Contents

1. Pressure and codons (Problem 85)

Dan Thompson gave a "Mini-presentation on Bowen Notebook Problem 85". This is the succinct problem/question:

Codon frequencies via equilibrium states for "some potential"?

Very briefly: in a recent paper, Thompson and Koslicki used a finite, small-in-large-scale version of topological pressure pressure to distinguish coding sequences in the genome of humans (and some other species) from the so-called "junk DNA" not directly transcribed to code proteins. Thompson also cited related work of a group including Mike Shub, and of Bruno Cessac.

See the mini-presentation for more.

2. The Entropy Conjecture (Problem 12)

Shub's famous and longstanding Entropy Conjecture holds that for any C^1 diffeomorphism of a compact manifold, the entropy is bounded below by $\log(\rho)$, where ρ is the spectral radius of the induced action on homology.

Mike Shub gave at the blackboard a presentation which followed fairly closely his succinct file, "Remarks on the history of the Entropy Conjecture". He described how the conjecture was born, and a significant comment of Bowen, which facilitated a key transition in the formulation of his conjecture.

See Shub's "Remarks ... " for more.

3. "Classify symbolic systems with specification" (Problem 32)

"Classify symbolic systems with specification": this was Bowen's Problem 32. Vaughn Climenhaga and Dominic Kwietniak gave presentations on this.

Climenhaga described some key advances on systems with specification since Bowen, due to Bertrand (1988), Fiebig and Fiebig (1992) and Thomsen (2006).

Kwietniak offered a program to show that classification of symbolic systems with specification up to topological conjugacy is maximal among the Borel equivalence relations with countable equivalence classes (under the preordering of Borel reducibility). Informally, this would imply that any dynamical classification problem with countable equivalence classes can be reduced to the problem of classifying systems with specification up to topological conjugacy. The idea is to find a class of systems with specification for which classification of their Markov boundaries (introduced by Klaus Thomsen) implies topological conjugacy.

In contrast, by Mike Hochman's work (as Kwietniak noted), given Bernoullicity of the unique measure of maximal entropy for subshifts with specification (with entropy then being a complete invariant of measurable conjugacy w.r.t. these measures), topological entropy is a complete invariant for the following relation: Borel conjugacy of the systems restricted to the complement of the periodic points. The Bernoullicity statement is a special case of Thm 1.1(iv) in Vaughn Climenhaga's paper "Specification and towers in shift spaces", https://arxiv.org/abs/1502.00931.

Discussion was lively. For example, a free thinker asked why specification was of interest. Another offered a classification program which could not succeed if Kwietniak's program can be carried through.

4. Relative equilibrium states

Jisang Yoo gave a presentation describing progress on the understanding of relative equilibrium staets for a factor code $\pi : X \to Y$ from an irreducible shift of finite type: π is a composition of a class degree one code followed by a finite to one code from a sofic shift. Consequetly, any potential function of sufficient regularity lifts to a unique measure of maximal relative entropy. Yoo asked if there is any generalization of this result for $\mathbb{Z}^d$ SFTs for $d > 1$.

For more detail, see Yoo's presentation, "Generalizing the uniqueness of equilibrium states in a conditional setting", the references cited there, and his paper "Decomposition of infinite-to-one factor codes and uniqueness of relative equilibrium states" (https://arxiv.org/abs/1705.00448).

5. Geodesic flows as Smale flows (Problem 6)

Bowen's Problem 6 ("Zeta function for Axiom A flows and systems") included a request for connection with geodesic results, and a question about meromorphic extension of the zeta function. For Axiom A flows, Pollicott proved in 1986 that the zeta function has a meromorphic extension to the region of the complex plane with real part greater than $h - \epsilon$, where h is the entropy of the flow. (Extension to the whole complex plane was obtained by Fried for analytic flows, and by Guilietti-Liverani-Pollicott for C^∞ contact flows, which includes the case of geodesic flow on a closed negative curvature Riemannian manifold).

Dan Thompson described his joint work with Constantine and Lafont studying geodesic flows of CAT(-1) spaces as Smale flows, which was being written up at the time of the presentation. For this class of geodesics flows, they prove the existence of the same type of symbolic dynamics that were obtained by Bowen in the case of geodesic flow on a closed negative curvature Riemannian manifold. This result allows Pollicott's results on the domain of the meromorphic extension of zeta functions for Axiom A flows to be extended to this more general setting. A key technical issue is to ensure that the roof function in the symbolic dynamics construction can be taken Lipschitz. This is not known for general Smale flows, but there is additional geometric structure that can be exploited in the CAT(-1) geodesic flow setting. This is required to apply Pollicott's work at full strength. As well as the results on zeta functions, the symbolic dynamics yields consequences such as the Bernoulli property and the Central Limit Theorem for the Bowen-Margulis measure (the measure of maximal entropy for the geodesic flow).

Thompson posed the problem of investigating rates of mixing for the Bowen-Margulis measure of these flows. Perhaps one should expect exponential mixing like in the Riemannian case (proved by Liverani for contact flows), but fundamental new theory would be required to prove this.

Previous work by Constantine-Lafont-Thompson showed that geodesic flows on CAT(-1) spaces have the weak specification property, and they used this to prove results on (weighted) equidistribution of periodic orbits and equilibrium states. For more information, see the paper "The weak specification property for geodesic flows on CAT(-1) spaces" (https://arxiv.org/abs/1606.06253). This paper was also heavily influenced by Bowen; he proved the specification property and equidistribution of periodic orbits for geodesic flow on negative curvature manifolds in work published in 1972, and established the existence of symbolic dynamics in that setting in work published in 1973. The Constantine-Lafont-Thompson project is following the same trajectory 45 years later in this more general setting.

6. Almost specification and unique measures of maximal entropy

Ronnie Pavlov gave background and (see below) one precise version of the question: how much can the specification property be relasxed and still guarantee there is a unique measure of maximal entropy (MME)?

A subshift X has *almost specfication* for a mistake function $g : \mathbb{N} \to \mathbb{N}$ if for all words $w_1, \dots, w_k$ in the $L(X)$ (the language of X) there exist words $v_1, \dots, v_k$ in $L(X)$ such that

1. For all i, v_i and w_i have equal length.
2. For all i, v_i and w_i differ in at most $g(|w_i|)$ letters (i.e., v_i is copied from w_i with at most $g(|w_i|)$ "mistakes").
3. The word $v_1 v_2 \dots v_k$ is in $L(X)$.

For example, β-shifts have almost specification with $g = 1$ (i.e., g is the constant function 1). Pavlov has shown there exist subshifts with $g = 4$ (!) which do not have a unique MME. Climenhga and Pavlov have shown $g = 1$ does guarantee a unique MME.

Pavlov asked for the boundary constant: does $g = 2$ guarantee a unique MME? $g = 3$?

Answering a question, Pavlov asserted he could adapt his construction to produce an arbitrarily large finite number of ergodic MMEs for a subshift satisfying almost specification with $g = 4$. This leaves another open question: which mistake functions g guarantee there are only finitely many ergodic MMEs?

For more, see the Climenhaga-Pavlov paper "One-sided almost specification and intrinsic ergodicity" (https://arxiv.org/abs/1605.05354) and its references.

7. Symbolic codings for Vershik maps

Karl Petersen posed the problem: when does a Vershik map on a Bratteli diagram X admit a symbolic coding?

Here, the lexicographic (Vershik) map is not required to have unique maximal and minimal paths, or to be continuous. A symbolic coding here means a factor map from the Vershik map to a subshift which is injective on the complement of a set which has measure zero for every nonatomic invariant Borel probability. Such a coding maps a point to its itinerary through a finite Borel partition, and the system is called "essentially expansive". Variants of the question might require the finite partition to be a partition of X into clopen sets, or to be adapted only to a single measure on X,

Petersen mentioned work by Xavier Méla (coding the Pascal graph with the left-right ordering, using the partition according to the first edge); generalizations by Frick; by Frick-Petersen-Shields (coding the Pascal with any ordering, using the first three edges); and by Downarowicz-Maass (a continuous Vershik map on a Bratteli diagram with a uniformly bounded number of vertices at each level defines either a subshift or an odometer). How can one determine from the graph whether the system is essentially expansive?

8. Hochman's Speed Date

Mike Hochman gave several open problems in a "speed date", giving not so much definition and background so as to give more problems.

(1) For $k < d$, the automorphism group of a full $\mathbb{Z}^k$ shift on two symbols embeds as a subgroup in the automorphism group of a full $\mathbb{Z}^d$ shift on two symbols. Is there an embedding if $k > d$?

(2) The set of nonexpansive directions for an infinite $\mathbb{Z}^2$ subshift can be any nonempty closed set of directions. But, what can this set be if the subshift is required to be minimal?

(3) (Question of B. Weiss) Suppose T is a homeomorphism of a compact metric space X, and for all (x, y) in $X \times X$, the point (x, y) is forward or backward recurrent under $T \times T$. Does this force the topological entropy of T to be zero? (If "forward or backward" is replaced with "forward", the answer is Yes.)

(4) Let X be a $\mathbb{Z}^2$ SFT with block gluing at seperation n^α: that is there exists a positive constant C such that for any pair of $n \times n$ words of X, there is a point of X in which they occur with separation at most Cn^α. In an arxiv post, Gangloff and Sablik produce a positive constant κ such that $\alpha < \kappa$ implies the language of X is decidable (there is a Turing machine which for all words on the alphabet of X will give a definite answer as to whether the word is in the language of X). Gangloff and Sablik show for $\alpha = 1$ this property is lost.

    What is the largest number κ such that for $\alpha < \kappa$, a $\mathbb{Z}^2$ SFT with block gluing at seperation n^α must have a decidable language ?

(5) Let X be the full $\mathbb{Z}$ shift on two symbols. Let Y be the $\mathbb{Z}$ SFT on symbols a, b, c defined by disallowing the words aa, bb, cc. Let X' be the complement in X of the periodic points. Let Y' be the complement in Y of the periodic points. With the restriction of the shift, these are self homeomorphisms of Polish spaces. There is a Borel isomorphism $\phi : X' \to Y'$ which conjugates these actions.

    Can ϕ be made continuous? (i.e., a topological conjugacy)

9. Pivot subshifts

Below is a detailed exposition of Mike Hochman for another problem he posed in the session.

A language $L \subseteq A^*$ has the k-pivot property if for every $a, b \in L$ of the same length, there is a seqeunce $a = a_1, a_2, \ldots, a_n = b$ with $a_i \in L$ and a_{i+1} differing from a_i at most in k sites.

A subshift X has the pivot property if its language $L(X)$ has the k-pivot property for some k. Note that if we required e.g. the 1-pivot property this would not be an isomorphism invariant.

Examples: full shifts; any shift with a "safe symbol".

Problem Do there exist non-strongly-mixing pivot subshifts? Do there exist minimal pivot subshifts?

Background: This problem came up while studying topological models for rigid ergodic transformations. An ergodic system $(X, \mathcal{B}, \mu, T)$ is rigid if there exists a sequence of times $n_k \to \infty$ such that $T^{n_k} f \to f$ in L^2 for every $f \in L^2$. A topological system (Y, S) is called (uniformly) rigid if there is a sequence $m_k \to \infty$ such that $T^{m_k} y \to y$ uniformly in $y \in Y$ (equivalently $T^{m_k} f \to f$ uniformly for $f \in C(Y)$). My student Uri Gabor recently proved that every rigid ergodic system has a rigid topological model. Shao and Donoso proved this independtly around the same time. Neither of the authors could prove that a minimal model Y exists.

Now, to construct a rigid system you cannot work with symbolic systems, because the only rigid symbolic systems are finite. But you can construct the "language" of the topological model inductively using ever-denser alphabets $A_n \subseteq [0,1]$, ending up with a realization on $[0,1]^{\mathbb{Z}}$. If y is a sequence in such an approximation we know that $d(T^{i+n}y, T^i y) < \varepsilon$ for some n and all i. If this system is transitive (and T^n is ergodic), then for every for word w in y appearing at i_0, we have the sequence of words appearing at $i_0 + kn$, $k \in \mathbb{N}$, and these change slowly (like in a pivot system!) and eventually reach every other word w'.

The problem is that it is hard to build languages with the property above which also are approximately minimal (give minimal subshifts in the end). The pivot property is a symbolic analog. The analogy is very imperfect - rigid systems have entropy 0 and full shifts have the pivot property! But the combinatorial challenges seem similar. So I would be interested to know what types of subshifts can be realized in this way.

10. Oriented expansive lines

John Franks noted that there are oriented and nonoriented notions of an expansive line for a $\mathbb{Z}^2$ action on a compact metric space, and that there are good reasons for considering the oriented version. He asked if there could be a classification of Z^2 SFTS for which there are finitely many nonexpansive oriented lines. (There are many algebraic systems satisfying this condition; the Ledrappier example is an examplar.)

Given the technology for constructions involving Turing machines, some pessimism was expressed by other participants about prospects for such a classification.

11. Bowen's Dream (Problem 7)

Bowen's Problem 7, "Structure of basic sets", has a part (a): "Classification via (R,A)". Mike Boyle referred to this as 'Bowen's Dream', still largely unrealized. What is it?

A square nonnegative integer matrix A can be used to define an SFT. Significant invariants of the SFT can be computed from A. Bowen noted that any expansive quotient (factor) of an SFT can be presented (up to topological conjugacy) by a pair (R, A). Here R is a relation $\sim$ on symbols of the SFT: under the factor map to the quotient system, points x and w have the same image if and only if for every n in $\mathbb{Z}$, $x_n \sim w_n$. The relation is reflexive and symmetric, but not transitive (when the quotient is not zero dimensional). The dream would be to compute properties of the quotient from (R, A) – classification? fundamental group? ... This is done for the zeta function, but (apart from the case of a zero dimensional quotient–i.e., a sofic shift) not for much more. David Fried studied these quotients in his paper "Finitely presented dynamical systems". See the comments on Problem 7 in the notebook for more, and for references.

Another question asks if the classification up to topological conjugacy from (R, A) is undecidable. This is open even in the case R is trivial – i.e., it is not known if the classification of SFTs is undecidable.