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Parabolic systems with rough initial data

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Contents:

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1.) Basics on parabolic equations

The heat equation

A function $u : \mathbb{R}^n \times \mathbb{R}_0^+ \to \mathbb{R}$ solves the heat equation with initial value $u_0 : \mathbb{R}^n \to \mathbb{R}$ if

$$(\partial_t - \Delta)u(x,t) = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,\infty) \quad \text{and}$$
$$u(\cdot,0) = u_0.$$

If for $f : \mathbb{R}^n \times \mathbb{R}_0^+ \to \mathbb{R}$

$$(\partial_t - \Delta)u(x,t) = f(x,t) \quad \text{and} \quad u(\cdot,0) = u_0$$

then u solves the inhomogeneous heat equation

Properties:

1) Fundamental solution $\Phi(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$ solves the heat equation with $u_0 = \delta_0$
(Note: For every $t > 0$, $\Phi(\cdot,t) \in C^\infty(\mathbb{R}^n,\mathbb{R})$)

2) Solution of the inhomogeneous equation
($u_0 \in C^0 \cap L^\infty(\mathbb{R}^n)$, $f \in C^2(\mathbb{R}^n \times [0,\infty))$)

$$\begin{aligned} u(x,t) =& \int_{\mathbb{R}^n} \Phi(x-y,t)u_0(y)dy \\ &+ \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)dy \; ds \\ =& Su_0(x,t) + Vf(x,t) \quad \text{(Duhamel's principle)} \end{aligned}$$

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3) Solutions are smooth for $t > 0$. In fact, for every $t > 0$ fixed, the map

$$x \to u(x,t)$$

is analytic. This is not true in general for the map

$$t \to u(x,t).$$

4) Let $T > 0$. Solutions are unique in the class of all functions $u \in C^{2,1}(\mathbb{R}^n \times (0,T]) \cap C^0(\mathbb{R}^n \times [0,T])$ satisfying

$$|u(x,t)| \leq Ae^{a|x|^2} \quad (x \in \mathbb{R}^n, \quad t \in [0,T]),$$

for constants $A, a > 0$.

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Constant coefficients:

Now consider the more general equation

$$(\partial_t - \sum_{i,j=1}^{n} a_{ij}\partial_{ij}^2)u(x,t) = f(x,t) \quad \text{and} \quad u(\cdot,0) = u_0,$$

with $a_{ij} = a_{ji} = const.$ and for some $\lambda > 0$ and all $\xi \in \mathbb{R}^n$

$$\sum_{ij} a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2.$$

Nash '58, Fabes & Stroock '86: The fundamental solution Φ_a satisfies (for a constant $\delta > 0$)

$$|\partial_t^l D^\gamma \Phi_a(x,t)| \leq c_{\gamma,l} t^{-\frac{n+2l+|\gamma|}{2}} e^{-\delta\frac{|x|^2}{t}}$$

$\Rightarrow$ all results from heat equation carry over to this case

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Fully nonlinear equations:

Consider the initial value problem

$$\begin{aligned} u_t =& F(x,t,u,Du,D^2u) \quad \text{in} \quad \mathbb{R}^n \times (0,T), \qquad (1) \\ u(\cdot,0) =& u_0, \end{aligned}$$

where $F : \mathbb{R}^n \times \mathbb{R}_0^+ \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$ is elliptic and smooth, i.e. there exists $\lambda > 0$ such that

$$\frac{\partial F}{\partial q_{ij}}(x,t,z,p,q)\xi_i\xi_j \geq \lambda|\xi|^2.$$

General result: For $u_0 \in C^\infty(\mathbb{R}^n)$ there exists a number $T > 0$ and a smooth solution u of (1).

Question: Can we do better than $u_0 \in C^\infty(\mathbb{R}^n)$?

2.) Geometric motivations

Smoothing of non-smooth geometric objects

a) Submanifolds

Let $F : \Sigma^n \to R^{n+1}$ be a smooth immersion (i.e. the differential is injective). We have the induced metric

$$g_x(v, w) = \langle DF(v), DF(w) \rangle$$

and the mean curvature

$$H\nu = \sum_{i,j} g^{ij} \partial^2_{ij} F,$$

where ν is the normal vector

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What about non-smooth immersions (not C^2)?

Example: A cone with vertex at the origin.

The standard immersion is Lipschitz but not C^2.
In this situation the mean curvature is only defined almost everywhere.

Question: Can we approximate a Lipschitz immersion F_0 by smooth ones?

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Idea: Use parabolic equation (here mean curvature flow)

$$(MCF) \quad \partial_t F = H\nu \quad \text{on} \quad \Sigma \times (0,T) \quad \text{and} \quad F(\cdot,0) = F_0$$

Problem: Not elliptic (invariant under tangential diffeomorphisms)

Here: $M^n = \mathbb{R}^n$, $F_0(x) = (x, f_0(x))$ $(f_0 : \mathbb{R}^n \to \mathbb{R})$
(MCF) equivalent to

$$(GMCF) \qquad \partial_t f = g^{ij}\partial^2_{ij} f, \quad f(\cdot,0) = f_0,$$

where $g_{ij} = \delta_{ij} + \partial_i f \partial_j f$.
Elliptic if $||\nabla f||_{L^\infty} \leq c$.

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Known results:

1) $m = 1$ Ecker-Huisken '91: f_0 locally Lipschitz $\Rightarrow$ $\exists$ global smooth solution (see also Evans-Spruck)
2) $m \geq 1$ M.T. Wang '04: M compact, $||\nabla f_0||_{L^\infty}$ locally small $\Rightarrow$ $\exists$ global smooth solution
3) $m = n$ Chau-Chen-He '09: $f_0 = \nabla u_0$, $-(1-\delta)id \leq \nabla^2 u_0 \leq (1-\delta)id$ $\forall$ $\delta > 0$ $\Rightarrow$ $\exists$ global smooth solution of Lagrangian MCF
4) for some n, $m > 1$ Lawson-Osserman '77: $\exists$ minimal graph which is Lipschitz but not C^1

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b) General manifolds

Let (M, g) be a smooth Riemannian manifold. Then we can define the (Ricci-) curvature of the manifold. In "nice" coordinates it is of the form

$$Ric_{ij} = -\frac{1}{2}\Delta_g g_{ij} + l.o.t.$$

Hence we need at least $g \in C^2$.

Question: Can we approximate L^∞-metrics by C^∞ ones?

Ricci-flow:

(RF) $\partial_t g = -2Ric_g$ on $M \times (0, T)$ and $g(\cdot, 0) = g_0 \in L^\infty$

Problem: Not elliptic (invariant under diffeomorphisms)

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Here: Ricci-DeTurck flow on $(\mathbb{R}^n, \delta)$

$$(RDF) \qquad \begin{aligned} \partial_t g_{ij} =& g^{ab}\nabla^2_{ab} g_{ij} + g^{-1} \star g^{-1} \star \nabla g \star \nabla g \\ g(\cdot, 0) =& g_0, \end{aligned}$$

where $g_0 \in L^\infty(\mathbb{R}^n)$.
Defining $h = g - \delta$ we calculate

$$(\partial_t - \Delta)h_{ij} = \nabla_a\Big((g^{ab} - \delta^{ab})\nabla_b h_{ij}\Big) + R_0[h],$$

where

$$R_0[h] = \Big(1 + (\delta + h)^{-1}(\delta + h)^{-1}\Big)\nabla h \star \nabla h$$

Previous results: Simon '04: local existence for L^∞-perturbations of C^2-metrics
Schnürer, Schulze & Simon '08: C^0-perturbation of δ on $\mathbb{R}^n$

3.) Formulation of results

Remember: Goal was to study for which initial data u_0 do we get a local solution $u : \mathbb{R}^n \times [0, T) \to \mathbb{R}$ of

(1) $\quad u_t = F(x, t, u, Du, D^2u) \;$ in $\; \mathbb{R}^n \times (0, T), \quad u(\cdot, 0) = u_0,$

where F is smooth and elliptic.

Case 1: compare with (GMCF)

$$F(x, t, u, Du, D^2u) = a_{ij}(x, t, u, Du)\partial^2_{ij}u + f(x, t, u, Du)$$

where $a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$ and $a_{ij}, f \in C^1$ bounded, with uniformly continuous and bounded derivatives.

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Theorem 1. *Let $u_0 \in C^1(\mathbb{R}^n)$ have a uniformly continuous and bounded first derivative. Then there exists $T = T(u_0) > 0$ and a unique, analytic solution $u : \mathbb{R}^n \times (0, T) \to \mathbb{R}$ of (1). Moreover u depends analytically on u_0.*

Bem.: Theorem 2 remains true for weighted Lipschitz-perturbations of u_0 as above. In particular this applies to $(GMCF)$ for arbitrary $m \geq 1$.

Theorem 2. *There exists $\varepsilon_0 > 0$, such that for all $u_0 \in C^{0,1}(\mathbb{R}^n)$ with $||u_0||_{C^{0,1}} < \varepsilon_0$ there exists a unique, global and analytic solution of (1).*

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<u>**Case 2:**</u> compare with (RDF), harmonic map flow

$$F(x,t,u,Du,D^2u) = \partial_i(a_{ij}(x,t,u)\partial_j u) + b_{ij}(x,t,u)\partial_i u \partial_j u + f(x,t,u)$$

a_{ij}, b_{ij}, f as in Case 1

Theorem 3. *Let u_0 be uniformly continuous with $|u_0(x) - u_0(y)| \leq c(|x-y|+1)\ \forall x, y \in \mathbb{R}^n$. Then there exists $T > 0$ and a unique, analytic soln $u : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ of (1).*

<u>Remark.:</u>

Theorem 4 true for small L^∞-perturbations of u_0 as above.

Theorem 4. $\exists\, \varepsilon_0 > 0$, *s.t.* $\forall\, u_0 \in L^\infty(\mathbb{R}^n)$ *with* $||u_0||_{L^\infty} < \varepsilon_0$ *there exists a unique, global, analytic solution of* (1).

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Case 3: general elliptic F with uniformly continuous and bounded second derivatives

Theorem 5. *Let $u_0 \in C^2$ with uniformly continuous and bounded 2nd derivatives. There exists $T > 0$ and a unique, analytic solution $u : \mathbb{R}^n \times (0, T) \to \mathbb{R}$ of (1).*

Remark.: Further results for systems of higher order (Willmore flow, surface diffusion, etc.)

4.) Proofs: Navier-Stokes (Koch-Tataru 2001)
Sketch of proof of Thm. 2 for (GMCF):

Rewrite (GMCF) as

$$(\partial_t - \Delta)f = (g^{ij} - \delta^{ij})\partial^2_{ij} f =: M[f], \quad f(\cdot, 0) = f_0$$

and consider the corresponding integral equation

$$f(x,t) = Sf_0(x,t) + VM[f](x,t),$$

where

$$Sf_0(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) f_0(y) dy,$$
$$VM[f](x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) M[f](y,s) dy \ ds.$$

Define Banach space

$$X = \{f| \ ||\nabla f||_{L^\infty(\mathbb{R}^n \times \mathbb{R}^+)} + \sup_{x\in\mathbb{R}^n} \sup_{R>0} R^{\frac{2}{n+4}} ||\nabla^2 f||_{L^{n+4}(B_R(x)\times(\frac{R^2}{2},R^2))} < \infty\}$$

<u>Remark:</u> X, $\dot{C}^{0,1}$ and (GMCF) invariant w.r.t. the scaling

$$f_\lambda(x,t) = \lambda^{-1} f(\lambda x, \lambda^2 t) \quad (\lambda > 0)$$

<u>Claim:</u> $\exists\ \delta, \varepsilon_0 > 0$ s.t. $\forall\ f_0$ with $||\nabla f_0||_{L^\infty} < \varepsilon_0$ the map

$$F_{f_0} : X^\delta \to X^\delta = \{f \in X| \ ||f||_X < \delta\}$$
$$F_{f_0}(f) = S f_0 + VM[f]$$

is a contraction.

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Step 1: $||Sf_0||_X \le c||\nabla f_0||_{L^\infty}$
Follows from the heat kernel estimates.
Step 2: For every $f \in X^\gamma$ $(\gamma < 1)$:

$$||M[f]||_Y \le c||f||_X^2,$$

where

$$Y = \{g| \sup_{x\in\mathbb{R}^n} \sup_{R>0} R^{\frac{2}{n+4}}||g||_{L^{n+4}(B_R(x)\times(\frac{R^2}{2},R^2))} < \infty\}.$$

Follows from

$$||(g^{ij} - \delta^{ij})\nabla^2_{ij}f||_Y \le c||g^{-1} - \delta||_{L^\infty}||f||_X \le c(||\nabla f||_{L^\infty})||f||_X^2.$$

Step 3: For every $M \in Y$ we have

$$||VM||_X \le c||M||_Y.$$

Remark: Theorem gives the existence of global unique and analytic self-similar solutions for self-similar Lipschitz initial data with small norm

Analyticity in x and t: (cf. Angenent '90)

Let $a \in \mathbb{R}^n$ be close to 0 and $\tau \in \mathbb{R}$ close to 1. Same arguments as above apply to the equation

$$(\partial_t - \Delta)f = \tau M[f] + (\tau - 1)\Delta f - a\nabla f$$

If f is the unique solution of (GMCF) $f_{a,\tau}(x,t) = f(x - at, \tau t)$ uniquely solves the above. By the analytic implicit function theorem $f_{a,\tau}$ depends analytically on a, τ $\Rightarrow$ Claim!

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Remarks on Thm. 1:
Proof uses almost the same Banach space and a localization argument + the estimate of Fabes & Stroock.

Current Projects:

1) Do everything on manifolds (compact or volume doubling, i.e. $vol(B_{2R}(x)) \leq C vol(B_R(x))$).
Applications: for example, Local well-posedness for mean curvature flow for initial data which are Lipschitz-perturbations of C^1-immersions

2) In case 2 allow BMO initial data (Harmonic map flow)