Stationary Statistical Properties of Dissipative Systems

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Stochastic and Probabilistic Methods in Ocean-Atmosphere Dynamics, Victoria, July 2008

Outline

Logistic map

$$T(x) = 4x(1-x), x \in [0,1]$$

Figure: Sensitive dependence

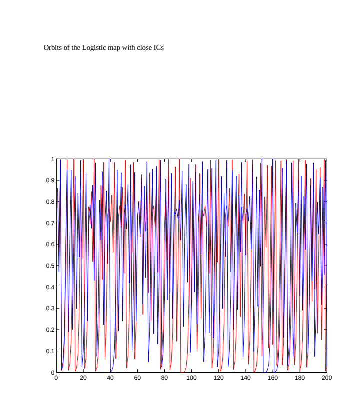
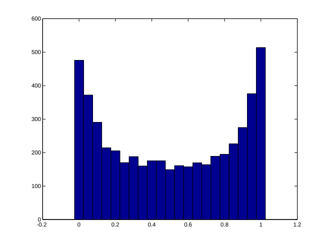


Figure: Statistical coherence



Lorenz 96 model

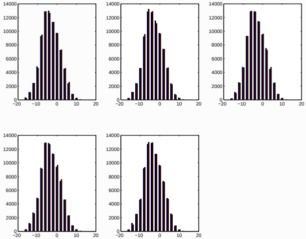
Edward Norton Lorenz, 1917-2008

$$\textit{Lorenz96}: \frac{du_j}{dt} = (u_{j+1} - u_{j-2})u_{j-1} - u_j + F$$

$$j = 0, 1, \cdots, J; \qquad J = 5, F = -12$$

Figure: Sensitive dependence

Figure: Statistical coherence



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(Sensitive dependence)

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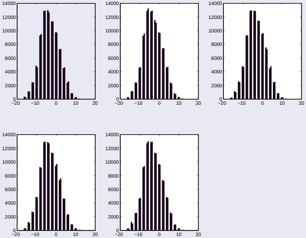

Statistical approach

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}), \quad \mathbf{u} \in H$$

- Long time average

$$< \Phi >= \lim_{T\to\infty} \frac{1}{T} \int_0^T \Phi(\mathbf{v}(t))\, dt$$

- Spatial averages

$$< \Phi >_t= \int_H \Phi(\mathbf{v})\, d\mu_t(\mathbf{v})$$

$\{\mu_t, t \geq 0\}$ statistical solutions

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Statistics Solutions

$$\{\mu_t, t \geq 0\}, \quad \frac{d\mathbf{v}}{dt} = \mathbf{F}(\mathbf{v}), \quad \{S(t), t \geq 0\}$$

- Pull-back

$$\mu_0(S^{-1}(t)(E)) = \mu_t(E)$$

- Push-forward (Φ: suitable test functional)

$$\int_H \Phi(\mathbf{v})\, d\mu_t(\mathbf{v}) = \int_H \Phi(S(t)\mathbf{v})\, d\mu_0(\mathbf{v})$$

- Finite ensemble example

$$\mu_0 = \sum_{j=1}^{N} p_j \delta_{\mathbf{v}_{0j}}(\mathbf{v}), \mu_t = \sum_{j=1}^{N} p_j \delta_{\mathbf{v}_j(t)}(\mathbf{v}), \mathbf{v}_j(t) = S(t)\mathbf{v}_{0j}$$

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Figure: Joseph Liouville, 1809-1882

Figure: Eberhard Hopf, 1902-1983

Liouville and Hopf equations

- Liouville type equation

$$\frac{d}{dt}\int_H \Phi(\mathbf{v})\, d\mu_t(\mathbf{v}) = \int_H < \Phi'(\mathbf{v}), \mathbf{F}(\mathbf{v}) > \, d\mu_t(\mathbf{v})$$

Φ good test functionals e.g.

$$\Phi(\mathbf{v}) = \phi((\mathbf{v}, \mathbf{v}_1), \cdots, (\mathbf{v}, \mathbf{v}_N))$$

- Liouville equation (finite d)

$$\frac{\partial}{\partial t} p(\mathbf{v}, t) + \nabla \cdot (p(\mathbf{v}, t)\mathbf{F}(\mathbf{v})) = 0$$

- Hopf's equation (special case of Liouville type)

$$\frac{d}{dt}\int_H e^{i(\mathbf{v},\mathbf{g})}\, d\mu_t(\mathbf{v}) = \int_H i < \mathbf{F}(\mathbf{v}), \mathbf{g} > e^{i(\mathbf{v},\mathbf{g})}\, d\mu_t(\mathbf{v})$$

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Stationary Statistics Solutions (IM)

- Invariant measure (IM) $\mu \in \mathcal{PM}(H)$

$$\mu(S^{-1}(t)(E) = \mu(E)$$

- Stationary statistical solutions: essentially

$$\int_H < F(\mathbf{v}), \Phi'(\mathbf{v}) > \, d\mu(\mathbf{v}) = 0, \forall \Phi$$

- Stationary statistical solutions /IM are not necessarily supported on steady state solutions or periodic orbits
- Uncertainty in both initial condition and parameter(s) (model error).

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Figure: George David Birkhoff, 1884-1944

Definition

μ is ergodic if $\mu(E) = 0, \;$ or 1 for all invariant sets E.

Theorem (Birkhoff's Ergodic Theorem)

If μ is invariant and ergodic, the temporal and spatial averages are equivalent, i.e.

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi(S(t)\mathbf{u})\, dt = \int_H \varphi(\mathbf{u})\, d\mu(\mathbf{u}), \; a$$

Maximum entropy for conservative case

- Maximum entropy principle: most probable pdf maximize the Shannon entropy $\mathcal{S} = -\int p \ln p$.
- Entropy is conserved in the deterministic case with Liouville property ($\nabla \cdot \mathbf{F} = 0$).
- $$\frac{d\vec{X}}{dt} = \vec{F}(\vec{X}) + \epsilon \frac{d\vec{W}}{dt}$$
- Fokker-Planck equation (Kolmogorov, Smoluchowski)
$$\frac{\partial p}{\partial t} + \vec{F} \cdot \nabla_{\vec{x}} p - \frac{\epsilon^2}{2} \Delta_{\vec{x}} p = 0$$
- Equation for the density of Shannon entropy $Q = -p \ln p$
$$\frac{\partial Q}{\partial t} + \nabla_{\vec{x}} \cdot (\vec{F} Q) - \frac{\epsilon^2}{2} \Delta_{\vec{x}} Q = \frac{\epsilon^2}{2p} |\nabla p|^2$$
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$$\frac{d}{dt} \mathcal{S}(p(t)) \geq 0.$$

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More on statistical theories for complex dynamical systems can be found:

- Majda, A.J. and Wang, X. *Nonlinear Dynamics and Statistical Theories for Basic Geophysical Flows*. Cambridge University Press, 2006.
- Majda, A.J., Abramov, R., and Grote, M., *Information theory and stochastics for multiscale nonlinear systems*, CRM monograph series, American Mathematical Society, 2005.
- Foias, C.; Manley, O.; Rosa, R.; Temam, R.; *Navier-Stokes Equations and Turbulence*, Cambridge University Press, Cambridge, UK, 2001.

Dissipative system

Definition

A dynamical system $S(t), t \geq 0$ on a phase space H is called dissipative if there exists a global attractor $\mathcal{A}$ such that

- $\mathcal{A}$ is invariant under $S(t)$.
- $\mathcal{A}$ is compact.
- $\mathcal{A}$ attracts any bounded set B in H, i.e.,

$$\lim_{t\to\infty} \operatorname{dist}(S(t)B, \mathcal{A}) = 0.$$

$\mathcal{IM}$ and attractors

Theorem ($\mathcal{IM}$ and the global attractors, W. 08)

1. *$\mathcal{IM}$ is a convex compact set (with respect to the weak topology)*
2. *supp*$\mu \subset \mathcal{A}, \forall \mu \in \mathcal{IM}$

- Singular nature of invariant measure
- Earlier work under existence of a compact absorbing set (Ciprian Foias et al)

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$\mathcal{IM}$ and time averages

Theorem ($\mathcal{IM}$ and time averages, W. 08)

Assume H reflexive, S dissipative.
$\forall \mathbf{u}_0 \in H, \forall LIM \Rightarrow \exists! \mu \in \mathcal{IM}$ *such that*

$$LIM_{T\to\infty}\frac{1}{T}\int_0^T \varphi(S(t)\mathbf{u}_0)\,dt = \int_H \varphi(\mathbf{u})\,d\mu(\mathbf{u}), \forall \varphi \in C(H).$$

Earlier work under existence of a compact absorbing set or smaller class of weakly continuous functionals (Foias et al)

Ergodicity and extremal points

Theorem (Ergodicity and extremal points, W. 08)

Let $\mathcal{IM}$ be the set of all invariant probability measures of a dissipative dynamical system $\{S(t), t \geq 0\}$. Then an invariant measure μ is ergodic if μ is an extreme point of $\mathcal{IM}$. Moreover, if the dynamical system is injective on the global attractor $\mathcal{A}$, then every ergodic invariant measure must be an extremal point of $\mathcal{IM}$.

Other versions with group (ODE) assumption is well-known.

Regular perturbation

Theorem (Conv. of $\mathcal{IM}$, regular version, W. 08)

Assume for $S(t, \epsilon)$

1. *(uniformly dissipativity) pre-compactness of* $K = \bigcup_{0<|\epsilon|\leq\epsilon_0} \mathcal{A}_\epsilon$
2. *(finite time u-conv.)* $\lim_{\epsilon\to 0} \sup_{\mathbf{u}\in\mathcal{A}_\varepsilon} \|S(t,\epsilon)\mathbf{u} - S(t,0)\mathbf{u}\|_H = 0, \forall t \geq 0.$

Then

- $\lim_{\epsilon\to 0} \mu_\epsilon = \mu_0, \mu_\epsilon \in \mathcal{IM}_\epsilon, \mu_0 \in \mathcal{IM}_0$

Beyond continuity, linear response ...

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Two time scale set-up

Two-time-scale problem: X_1, X_2: Hilbert spaces

$$
\begin{aligned}
\varepsilon(\frac{d\mathbf{u}_1}{dt} + g(\mathbf{u}_1, \mathbf{u}_2)) &= f_1(\mathbf{u}_1, \mathbf{u}_2), \ \mathbf{u}_1(0) = \mathbf{u}_{10}, \\
\frac{d\mathbf{u}_2}{dt} &= f_2(\mathbf{u}_1, \mathbf{u}_2), \ \mathbf{u}_2(0) = \mathbf{u}_{20},
\end{aligned}
$$

Limit problem ($\varepsilon = 0$)

$$
\begin{aligned}
0 &= f_1(\mathbf{u}_1^0, \mathbf{u}_2^0), \\
\frac{d\mathbf{u}_2^0}{dt} &= f_2(\mathbf{u}_1^0, \mathbf{u}_2^0), \mathbf{u}_2^0(0) = \mathbf{u}_{20}.
\end{aligned}
$$

$y = f_1(\mathbf{u}_1, \mathbf{u}_2) \Leftrightarrow \mathbf{u}_1 = F_1(\mathbf{u}_2, y)$

Conv. of stat. prop.: 2 time scale case

Theorem (Conv. of $\mathcal{IM}$, singular version, W. 08)

Assume

1. *(uniform dissipativity) pre-compactness of* $K = \bigcup_{0<\varepsilon<\varepsilon_0} \mathcal{A}_\varepsilon$
2. *(dissipativity of the limit system)* $\mathcal{A}_0$ *in* X_2.
3. *(conv. of the slow variable)* $\lim_{\varepsilon\to 0} \sup_{\mathbf{u}_2 \in \mathcal{P}_2\mathcal{A}_\varepsilon} \|\mathcal{P}_2 S(t,\varepsilon)(F_1(\mathbf{u}_2, 0), \mathbf{u}_2) - S(t,0)\mathbf{u}_2\|_{X_2} = 0, \ \forall\, t \geq 0.$
4. *(smallness of the perturbation)* $\lim_{\varepsilon\to 0} \sup_{(\mathbf{u}_1,\mathbf{u}_2)\in\mathcal{A}_\varepsilon} \|\varepsilon(\frac{d\mathbf{u}_1}{dt} + g(\mathbf{u}_1, \mathbf{u}_2))\|_{X_1} = 0.$
5. *(continuity of the slave relation)* $\mathbf{u}_1 = F_1(\mathbf{u}_2, y)$

Then

$$\mu_\varepsilon \rightharpoonup \mathcal{L}\mu_0,$$

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Rayleigh and Bénard

Figure: Lord Rayleigh (John William Strutt) 1842-1919

Figure: Henri Bénard (left), 1874-1939

Application to RBC, W., CPAM 07

Boussinesq system for Rayleigh-Bénard convection

$$\begin{aligned}\frac{1}{Pr}\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right)+\nabla p &= \Delta\mathbf{u}+Ra\,\mathbf{k}\theta, \quad \nabla\cdot\mathbf{u}=0, \quad \mathbf{u}|_{z=0,1}=0,\\ \frac{\partial\theta}{\partial t}+\mathbf{u}\cdot\nabla\theta-u_3 &= \Delta\theta, \quad \theta|_{z=0,1}=0,\end{aligned}$$

Infinite Prandtl number model for convection

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RBC set-up and numerics

Figure: RBC set-up

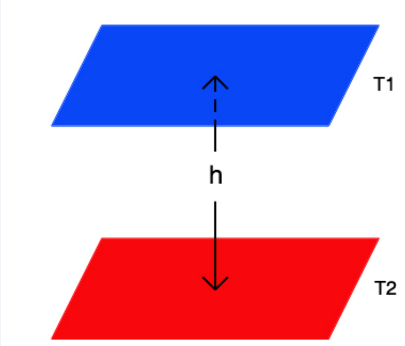

Figure: Numerical simulation

(∞ Pr. simulation)

Possible deficiency of classical schemes

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}), \quad \mathbf{u} \in H$$

- classical scheme of order m

$$\|\mathbf{u}(n\Delta t) - \mathbf{u}^n\| \leq C(n\Delta t)\Delta t^m$$

- Dependence on T

$$C(T) = \exp(\alpha T)$$

- Error in approximation of long time averages

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Difficulty with large Rayleigh number

Infinite Pr number model

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A: Stokes operator

Alternative form with $s = Ra\, t$

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Semi-implicit scheme

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Nusselt number (recast)

- For the infinite Prandtl number model

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\begin{aligned}
Nu &= \sup_{\theta_0 \in L^2} \limsup_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t \int_\Omega |\nabla T(\mathbf{x},s)|^2 \, d\mathbf{x}ds, \\
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- Convergence of Nusselt number

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