(\nabla^{\perp} \psi_{\Lambda} \cdot \nabla(\omega_{\Lambda} + h_{\Lambda}) \right)$$

$$\frac{\partial \hat{\omega}_{\vec{k}}}{\partial t} - \sum_{\substack{\vec{l} + \vec{m} = \vec{k}, \\ |\vec{l}|^{2} \le \Lambda, \, |\vec{m}|^{2} \le \Lambda}} \frac{\vec{l}^{\perp} \cdot \vec{m}}{|\vec{l}|^{2}} \hat{\omega}_{\vec{l}} (\hat{\omega}_{\vec{m}} + \hat{h}_{\vec{m}}) = 0.$$

Wang, Xiaoming wxm@math.fsu.edu Complete Statistical Mechanics and Emergence of Large Scale Coherent Structure

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Conservation of truncated energy and enstrophy and Liouville's property

• Truncated energy and enstrophy

$$\begin{split} E_{\Lambda} &= \frac{1}{2|\Omega|} \int |\nabla^{\perp}\psi_{\Lambda}|^2 \, d\vec{x} = \frac{1}{2} \sum_{\substack{1 \le |\vec{k}|^2 \le \Lambda}} |\vec{k}|^2 |\hat{\psi}_{\vec{k}}|^2, \\ \mathcal{E}_{\Lambda} &= \frac{1}{2|\Omega|} \int (\omega_{\Lambda} + h_{\Lambda})^2 \, d\vec{x} = \frac{1}{2} \sum_{\substack{1 \le |\vec{k}|^2 \le \Lambda}} |-|\vec{k}|^2 \hat{\psi}_{\vec{k}} + \hat{h}_{\vec{k}}|^2. \end{split}$$

• Liouville property (detailed)

$$F_j(\vec{X}) = F_j(X_1, \cdots, X_{j-1}, X_{j+1}, \cdots, X_{2M}),$$

$$\begin{split} \vec{X} &\equiv (\operatorname{Re} \hat{\psi}_{\vec{k_1}}, \operatorname{Im} \hat{\psi}_{\vec{k_1}}, \cdots, \operatorname{Re} \hat{\psi}_{\vec{k_M}}, \operatorname{Im} \hat{\psi}_{\vec{k_M}}), \\ S &= \left\{ \vec{k_1}, \cdots, \vec{k_M} \right\}: \text{ defining set of modes such that } \\ \vec{k} \in S \Rightarrow -\vec{k} \notin S, \quad S \cup (-S) = \{ 1 \leq |\vec{k}|^2 \leq \Lambda \}. \end{split}$$

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Prediction of the truncated system

Gibbs measure

$$\begin{aligned} \mathcal{G}_{\alpha,\mu,\Lambda} &= \mathcal{Z}_{\alpha,\mu,\Lambda}^{-1} \exp(-\frac{\alpha}{2} \sum_{1 \le |\vec{k}|^2 \le \Lambda} |-|\vec{k}|^2 \hat{\psi}_{\vec{k}} + \hat{h}_{\vec{k}}|^2) - \frac{\theta}{2} \sum_{1 \le |\vec{k}|^2 \le \Lambda} |\vec{k}|^2 |\hat{\psi}_{\vec{k}}| \\ &= \mathcal{Z}_{\alpha,\mu,\Lambda} \exp(-\frac{\alpha}{2} \sum_{1 \le |\vec{k}|^2 \le \Lambda} |\vec{k}|^2 (|\vec{k}|^2 + \mu) (\hat{\psi}_{\vec{k}} - \overline{\psi}_{\vec{k}})^2)) \\ &= \Pi_{j=1}^{2M} \mathcal{G}_{\alpha,\mu,\Lambda}^j (X_j) \end{aligned}$$

 α, θ Lagrange multipliers for enstrophy and energy, $\mu = \frac{\theta}{\alpha}$. • Mean state and equation

$$\begin{split} \overline{\widehat{\psi}}_{\vec{k}} &= \frac{\widehat{h}_{\vec{k}}}{\mu + |\vec{k}|^2}, \quad \overline{\psi}_{\Lambda}(\vec{x}, t) = \sum_{1 \le |\vec{k}|^2 \le \Lambda} \overline{\widehat{\psi}}_{\vec{k}} e^{i\vec{x} \cdot \vec{k}} \\ \mu \overline{\psi}_{\Lambda} - \Delta \overline{\psi}_{\Lambda} &= h_{\Lambda} \end{split}$$

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 α, θ Lagrange multipliers for enstrophy and energy, $\mu = \frac{\theta}{\alpha}$. • Mean state and equation

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Continuum limit

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Asymptotic constraints

$$\lim_{\Lambda \to \infty} \left\langle E_{\Lambda} \right\rangle_{\mathcal{G}} = E_0, \lim_{\Lambda \to \infty} \left\langle \mathcal{E}_{\Lambda} \right\rangle_{\mathcal{G}} = \mathcal{E}_0$$

$$\begin{split} \langle E_{\Lambda} \rangle_{\mathcal{G}} &= \overline{E}_{\Lambda} + E_{\Lambda}' = \frac{1}{2} \sum_{1 \le |\vec{k}|^2 \le \Lambda} \frac{|\vec{k}|^2 |\hat{h}_{\vec{k}}|^2}{(\mu + |\vec{k}|^2)^2} + \frac{\alpha^{-1}}{2} \sum_{1 \le |\vec{k}|^2 \le \Lambda} \frac{1}{\mu + |\vec{k}|^2} \\ \langle \mathcal{E}_{\Lambda} \rangle_{\mathcal{G}} &= \overline{\mathcal{E}}_{\Lambda} + \mathcal{E}_{\Lambda}' = \frac{1}{2} \sum_{1 \le |\vec{k}|^2 \le \Lambda} \frac{\mu^2 |\hat{h}_{\vec{k}}|^2}{(\mu + |\vec{k}|^2)^2} + \frac{\alpha^{-1}}{2} \sum_{1 \le |\vec{k}|^2 \le \Lambda} \frac{|\vec{k}|^2}{\mu + |\vec{k}|^2} \end{split}$$

$$\mu_{\Lambda} \to \mu_{\infty}, \alpha_{\Lambda} \to \infty.$$

 $\Delta \overline{\psi}_{\mu} + h = \mu_{\infty} \psi_{\mu}$

$$\mu_{\infty} \in (-1,\infty)$$
: $E(\overline{\psi}_{\mu_{\infty}}) = E_0.$

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Large coherent structure



 Understand the emergence and persistence of such large scale coherent structure

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Prediction

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Prediction

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Prediction

Wang, Xiaoming wxm@math.fsu.edu Complete Statistical Mechanics and Emergence of Large Scale Coherent Structure

Mathematical model

One layer model (Two dimensional fluid system for potential vorticity)

$$\frac{\partial \boldsymbol{q}}{\partial t} + \nabla^{\perp}\psi\cdot\nabla \boldsymbol{q} = \mathcal{D}(-\Delta)\psi + \mathcal{F}, \ \boldsymbol{q} = \Delta\psi + \beta \boldsymbol{y} - \boldsymbol{F}\psi + h \ \mathcal{D}(-\Delta)\psi = \sum_{j\geq 1} d_j(-\Delta)^j\psi$$

 d_1 : Ekman damping, d_2 :Newtonian viscosity, $d_j, j \ge 3$: hyper-viscosity

- Undamped/unforced setting customary
- Information theoretical approach: maximize Shannon entropy with given information
- Conserved quantity becomes constraints on ρ
- Mean field equation

$$\bar{q} = \mathcal{G}(\bar{\psi})$$

- Most of them are stable under appropriate assumptions
- Majda and W., Nonlinear Dynamics and Statistical Theories for Basic Geophysical Flows, CUP, 2006

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- Selective decay (decaying flow) (Foias-Saut, Majda-Shim-W., Montgomery, McWilliam etc)
- Large scale structure: ground energy shell

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Damped driven environment

- unresolved small scale in forcing (small scale convection on Jupiter weather layer, storms for the oceans' mixing layer)
- random small scale forcing (in Jupiter's case: predominantly positive)
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Simple model

Two dimensional Navier-Stokes equation (vorticity-stream function)

$$\begin{aligned} \frac{\partial \boldsymbol{q}}{\partial t} + \nabla^{\perp} \psi \cdot \nabla \boldsymbol{q} &= \nu \Delta \boldsymbol{q} + \mathcal{F}, \\ \Delta \psi &= \boldsymbol{q}, \\ \boldsymbol{q}|_{t=0} &= \boldsymbol{q}_0 (\geq \boldsymbol{0}) \\ \psi &= \boldsymbol{q} &= \boldsymbol{0}, \text{ on } \partial \boldsymbol{Q} \end{aligned}$$

 $\pmb{Q} = [\pmb{0},\pi]\times[\pmb{0},\pi]$



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Impulse(kick) random small scale forcing



 $\mathcal{F} = \sum_{j=1}^{\infty} \delta(t - j \Delta t) A \omega_r(\mathbf{x} - \mathbf{x}_j)$

$$\omega_r(\mathbf{x}) = \begin{cases} \left(1 - |\mathbf{x} - \mathbf{x}_j|^2 / r^2\right)^2 \\ 0, \end{cases}$$

• \mathbf{x}_j : uniform distribution on $Q_{r_0} = [r_0, \pi - r_0] \times [r_0, \pi - r_0]$

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Prediction via statistical theory (Grote-Majda)

- EEST leads to the ground state sin x sin y
- PVST or ESTP leads to sinh-Poisson

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• crude closure (tracking energy and circulation only) works very well

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Numerical results

Figure: Contour

Figure: Vorticity



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Complete Statistical Mechanics and Emergence of Large Scale Coherent Struct
Numerical results (correlation, Dirichlet quotient, energy)

Figure: Correlation and D quotient

Figure: Energy



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Decomposition of the kick as mean plus fluctuation

$$\omega_r = \bar{\omega}_r + \omega'_r, \quad \bar{\omega}_r = \mathbb{E}\omega_r$$

• cumulative forcing effect (deterministic part)

$$\lfloor \frac{t}{\Delta t} \rfloor A \bar{\omega}_r$$

• deterministic part remain order one requires

$$A \approx \Delta t$$
, or $A = c_r \Delta t$

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stochastic forcing (fluctuation part)

• cumulative forcing effect (fluctuation part)

$$\int_0^t \mathcal{F}' = A \frac{\omega_r'(1) + \dots + \omega_r'(\lfloor \frac{t}{\Delta t} \rfloor)}{\sqrt{\lfloor \frac{1}{\Delta t} \rfloor}} \sqrt{\frac{1}{\Delta t}}$$

Donsker's invariance principle

$$\int_0^t \mathcal{F}' \approx \frac{A}{\sqrt{\Delta t}} G(t) = c_r \epsilon G(t)$$
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Stochastic continuous version

• The continuous equation

$$\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q = \nu \Delta q + c_r \bar{\omega}_r + c_r \epsilon \frac{dG}{dt},$$
$$q = \Delta \psi$$

 existence and uniqueness of solutions well known, existence of invariant measure, random dynamical system, existence of random attractor well-known (Benssouson-Temam, Vishik-Fursikov, Schmalfuss, Crauel-Debussche-Flandoli...)

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• heuristic limit as $\epsilon \rightarrow 0$

$$\frac{\partial q^{0}}{\partial t} + \nabla^{\perp}\psi^{0} \cdot \nabla q^{0} = \nu \Delta q^{0} + c_{r}\bar{\omega}_{r},$$
$$q^{0} = \Delta \psi^{0}$$

• limiting behavior in time for relatively small $c_r \bar{\omega}_r$

$$\nabla^{\perp}\psi^{0}\cdot\nabla q^{0}=\nu\Delta^{2}\psi^{0}+c_{r}\bar{\omega}_{r}$$

• limiting behavior as $c_r \rightarrow 0$

$$q^0 pprox rac{C_r}{
u} (-\Delta)^{-1} (ar{\omega}_r)$$

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Heuristic limit (approximation)



 $corr(\sin x \sin y, (-\Delta)^{-1}(1)) \approx 0.99$

Wang, Xiaoming wxm@math.fsu.edu Complete Statistical Mechanics and Emergence of Large Scale Coherent Structure

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Pathwise convergence (Majda-W. 2006)

Theorem

$$\|q-q^0\|_{L^\infty(0,T;L^2(\Omega))}
ightarrow 0,$$
 a.s.

$$\frac{\partial q}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q = \nu \Delta q + c_r \bar{\omega}_r + c_r \epsilon \frac{dG}{dt}$$

• For $\tilde{q} = q - c_r \epsilon G$

$$= \frac{\partial \tilde{q}}{\partial t} + \nabla^{\perp} (\tilde{\psi} + c_r \epsilon \Delta^{-1} G) \cdot \nabla (\tilde{q} + c_r \epsilon G)$$
$$= \nu \Delta \tilde{q} + c_r \bar{\omega}_r + \nu c_r \epsilon \Delta G$$

• For $q' = \tilde{q} - q^0$

$$\frac{\partial q'}{\partial t} + \nabla^{\perp}\psi \cdot \nabla q' + \nabla^{\perp}(\psi' + c_r \epsilon \Delta^{-1}G) \cdot \nabla q^0$$

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Rate of convergence (Majda-W., 2006)

• Theorem

$$\mathbb{E}(\|\boldsymbol{q}-\boldsymbol{q}^0\|_{L^2}^2) \leq \kappa \varepsilon^2$$

• $q' = q - q^0$

 $dq' + (-\nu \Delta q' + \nabla^{\perp} \psi \cdot \nabla q' + \nabla \psi' \cdot \nabla q^{0}) dt = c_r \epsilon dG$

• Ito's formula \Rightarrow

$$\frac{d}{dt}\mathbb{E}(\|q'\|_{L^2}^2) \leq -(2\nu - \frac{c}{\nu^3}\|q^0\|_{L^2}^8)\mathbb{E}(\|q'\|_{L^2}^2) + c_r^2\epsilon^2\sum b_{\vec{k}}^2$$

where

$$G(\mathbf{x},t) = \sum b_{\mathbf{k}} e_{\mathbf{k}}(\mathbf{x}) \beta_{\mathbf{k}}(t)$$

 $\{e_{\mathbf{k}}(\mathbf{x})\}$ o.n.b., $\{\beta_{\mathbf{k}}(t)\}$ Brownians

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Convergence of attractors (Majda-W., 2006)

Theorem

$$\lim_{\epsilon \to 0} dist(\mathcal{A}_{\epsilon}(\omega), \mathcal{A}_{0}) = 0, \ a.s.$$

random dynamical system

$$\varphi: R^{+} \times \Omega \times H \to H, (t, \omega, u) \mapsto \varphi(t, \omega)u$$
$$\varphi(0, \omega) = id, \quad \varphi(t + s, \omega) = \varphi(t, \theta_{s}\omega) \circ \varphi(s, \omega)$$
$$(\Omega, \mathcal{F}, P, (\theta_{t})_{t \in R}),$$

 θ_t measure preserving, $\theta_0 = id$, $\theta_{t+s} = \theta_t \theta_s$

• random attractor $\mathcal{A}(\omega)$ (compact, measurable)

$$\varphi(t,\omega)\mathcal{A}(\omega)=\mathcal{A}(\theta_t\omega)$$

$\lim_{t\to\infty} dist(\varphi(t,\theta_{-t}\omega)B,\mathcal{A}(\omega)) = 0$

generalization of Caraballo-Langa-Robinson

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Convergence of attractors (Majda-W., 2006)

Theorem

$$\lim_{\epsilon \to 0} dist(\mathcal{A}_{\epsilon}(\omega), \mathcal{A}_{0}) = 0, \ a.s.$$

random dynamical system

$$\varphi: \mathbf{R}^+ \times \Omega \times \mathbf{H} \to \mathbf{H}, (t, \omega, \mathbf{u}) \rightarrowtail \varphi(t, \omega) \mathbf{u}$$

$$\varphi(0,\omega) = id, \quad \varphi(t+s,\omega) = \varphi(t,\theta_s\omega) \circ \varphi(s,\omega)$$
$$(\Omega, \mathcal{F}, P, (\theta_t)_{t\in R}),$$

 θ_t measure preserving, $\theta_0 = id$, $\theta_{t+s} = \theta_t \theta_s$

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Complete Statistical Mechanics Emergence of large scale coherent structure

Commutative diagram (Majda-W., 2006)

Theorem

Wang, Xiaoming wxm@math.fsu.edu Complete Statistical Mechanics and Emergence of Large Scale Coherent Structure

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• Invariant measure is unique for small data

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$$q' = q^2 - q^1$$

$$\frac{d}{dt} \|q'\|^2 \le (-2\nu - c\frac{\|\nabla q^1\|^2}{\nu}) \|q'\|^2$$

- Main ingredient: contraction, Ito+Burkholder (with mean forcing and dependent Brownian motion)
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- Large structures well predicted by equilibrium statistical theory
- Random bombardment could alter sign as long as the mean is not zero
- Different large coherent structure could emerge depending on different distribution of small scale forcing
- Generalizes to other geometry and more general one layer system, or multi-layer system
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- What if smallness assumption is violated?
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