On Securitization, Market Completion and Equilibrium Risk Transfer*

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Abstract

We propose an equilibrium framework within to price financial securities written on non-tradable underlyings such as temperature indices. We analyze a financial market with a finite set of agents whose preferences are described by a convex dynamic risk measure generated by the solution of a backward stochastic differential equation. The agents are exposed to financial and non-financial risk factors. They can hedge their financial risk in the stock market and trade a structured derivative whose payoff depends on both financial and external risk factors. We prove an existence and uniqueness of equilibrium result for bond prices and characterize the equilibrium market price of risk in terms of a solution to a non-linear BSDE.

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1 Introduction and overview

In recent years there has been an increasing interest in *structured derivatives* at the interface of finance and insurance. Mortgage backed securities, risk bonds, and weather derivatives are the end-product of a financial process called securitization that transforms non-tradable risks into tradable assets. Developed in the 1970s, the original goal of securitization was to create securities based on financial assets such as receivables on mortgage or auto loans. The idea of pooling and selling insurance risk that cannot be hedged through investments in the capital markets alone has since been extended to an array of non-financial assets including aircrafts and buildings and - more recently - external risk factors such as weather and climate phenomena.

Insurance companies traditionally share their underwriting risk with re-insurers. Reinsurance can be viewed as a form of *risk sharing* where insurers conclude agreements with the purpose of an efficient redistribution of the commitments among them. The problem of efficient risk allocation in a reinsurance market has first been studied in a seminal paper by Borch (1962). His model has recently been extended in several important ways by, e.g., Hu, Imkeller & Müller (2005), Filopovic & Kupper (2006) and Barrieu & El Karoui (2005). The latter also clarify the link between risk sharing and indifference valuation for certain classes of translation invariant utilities.

Nowadays insurance companies and other institutional investors increasingly transfer their exposure to non-financial risk to the capital markets by selling structured products to customers that seek financial securities with low correlation with stock indices as additions to diversified portfolios. Structured products are typically written on non-tradable underlyings so the market for such products is generally incomplete. Typically, it is also quite illiquid with only a relatively small number of market participants. In such an environment utility indifference valuation is not always an appropriate pricing scheme: for the indifference value to be a transaction rather than mere benchmark price price the demand for an asset must come form many individuals with identical preferences and endowments while the supply must come from a single agent.

In this paper we propose a general equilibrium framework within which to price structured derivatives. Our model is flexible enough to capture both the traditional case of *risk sharing* and the more involved case of *risk transfer*. In a model of risk sharing individual risk exposures are pooled and redistributed using an auxiliary asset that follows a *forward dynamics* with a *given volatility*. In a model of risk transfer the exchange of risk takes place through trade in a derivative security that follows a *backward dynamics* with an *exogenous volatility*.

We consider a partial equilibrium model with a finite set of economic agents that have dynamic monetary utility functions, i.e., the utility of any cash amount added to a position is the cash amount itself. Following the approach of Barrieu & El Karoui (2005) we consider utility functions generated by the solution to a backward stochastic differential equation (BSDE). We assume that the agents can trade a stock and a financial security ("risk bond") whose payoff depends on a non-financial risk factor and that agents maximize their utility from terminal consumption.

Asset prices will follow an exogenous diffusion possibly depending on the external risk factor; by contrast the risk bond will be priced *endogenously* by an equilibrium condition. In order to prove existence and uniqueness of equilibrium prices, we first characterize a set of market prices of risk that are consistent with the assumption of no-arbitrage in the stock and bond market. For any such price of risk we then solve the agents' optimization problem under the assumption that the bond completes the market. The assumption of market completeness is always satisfied in a model of risk sharing; for the case of risk transfer it imposes an additional restriction that needs verification. Subsequently we characterize a candidate for the equilibrium market price of risk in terms of a solution to a BSDE. For the special case of exponential utility functions and for market prices of financial risk that satisfy a version of the Novikov condition we use the Clark-Haussmann formula to obtain an explicit representation of the equilibrium bond price volatility. From this representation we see that the market is indeed complete in equilibrium if the risk bond's payoff depends monotonically on the external risk factor. A similar condition has been derived by Horst & Müller (2006) using a different method. Our approach via the Clark-Haussmann formula appears more direct and requires weaker conditions.

Our analysis of equilibrium is based on a representative agent approach. The idea is to consider a social planer and let her share the aggregate risk among individual agents in a Pareto optimal way is standard in the framework of complete markets. A landmark paper that uses the representative agent approach to establish existence and uniqueness of the equilibrium results in a stochastic consumption/production economy with continuous trade is Karatzas et al. (1990). In their model the representative agent's preference is given by a weighted average of the individual agents' utilities where the weights are part of the solution of the equilibrium. The weights are chosen so that markets clear; the existence/uniqueness of an equilibrium is then achieved by a fixed point argument.[1] The analysis is much simpler when preferences are monetary. In that case the representative agent turns out to be independent of the equilibrium to be supported.

The application of the representative agent approach to the case of dynamic monetary utilities is new to the best of our knowledge. Filipovic & Kupper (2006) studied a static model; Barrieu & El Karoui (2004, 2005) and Hu, Imkeller & Müller (2005) considered dynamic models, but restricted themselves to the simpler case of risk sharing. Horst & Müller (2006) analyzed the equilibrium for a particular class of dynamic monetary utilities, the entropic one. Their method relied on the closed form representation of the optimal solution of an agent's optimization problem as computed by duality methods. We characterize the representative agent's preferences in terms of a convolution of individual utility functions, solve her optimization problem, and show that equilibrium prices in the representative agent economy are equilibrium prices of the underlying

[1] We notice that Karatzas et al. (1990) studied the problem of reallocating a perishable good by trading a financial security in zero net supply. Hence their model cannot be applied to our framework of risk transfer where the goal is to price a dividend paying asset in unit net supply.

competitive economy. It turns out that an equilibrium market price of risk can be characterized in terms of a solution to a BSDE. For the case of dynamic entropic risk measures we show that the equilibrium market price is risk can be characterized in terms of a solution to a BSDE with quadratic growth[2]. We establish boundedness properties of the integrand and a representation of the integrand in terms of a Lipschitz continuous function of the forward process from which we then deduce uniqueness of equilibrium in a certain class.

The model of this paper is much more general than that of Horst & Müller (2006). They considered a benchmark case with a deterministic market price of financial risk. The representative agent framework proposed in this paper allows us to go a lot further. Our approach is flexible enough to capture market prices of financial risk that are bounded functions of the stock price. It also applies to certain class of stochastic volatility models with unbounded market prices of financial risk and to the case where the market price of financial risk depends on the external risk factor. The latter captures a dependence of asset prices on non-financial sources of risk. To deal with these cases we establish existence and differentiability of quadratic BSDE results with unbounded terminal values that extend some of the results of Anlirchner, Imkeller & dos Reis (2007). We also report an array of numerical results that shed additional light on the structure of our equilibrium pricing problem and the quantitative differences between risk sharing and risk transfer. Our simulations suggest that, at least within our framework, risk transfer is more beneficial for the agents than risk sharing and that a dependence of the market price of financial risk on the external risk factor may generate a significant dependence of stock prices on non-financial risk factors.

The remainder of this paper is organized as follows. We specify our microeconomic setup in Section 2 and solve the individual agent's optimization problem in Section 3. For the special case of exponential utility function the solution is given in closed form in Section 4. Section 5 performs the representative agent analysis and gives sufficient conditions for existence and uniqueness (in a certain class) of competitive equilibria. A model with an unbounded market price of financial risk where all these assumptions can be verified is studied in Section 6. Numerical simulations of equilibrium prices and optimal utilities the are reported in Section 7; Section 8 concludes.

2 The microeconomic setup

We consider a partial equilibrium model with a finite set $\mathbb{A}$ of economic *agents*. The agents can trade a stock whose price process (S_t) follows an exogenous diffusion and a structured derivative whose payoff depends on a non-tradable risk factor described by the stochastic process (R_t). The structured derivative is in fixed supply and will be priced by equilibrium considerations. The

[2]The dynamics of an agent's optimal wealth for given pricing scheme, i.e., a given market price of external risk follows a linear BSDE. In equilibrium this *linear* BSDE turns into a *quadratic* BSDE so the analysis of equilibrium is mathematically much more involved than the utility optimization problem.

processes (S_t) and (R_t) are driven by independent Brownian motions W^S and W^R defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. More precisely, we assume that

$$\frac{dS_t}{S_t} = \mu_t^S dt + \sigma_t^S dW_t^S \tag{1}$$

where the diffusion coefficients μ^S and σ^S are adapted to the filtration generated by W^S and W^R. This allows for a feedback from the external risk factor into the dynamics of asset prices. The external risk process (R_t) is assumed to follow a Brownian motion with deterministic drift and volatility driven by W^R:

$$dR_t = a_t dt + b dW_t^R. \tag{2}$$

2.1 Payoffs and preferences

We think of the structured derivative ("bond") as being issued by an institutional investor with the intention of shifting some of its exposure to non-financial risk to the capital markets. The exposure can be direct through, for instance, a dependence of an insurer's underwriting risk on weather derivatives or indirect through a dependence of its financial risk on external events.

2.1.1 Payoffs

The bond's terminal payoff (h^I) and "yield curve" (φ^I) may depend on both S and R. This allows the issuer to adjust coupon payments in reaction to financial losses due to external events. Since the bond is in fixed supply, accumulated interest payments cannot be reinvested in the bond market in the same way interest payments are usually reinvested in a savings account. Under the simplifying assumption that the risk-free interest rate equals zero the bond's accumulated payoff H^I up to maturity is thus given by

$$H^I = h^I(T, R_T, S_T) + \int_0^T \varphi^I(u, R_u, S_u)\, du. \tag{3}$$

The income H^a of the agent $a \in \mathbb{A}$ may also be exposed to financial and non-financial risk. More specifically, we assume that

$$H^a = h^a(T, R_T, S_T) + \int_0^T \varphi^a(u, R_u, S_u) du. \tag{4}$$

Assumption 2.1. *All the payoff functions are bounded with uniformly bounded Lipschitz derivatives with respect to the non-financial risk factor.*

2.1.2 Preferences

We assume that the agents have monetary utility functions. This allows us to measure their well-being on a common scale. More specifically, we assume that - up to a change of sign - the utility function of the agent $a \in \mathbb{A}$ is given by a dynamic convex risk measure generated by the solution to a backward stochastic differential equation (BSDE); for more details on the link between BSDEs and convex measures of risk see Section 6.2 in Barrieu & El Karoui (2005)).

Definition 2.2. *(i) A backward stochastic differential equation is an equation of the form*

$$Y_t = H - \int_t^T Z_s dW_s + \int_t^T F(s, Y_s, Z_s) ds \quad (0 \leq t \leq T) \tag{5}$$

where W is a standard n-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the standard Brownian filtration $(\mathcal{F}_t)$. The time T is called the terminal time, *while the random function F and the $\mathcal{F}_T$-measurable random variable H are referred to as the* driver *and* terminal condition, *respectively.*

(ii) A solution *consists of an adapted process Y and an adapted integrand Z that satisfy the integral equation (5).*

Backward stochastic differential equations have recently attracted much attention as a mathematical framework for modelling the evolution of risk assessments and expected utilities when an agent's income at maturity is a known function of some underlying diffusion process (called the forward process). We shall assume that the dynamics of the risk assessment (Y_t^a) of the agent $a \in \mathbb{A}$ can be described by a BSDE with *driver* g^a and *terminal condition* ξ^a. More precisely,

$$-dY_t^a = g^a(t, Z_t)dt - Z_t^a dW_t \quad \text{with} \quad Y_T = -\xi^a. \tag{6}$$

The driver g^a completely specifies the agent's risk preferences. It is assumed to be convex and differentiable in z. In the absence of any trading opportunities the terminal condition $-\xi^a$ is given by the agent's initial endowment $-H^a$. In general, it will be given by the sum of the agent's random income and her gains or losses from trading.

2.2 Pricing rules and the market price of external risk

The bond price process will be derived *endogenously* by an equilibrium condition. Pricing the bond by market considerations requires the price process to be determined by a *linear* pricing rule - otherwise the agents have an incentive to trade the bond on a secondary market. A linear pricing scheme on L^2 can be identified with an 2-dimensional predictable process θ that makes the process (Z_t^θ) defined by

$$Z_t^\theta = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right) \tag{7}$$

a uniformly integrable martingale; see, e.g. Horst & Müller (2006). For any such process we denote by $\mathbb{P}^\theta$ a probability measure equivalent to $\mathbb{P}$ with density Z^θ and introduce the $\mathbb{P}^\theta$-Brownian motion

$$W_t^\theta = W_t + \int_0^t \theta_s \, ds.$$

Remark 2.3. *The density process Z^θ is a uniformly integrable martingale if, for instance, the process θ satisfies Novikov's condition or, more generally, if it belongs to the class of $\mathbb{P}$-BMO processes. Loosely speaking this means that the process θ defines a good measure change. We refer to Kazamaki (1994) for an extensive discussion of BMO-martingales.*

Following Horst & Müller (2006) we refer to the first component (θ^S) of the vector $\theta = (\theta^S, \theta^R)$ as the market price of financial risk. This process is exogenous because the agents' demand does not affect the dynamics of stock prices. It is given by

$$\theta_t^S = \frac{\mu_t^S}{\sigma_t^S}. \tag{8}$$

The second component (θ^R) is called the *market price of external risk*. It will be derived by an equilibrium approach. For a given price of external risk the initial bond price is $B_0^\theta = \mathbb{E}^\theta[H^I]$. This expectation makes sense because H^I is bounded. To exclude arbitrage opportunities bond prices (B_t^θ) need to be defined as the conditional discounted expected payoffs under $\mathbb{P}^\theta$ so

$$B_t^\theta = \mathbb{E}^\theta \left[h^I(T, R_T, S_T) + \int_0^T \varphi^I(s, R_s, S_s) ds | \mathcal{F}_t \right].$$

Representing the random variables B_t^θ as stochastic integrals with respect to the $\mathbb{P}^\theta$-Brownian motion W^θ yields an adapted process $\kappa^\theta = (\kappa^{\theta,S}, \kappa^{\theta,R})$ such that

$$\begin{aligned} B_t^\theta &= \mathbb{E}^\theta[H^I] + \int_0^t \kappa_s^\theta dW_s^\theta \\ &= \mathbb{E}^\theta[H^I] + \int_0^t \kappa_s^{\theta,S}(dW_s^S + \theta_s^S ds) + \int_0^t \kappa_s^{\theta,R}(dW_s^R + \theta_s^R ds). \end{aligned} \tag{9}$$

2.3 Equilibrium pricing rules

Our goal is to characterize a class of *equilibrium pricing rules* for the risk bond. We shall distinguish two notions of equilibrium: equilibrium in a model of *risk sharing* and equilibrium in models of *risk transfer*. In models of risk sharing the agents pool their risks and redistribute them by trading a fictitious non-dividend paying asset. In a model of risk transfer income streams are exchanged through decentralized markets by trading a dividend-paying risk bond[3].

2.3.1 Equilibrium in a model of risk sharing

Much of the mathematical finance literature is concerned with models of optimal or equilibrium risk sharing. In this section we show how this case fits within our framework. To this end, we introduce, for any given market price of risk θ a *forward* price process

$$d\bar{B}_t^\theta = dW_t^R + \theta_t^R dt, \qquad \bar{B}_0^\theta = 0. \tag{10}$$

[3]The fictitious asset can be viewed as a form of mutual contract through which the exchange of risk exposures takes place so it is natural to assume that it is in zero net supply.

We view the process $(\bar{B}_t^\theta)$ as the price process of a non-dividend paying (auxiliary, fictitious) asset in zero net supply and denote the holdings of the agent $a \in \mathbb{A}$ in the stock and fictitious asset at time t by $\pi_t^{a,1}$ and $\pi_t^{a,2}$, respectively. Introducing the additional asset completes the market because its volatility is strictly positive; in fact, it is with no loss of generality normalized to one. The gains or losses from trading are then given by

$$\bar{V}_t^{a,\theta}(\pi^a) = \int_0^t \pi_s^{a,1} dS_s + \int_0^t \pi_s^{a,2} d\bar{B}_s^\theta \tag{11}$$

and the dynamics of the resulting risk process is described by a backward stochastic differential equation

$$-d\bar{Y}_t^a(\pi^a) = g^a(t, Z_t^a)dt - Z_t^a dW_t \quad \text{with} \quad \bar{Y}_T^a(\pi^a) = -H^a - \bar{V}_T^{a,\theta}(\pi^a). \tag{12}$$

The following notion of admissible trading strategies guarantees that there is no arbitrage in the extended market and that the agent's utility for an admissible trading strategy is well defined.

Definition 2.4. *For a given pricing measure $\mathbb{P}^\theta$ we denote by $\bar{\mathbb{S}}^\theta$ the class of all trading strategies $\bar{\pi}$ that are $(\mathcal{F}_t)$-adapted and satisfy*

$$\mathbb{E}\int_0^T ||\bar{\pi}_s||^2 d\langle \bar{B}^\theta, S\rangle_s < \infty,$$

where $||\cdot||$ stands for the Euclidean norm. A trading strategy $\bar{\pi} \in \bar{\mathbb{S}}^\theta$ is admissible if the BSDE (12) has a unique solution. The set of all admissible strategies is denoted $\bar{\mathcal{S}}^\theta$.

The agent's goal is to minimize her terminal risk (maximize her terminal utility) from trading in the stock and auxiliary market. Her optimization problem is thus given by

$$\min_{\bar{\pi}^a \in \bar{\mathcal{S}}^\theta} Y_0^a(\bar{\pi}^a). \tag{13}$$

We call θ^* an equilibrium market price of risk if the optimal utilities are attained and the market clearing condition of zero total excess demand in the auxiliary asset holds at any point in time.

Definition 2.5. *A market price of external risk θ^* that makes the density process in (7) a uniformly integrable martingale is an* equilibrium market price of risk in a model of risk sharing *if there exist optimal admissible trading strategies $\tilde{\pi}^a$ for all agents $a \in \mathbb{A}$ such that the auxiliary market clears at all times in all the states of the world, i.e., almost surely*

$$\sum_{a\in\mathbb{A}} \tilde{\pi}_t^{a,2} \equiv 0 \quad (0 \le t \le T).$$

2.3.2 Equilibrium in a model of risk transfer

The goal of this paper is to go a step further and address the more involved problem of *risk transfer*. Here the exchange of risk exposures takes place through trade in a dividend-paying risk bond. In this case the bond price process follows a backward (not forward) dynamics and its volatility is endogenously determined by the equilibrium condition[4]. We denote the holdings of the agent $a \in \mathbb{A}$ in the stock and bond market at time t by $\pi_t^{a,1}$ and $\pi_t^{a,2}$, respectively, so in this case her gains from trading are given by

$$V_t^{a,\theta}(\pi^a) = \int_0^t \pi_s^{a,1} dS_s + \int_0^t \pi_s^{a,2} dB_s^\theta. \tag{14}$$

The associated risk process evolves according to the BSDE

$$-dY_t^a(\pi^a) = g^a(t, Z_t)dt - Z_t dW_t \quad \text{with} \quad Y_T^a(\pi^a) = -H^a - V_T^{a,\theta}(\pi^a). \tag{15}$$

Definition 2.6. *For a given pricing measure $\mathbb{P}^\theta$ we denote by $\mathbb{S}^\theta$ the class of all trading strategies π that are $(\mathcal{F}_t)$-adapted and satisfy*

$$\mathbb{E} \int_0^T ||\pi_s||^2 d\langle B^\theta, S\rangle_s < \infty.$$

A trading strategy $\pi \in \mathbb{S}^\theta \in$ is called admissible if the BSDE (15) has a unique solution. The set of all admissible strategies is denoted $\mathcal{S}^\theta$.

The process θ^* will be called an equilibrium market price of risk in a model of risk transfer if at any point in time the total demand for the bonds equals one, i.e., if the bond market clears.

Definition 2.7. *A market price of external risk θ^* that makes the density process in (7) a uniformly integrable martingale is an* equilibrium market price of risk in the model of risk transfer *if there exist optimal admissible trading strategies $\tilde{\pi}^a$ such that*

$$\sum_{a \in \mathbb{A}} \tilde{\pi}_t^{a,2} \equiv 1 \quad (0 \le t \le T).$$

In order to have a unified presentation and to simplify the exposition of the results we shall from now on use the notation of this subsection with the understanding that $\kappa^{\theta,S} \equiv 0$ and $\kappa^{\theta,R} \equiv 1$ in (9) in the case of risk sharing.

3 The individual optimization problems

In order to establish the existence of a competitive equilibrium we first consider a single agent's optimization problem. To this end, *we fix a market price of risk θ* and denote by $\mathbb{P}^\theta$ the corresponding pricing measure. For the optimization problem to have a solution we need to assume

[4] In a model of risk sharing it is normalized to unity.

that fluctuations in the external risk process translate into fluctuations of bond prices, i.e., that the bond price volatility does not vanish. Thus, for the rest of this section we work under the following assumption:[5]

Assumption 3.1. *After the bond has been issued the market is complete, i.e., the volatility process* $\{\kappa_t^{\theta,R}\}_{t\in[0,T]}$ *is* $\mathbb{P}$*-a.s. strictly positive.*

We consider examples where the preceding assumption can be verified for the equilibrium market price of risk below. At this point our goal is to characterize a candidate for the agents' optimal trading strategy in terms of a solution to a backward equation. To this end, we first observe that the problem of risk minimization is equivalent to minimizing the *residual risk*. For an admissible trading strategy π^a the associated residual risk $\hat{Y}_t^a(\pi^a)$ at time $t \in [0,T]$ is given by

$$\hat{Y}_t^a(\pi^a) = Y_t^a(\pi^a) + V_t^{a,\theta}(\pi^a). \tag{16}$$

With the change of variables

$$\bar{Z}_s \triangleq Z_s + \pi_s^{a,1}\begin{pmatrix} \sigma_s^S S_s \\ 0 \end{pmatrix} + \pi_s^{a,2}\kappa_s^\theta \tag{17}$$

its dynamics is captured by the backward equation

$$\hat{Y}_t^a(\pi^a) = -H^a + \int_t^T G^a(s,\pi_s,\bar{Z}_s)\,ds - \int_t^T \bar{Z}_s\,dW_s,$$

with terminal condition $-H^a$ and driver

$$G^a(s,\pi_s,\bar{Z}_s) \triangleq g^a\left(s,\bar{Z}_s - \pi_s^{a,1}\begin{pmatrix} \sigma_s^S S_s \\ 0 \end{pmatrix} - \pi_s^{a,2}\kappa_s^\theta\right) - \pi_s^{a,1}\sigma_s^S S_s\theta_s^S - \pi_s^{a,2}\kappa_s^\theta\cdot\theta_s. \tag{18}$$

Since $g^a(t,\cdot)$ depends in a strictly convex manner on the space variable, it is almost everywhere differentiable. We denote by g_{z^i} the derivative with respect to the corresponding component of the space variable $z = (z^1, z^2)$ and put

$$\tilde{\pi}_s^a = \arg\min_{\pi\in\mathbb{R}^2} G^a(s,\pi,z).$$

A candidate for the optimal trading strategy is now defined implicitly in terms of the following *first order conditions*:

$$g_{z^1}^a\left(s, z - \tilde{\pi}_s^{a,1}\begin{pmatrix} \sigma_s^S S_s \\ 0 \end{pmatrix} - \tilde{\pi}_s^{a,2}\kappa_s^\theta\right) = -\theta_s^S, \tag{19}$$

[5]This assumption is void in the case of risk sharing; it will be verified for certain classes of risk transfer models below.

and

$$g^a_{z^2}\left(s, z - \tilde{\pi}^{a,1}_s \begin{pmatrix} \sigma^S_s S_s \\ 0 \end{pmatrix} - \tilde{\pi}^{a,2}_s \kappa^\theta_s\right) = -\theta^R_s. \tag{20}$$

The two-dimensional process defined by the previous conditions is of the form $\tilde{\pi}^a_t = \Pi_t(z)$. Plugging the random function $\Pi_t(z)$ into the driver G^a we obtain that

$$G^a(t, \Pi_t(z), z) \leq G^a(t, \pi_t, z) \quad \text{for any pair } (\pi, z) \in \mathbb{R}^4. \tag{21}$$

Let us now assume that the BSDE with driver $G^a(t, \Pi_t(z), z)$ and terminal condition $-H^a$ has a unique solution (Y^a, Z^a) and define a trading strategy $\tilde{\pi}^a_t$ by

$$\tilde{\pi}^a_t = \Pi_t(Z^a_t). \tag{22}$$

This yields an optimal trading strategy provided that the BSDEs associated with the agents' optimization problems satisfy the following comparison principle.

Definition 3.2. *The BSDEs associated with the risk minimization problems satisfy a comparison principle if for any admissible trading strategy π the solutions $(\hat{Y}^a, Z^a)$ and $(\hat{Y}^a(\pi^a), Z^a(\pi^a))$ of the BSDEs with drivers $G^a(t, \Pi_t(z), z)$ and $G^a(t, \pi_t, z)$, respectively, satisfy*

$$\hat{Y}^a_0 \leq \hat{Y}^a_0(\pi^a).$$

We are now ready to state the main assumption on the agents' preferences which guarantee that the risk minimization problem has a solution.

Assumption 3.3. *(i) The first order conditions (19) and (20) have a solution.*

(ii) The BSDE with driver $G^a(t, \Pi_t(z), z)$ and terminal condition $-H^a$ has a unique solution.

(iii) The candidate process $\tilde{\pi}^a$ defined by (22) is admissible.

(iv) The BSDEs associated with an agent's optimization problem satisfy a comparison principle.

The main result of this section now follows.

Proposition 3.4. *If Assumption 3.3 is satisfied, then the agents' optimization problems have a unique solution.*

The preceding considerations suggest that a first step in solving the problem of existence of equilibria consists in establishing a-priori integrability conditions on the equilibrium market price of risk that guarantee that the BSDE with driver $G^a(t, \pi_t(z), z)$ has a unique solution. Subsequently one needs to impose restrictions on the trading strategies that guarantee a comparison principle. A final step consists of a characterization of an equilibrium market price of external risk and in verifying the a priori conditions on θ. For the entropic case the solution to the optimization problem can be given in closed form; this is the topic of the next section.

4 The case of entropic utilities

In this section we illustrate how to solve the individual optimization problem in a situation where the agents' risk preferences are described by dynamic entropic risk measures, i.e., where the drivers g^a take the quadratic form

$$g^a(t,z) = \frac{1}{2\gamma_a}||z||^2. \tag{23}$$

Here $\gamma_a > 0$ is the agent's coefficient of risk tolerance. For a given trading strategy π the associated backward equation is given by

$$-dY_t^a(\pi) = \frac{1}{2\gamma_a}||Z_t||^2 dt - Z_t dW_t \quad \text{with} \quad Y_T^a(\pi) = -H^a - V_T^{a,\theta}(\pi). \tag{24}$$

For bounded market prices of financial risk the problem of risk minimization in an entropic framework has previously been studied by, e.g. Hu, Imkeller & Müller (2005). Several other authors including Barrieu & El Karoui (2005) have studied this problem under boundedness assumption on the terminal condition, i.e., under boundedness conditions on the gains process. As soon as one relaxes these conditions the analysis becomes more involved as we show in the sequel.

4.1 The candidate strategy

In a first step we are now going to characterize a set of admissible trading strategies and identify the candidate for the optimal trading strategy.

4.1.1 The set of admissible strategies

We divide the backward equation (24) by the parameter of risk tolerance and perform the change of variable $z \to \gamma_a z$ to get a new BSDE:

$$-dY_t^a(\pi) = \frac{1}{2}||Z_t||^2 dt - Z_t dW_t \quad \text{with} \quad Y_T^a(\pi) = \gamma_a^{-1}(-H^a - V_T^{a,\theta}(\pi)).$$

With the exponential change of variables $y = \exp(Y)$ we can rewrite this as

$$y_t = e^{\zeta} - \int_t^T z_s \, dW_s \quad \text{where} \quad \zeta = \gamma_a^{-1}(-H^a - V_T^{a,\theta}(\pi)).$$

The preceding equation has a unique solution (in a certain space) given by the martingale representation theorem if e^{ζ} is square integrable. In this case

$$y_t = \mathbb{E}[e^{\zeta}|\mathcal{F}_t].$$

Moreover, the process z belongs to $L^2(dt \otimes d\mathbb{P})$ and by the Burkholder-Davis-Gundy inequality

$$\mathbb{E}\left[\sup_{u\in[0,T]} y_u\right]^2 < \infty.$$

Thus, by the boundedness of H^a the following is a set of admissible trading strategies:

$$\mathcal{S}^\theta = \left\{\pi \in \mathbb{S}^\theta : \quad \mathbb{E}\left[\exp(-2\gamma_a^{-1} V_T^{a,\theta}(\pi))\right] < \infty\right\}. \tag{25}$$

In the next subsection we identify the candidate strategy $\tilde{\pi}^a$. Subsequently we give a sufficient condition on the market price of risk that guarantees that $\tilde{\pi}^a$ belongs to $\mathcal{S}^\theta$.

4.1.2 The candidate for the optimal trading strategy

By the change of variables (17), the BSDE (24) can be transformed into a backward equation that describes the dynamics of the residual risk. Its driver is given by

$$\begin{aligned} G^a(t, \pi_t, Z_t) \quad &= \quad \frac{1}{2\gamma_a}||(z_1 - \pi^{a,1}\sigma^S S_s - \pi^{a,2}\kappa_s^{\theta,S}, z_2 - \pi^{a,2}\kappa_s^{\theta,R})||^2 \\ &\quad - \left(\pi_s^{a,1}\sigma_s^S S_s\theta_s^S + \pi_s^{a,2}\kappa_s^{\theta,S}\theta_s^S + \pi_s^{a,2}\kappa_s^{\theta,R}\theta_s^R\right). \end{aligned}$$

The corresponding first order conditions yield

$$\tilde{\pi}_s^{a,1} = \frac{\gamma_a[\theta_s^S\kappa_s^{\theta,R} - \theta_s^R\kappa_s^{\theta,S}] + \bar{z}_s^1\kappa_s^{\theta,R} - \bar{z}_s^2\kappa_s^{\theta,S}}{\sigma_s^S S_s\kappa_s^{\theta,R}} \quad \text{and} \quad \tilde{\pi}_s^{a,2} = \frac{\gamma_a\theta_s^R + \bar{z}_s^2}{\kappa_s^{\theta,R}}, \tag{26}$$

with $\bar{Z} = (\bar{z}^1, \bar{z}^2)$ being the integrand part of the unique solution to the BSDE with terminal condition $-H^a$ and linear driver

$$G^a(s, \Pi_s(Z), Z) = -z^1\theta_s^S - z^2\theta_s^R - \frac{\gamma_a}{2}[(\theta_s^S)^2 + (\theta_s^R)^2]. \tag{27}$$

4.2 Sufficient conditions for admissibility

We are now going to give sufficient conditions on the market price of risk that guarantee that the candidate for the optimal trading strategy is admissible. This means that the BSDE with the linear driver (27) and terminal condition $-H^a$ has a unique solution and that the associated terminal wealth satisfies an exponential integrability condition. To this end, we first derive the closed form representation of the optimal gains/losses from trading. The straightforward way to do this is by means of duality but we prefer to use a BSDE approach. In the light of (26) we have

$$\begin{aligned} V_T^{a,\theta}(\tilde{\pi}^a) \quad &\triangleq \quad \int_0^T \tilde{\pi}_s^{a,1} dS_s + \int_0^T \tilde{\pi}_s^{a,2} dB_s^\theta \\ &= \quad \int_0^T (\gamma_a\theta_s^S + \bar{z}_s^1) dW_s^S + \int_0^T (\gamma_a\theta_s^R + \bar{z}_s^2) dW_s^R \\ &\quad + \int_0^T \gamma_a[[(\theta_s^S)^2 + (\theta_s^R)^2] + \bar{z}^1\theta_s^S + \bar{z}^2\theta_s^R] ds. \end{aligned}$$

In view of (27)

$$\int_0^T \bar{z}_s^1 dW_s^S + \int_0^T \bar{z}_s^2 dW_s^R = -H^a + \text{const} + \int_0^T \left[-\bar{z}^1\theta_s^S - \bar{z}^2\theta_s^R - \frac{\gamma_a}{2}[(\theta_s^S)^2 + (\theta_s^R)^2]\right] ds$$

so the optimal terminal wealth is given by

$$V_T^{a,\theta}(\tilde{\pi}^a) = -H^a + \text{const} + \gamma_a \int_0^T \theta_u^S \left(dW_u^S + \frac{1}{2}\theta_u^S \, du \right) + \gamma_a \int_0^T \theta_u^R \left(dW_u^R + \frac{1}{2}\theta_u^R \, du \right), \quad (28)$$

provided the candidate strategy is admissible. In order to give a condition for admissibility, we set

$$N_T \triangleq \int_0^T \theta_u^S \left(dW_u^S + \frac{1}{2}\theta_u^S \, du \right) + \int_0^T \theta_u^R \left(dW_u^R + \frac{1}{2}\theta_u^R \, du \right).$$

From (28) we see that

$$\gamma_a^{-1}(-H^a - V_T^{a,\theta}(\tilde{\pi}^a)) = -N_T + \text{const}.$$

Thus, for $\tilde{\pi}^a$ to belong to the class (25) of admissible strategies we should have that

$$\mathbb{E}\left[\exp(-2N_T)\right] < \infty. \quad (29)$$

Lemma 4.1. *A sufficient condition for (29) is*

$$\mathbb{E}\left[\exp\left(6\int_0^T \|\theta_s\|^2 ds\right)\right] < \infty. \quad (30)$$

Proof. We have that $\exp(-2N_T) = \zeta_T \exp\left(3\int_0^T ((\theta^S)_u^2 + (\theta^R)_u^2) \, du\right)$, where the process (ζ_t) is defined by

$$\zeta_t = \exp\left(\int_0^t (-2\theta_u^S)\left(dW_u^S - 2\theta_u^S \, du\right) + \int_0^t (-2\theta_u^R)\left(dW_u^R - 2\theta_u^R \, du\right)\right).$$

The process (ζ_t^2) is a supermartingale (as a positive local martingale) so

$$\mathbb{E}[\zeta_T^2] \leq \zeta_0^2 = 1.$$

Therefore condition (30) and Hölder's inequality prove (29). □

Under the assumption that (30) is satisfied, the Novikov condition holds so the BSDE with driver (27) has a unique solution (in a certain class), given by the conditional expectation of the terminal condition under the equivalent measure $\mathbb{P}^\theta$ with respect to which

$$W_t^\theta = W_t + \int_0^t \theta_s ds,$$

is a Brownian motion; the integrand comes from the martingale representation theorem under $\mathbb{P}^\theta$. Hence (30) is indeed sufficient for the candidate strategy to be admissible.

4.3 Optimality of the trading strategy

In order to establish optimality of the trading strategy $\tilde{\pi}^a$ under (29) it is convenient to work under $\mathbb{P}^\theta$. Since the Novikov condition is satisfied, W^θ is a Brownian motion with respect to the original filtration so it makes sense to consider the BSDE that describes the dynamics of the agent's risk assessment under $\mathbb{P}^\theta \approx \mathbb{P}$. Under the new measure the driver is given by

$$G^{\theta,a}(t,\pi,z) = G^a(t,\pi,z) + < z, \theta > .$$

Since the functions $\pi \mapsto G^{\theta,a}(t,\pi,z)$ and $\pi \mapsto G^a(t,\pi,z)$ have the same minimizers it is enough to establish the comparison principle under $\mathbb{P}^\theta$. The key observation is that after plugging in the optimal strategy, the resulting driver does not depend on Z. Thus, given the solution $(\tilde{Y}, \tilde{Z})$ associated with the trading strategy $\tilde{\pi}$ and some solution $(Y(\pi), Z(\pi))$ coming from another admissible strategy π, we have that

$$G^a(t,\tilde{\pi},\tilde{Z}_t) = G^a(t,\tilde{\pi}_t, Z_t(\pi)) \leq G^a(t,\pi_t,Z_t(\pi)).$$

As a result,

$$\left(\tilde{Y}_0 - Y_0(\pi)\right) \leq \int_0^T \left(\tilde{Z}_u - Z_u(\pi)\right) dW_u^\theta \qquad \mathbb{P}^\theta\text{-a.s.} \tag{31}$$

Since all the payoff functions are bounded, the process $\left(\int_0^\cdot \tilde{Z} dW^\theta\right)$ is a (true) $\mathbb{P}^\theta$-martingale. The following lemma shows that $\left(\int_0^\cdot Z(\pi) dW^\theta\right)$ is also a $\mathbb{P}^\theta$-martingale. Taking expectation with respect to $\mathbb{P}^\theta$ in (31) therefore shows that

$$\tilde{Y}_0 \leq Y_0(\pi) \qquad \mathbb{P}^\theta\text{-a.s.} \quad \text{and hence} \quad \mathbb{P}\text{-a.s.}$$

so that the comparison principle is indeed satisfied.

Lemma 4.2. *The process $M_t^\theta \triangleq \int_0^t Z_s(\pi)\, dW_s^\theta$ is a $\mathbb{P}^\theta$-martingale.*

Proof. Let (Y, Z) be the unique solution to the BSDE (24) and put

$$M_t \triangleq \int_0^t Z_s\, dW_s \quad \text{and} \quad M_t^* \triangleq \sup_{u\in[0,t]} |M_u|.$$

In order to prove the assertion it suffices to show that

$$\mathbb{E}^\theta \sqrt{\langle M^\theta \rangle_T} < \infty.$$

In fact, it is enough to show that

$$\mathbb{E}^\theta \sqrt{\langle M \rangle_T} < \infty \tag{32}$$

because $\langle M \rangle = \langle M^\theta \rangle$. To this end, we observe first that due to the Burkholder-Davis-Gundy inequality there exists a constant k such that

$$\mathbb{E}\langle M \rangle_T \leq k\mathbb{E}[M_T^*]^2.$$

In order to see that the latter quantity is finite let us put $\xi^a = -H^a - V^{a,\theta}(\pi^a)$ so

$$\xi^a - Y_0^a = \int_0^T \frac{1}{2\gamma_a} |Z_s|^2 \, ds - \int_0^T Z_s \, dW_s.$$

Hence for some positive constants k_i $(i = 1, ..., 4)$ we have that

$$\begin{aligned} \langle M \rangle_T^2 &= \left[\int_0^T |Z_s|^2 \, ds \right]^2 \\ &\leq k_1 + k_2 (\xi^a)^2 + k_3 [M_T]^2 \\ &\leq k_1 + k_2 (\xi^a)^2 + k_3 [M_T^*]^2. \end{aligned}$$

Moreover $\mathbb{E}\left[(\xi^a)^2 \right] < \infty$ because the trading strategy is admissible, and by the Burkholder-Davis-Gundy inequality

$$\mathbb{E}[M_T^*]^4 \leq k_4 \mathbb{E} \langle M \rangle_T^2.$$

Hereby

$$\mathbb{E}[M_T^*]^2]^2 \leq \mathbb{E}[M_T^*]^4 \leq k_4 \mathbb{E} \langle M \rangle_T^2 \leq k_4 [k_1 + k_2 \zeta^2 + k_3 \mathbb{E}[M_T^*]^2],$$

so that

$$\mathbb{E}[M_T^*]^2 < \infty.$$

Moreover by (30) the expectation of the squared stochastic exponential of the integral $-\int_0^T \theta dW$ is finite so by Hölder's inequality (32) holds true. □

The preceding considerations show that Assumption 3.3 is satisfied for the case of entropic utilities. In the sequel we characterize the equilibrium market price of risk as a solution to a backward stochastic differential equation. Our analysis is based on a representative agent approach. In the framework of the preceding example the representative agent's BSDE will be given by (27) with a different coefficient of risk tolerance and a different terminal condition. It turns out that an equilibrium market price of risk can be obtained from that BSDE by replacing the process θ^R by $-\frac{z^2}{\gamma_R}$. This translates a linear into a quadratic BSDE.

5 The representative agent

It is well known from the theory of general equilibrium that in complete markets Pareto optimal allocations (and hence competitive equilibria) can be supported as equilibria of a representative agent economy. The preferences and endowments of the representative agent are typically given in terms of a weighted average of the individual agents' characteristics with the weights depending on the equilibrium to be supported. This dependence of the weights on equilibrium allocations renders the general analysis of equilibrium rather involved. The situation is considerably simpler when the agents have monetary utilities. In this case the definition of the representative agent does not depend on the equilibrium. Our approach in finding equilibrium prices is thus to consider a representative agent, give her the total endowment including the bond, and then let her act to minimize her risk from trading in the financial and bond market. The equilibrium market price of risk is characterized by the fact that her optimal demand for the bond is identically equal to zero for the model with risk sharing and one for risk transfer.

5.1 The representative agent's preferences

In order to simplify the exposition we restrict ourselves to an economy with two agents, $\mathbb{A} = \{a, b\}$; the general case follows from straightforward modifications. The representative agent's goal at time t is to minimize the aggregate (residual) risk. Her utility function on the level of contingent claims (rather than trading strategies) thus takes the form

$$\inf_F \left\{ Y_t^a(\zeta_T - F) + Y_t^b(F) \right\} \tag{33}$$

for some total endowment $-\zeta_T$ where the infimum is taken over a suitable (e.g., square integrable or bounded) subset of $\mathcal{F}_t$-measurable random variables. On the level of backward equations such risk preferences can be represented by the solution of a BSDE with driver $g^{ab} = g^a \square g^b$, where $\square$ refers to the usual inf-convolution operation:

$$g^{ab}(t, z) = g^a \square g^b(t, z) = \inf_x \{g^a(t, z - x) + g^b(t, x)\}. \tag{34}$$

Assumption 5.1. *The mapping $z \mapsto g^{ab}(t, z)$ is strictly convex and the infimum is attained.*

The convexity condition on g^{ab} guarantees that the first order conditions associated with the representative agent's optimization problem have a solution. Sufficient conditions in terms of the individual agents' preferences are given in Barrieu & El Karoui (2005).

Remark 5.2. *The unique minimizer x^* of the mapping $x \mapsto g^a(t, z-x)+g^b(t, x)$ is characterized by the equations*

$$g^a_{z_i}(t, z - x^*) = g^b_{z_i}(t, x^*) \quad (i = 1, 2).$$

This means that

$$g^{ab}(t, z) = g^a(t, x) + g^b(t, y)$$

if and only if

$$g^a_{z_i}(t, x) = g^b_{z_i}(t, y) \quad \text{and} \quad x + y = z.$$

We shall use this characterization later in order to state sufficient conditions for the existence of competitive equilibria.

In a model with risk transfer the representative agent's initial endowment is given by the agents' combined endowment plus the bond's payoff, i.e., by $-H^a - H^b - H^I$. Given a trading strategy π in the financial and bond market her risk exposures follow the backward dynamics

$$-dY_t^{ab}(\pi) = g^{ab}(t, Z_t)dt - Z_t dW_t \quad \text{with} \quad Y_T^{ab}(\pi) = -H^a - H^b - H^I - V_T^{ab,\theta}(\pi).$$

The agent's goal is to minimize her risk over the set of admissible trading strategies:

$$\min_{\pi \in \mathcal{S}^\theta} Y_0^{ab}(\pi).$$

Arguing as in the case of the single agent, the convexity condition on the driver g^{ab} yields the following characterization of the optimal portfolio:

$$g_{z_1}^{ab}\left(s, \bar{Z}_s - \tilde{\pi}_s^{ab,1}\begin{pmatrix} \sigma^S S_s \\ 0 \end{pmatrix} - \tilde{\pi}_s^{ab,2}\kappa_s^\theta\right) = -\theta_s^S, \tag{35}$$

and

$$g_{z_2}^{ab}\left(s, \bar{Z}_s - \tilde{\pi}_s^{ab,1}\begin{pmatrix} \sigma^S S_s \\ 0 \end{pmatrix} - \tilde{\pi}_s^{ab,2}\kappa_s^\theta\right) = -\theta_s^R. \tag{36}$$

Thus, if the BSDEs with driver (34) and terminal conditions

$$-H^a - H^b - H^I - V_T^{ab,\theta}(\pi)$$

satisfy a comparison principle and if the trading strategy defined implicitly by the first order conditions as in (22) is admissible, her optimization problem has a unique solution.

5.2 From the representative agent to equilibrium

In the representative agent economy the bond market is in equilibrium if the agent's optimal trading strategy in the bond market is

$$\tilde{\pi}_t^{ab,2} \equiv 0.$$

The following theorem characterizes the equilibrium market price of external risk θ^{*R} in terms of a solution to the BSDE associated with the representative agent's utility optimization problem. It also shows that θ^{*R} is an equilibrium in the underlying competitive economy, given that the bond price volatility does not vanish.

Theorem 5.3. *Assume that there exists a solution $(\hat{Y}^{ab}, \bar{Z})$ of the backward stochastic differential equation*

$$\hat{Y}_t^{ab}(\pi) = -H^a - H^b - H^I + \int_t^T G^{ab}(s, \bar{Z}_s)\, ds - \int_t^T \bar{Z}_s\, dW_s, \tag{37}$$

with driver

$$G^{ab}(s,\bar{Z}_s) \triangleq g^{ab}\left(s, \bar{Z}_s - \tilde{\pi}_s^{ab,1}(\bar{Z}_s)\begin{pmatrix} \sigma^S S_s \\ 0 \end{pmatrix}\right) - \tilde{\pi}_s^{ab,1}(\bar{Z}_s)\sigma^S S_s \theta_s^S,$$

where $\tilde{\pi}_s^{ab,1} = \tilde{\pi}_s^{ab,1}(z)$ *is the unique solution of the equation in* x

$$g_{z_1}^{ab}\left(s, z - x\begin{pmatrix} \sigma^S S_s \\ 0 \end{pmatrix}\right) = -\theta_s^S. \tag{38}$$

Assume furthermore that Assumption 5.1 is satisfied. If the process $\theta^{*R} = \theta^{*R}(z)$ *defined by*

$$-\theta_s^{*R}(z) = g_{z_2}^{ab}\left(s, z - \tilde{\pi}_s^{ab,1}(z)\begin{pmatrix} \sigma^S S_s \\ 0 \end{pmatrix}\right) \tag{39}$$

along with the market price of financial risk θ^S *makes the density process in (7) a uniformly integrable martingale and if Assumptions 3.1 and 3.3 hold under* $\theta^* = (\theta^S, \theta^{*R}(\bar{Z}))$*, then* θ^{*R} *is an equilibrium market price of external risk.*

Proof. We proceed in three steps.

(i) For the given market price of external risk $\theta^{*R} = \theta^{*R}(\bar{Z})$ we obtain from equations (38), (39) and the convexity of $\pi \to G^{ab}(s,\pi,z)$ that

$$(\tilde{\pi}^{ab,1}(z), 0) = \arg\min_{\pi\in\mathbb{R}^2} G^{ab}(s,\pi,z).$$

Hence in view of the Assumption 3.3 it turns out that $\tilde{\pi}^{ab,2} = 0$ is optimal.

(ii) In order to show that θ^{*R} yields an equilibrium of the underlying competitive economy we denote by $\tilde{\pi}^a$ and $\tilde{\pi}^b$ the agents' unique optimal trading strategies associated with θ^{*R}. Our goal is then to show that

$$\tilde{\pi}^{ab} = \tilde{\pi}^a + \tilde{\pi}^b - (0,1)\,.$$

To this end, we recall that for $i \in \{a,b\}$ the optimality of the strategy $\tilde{\pi}^i$ comes from the first order conditions

$$g_{z_1}^i\left(s, \bar{Z}_s^i - \tilde{\pi}_s^{i,1}\begin{pmatrix} \sigma_s^S S_s \\ 0 \end{pmatrix} - \tilde{\pi}_s^{i,2}\kappa_s^\theta\right) = -\theta_s^S,$$

and

$$g_{z_2}^i\left(s, \bar{Z}_s^i - \tilde{\pi}_s^{i,1}\begin{pmatrix} \sigma_s^S S_s \\ 0 \end{pmatrix} - \tilde{\pi}_s^{i,2}\kappa_s^\theta\right) = -\theta_s^R.$$

Thus, by Remark 5.2 we have the following link between the drivers g^a, g^b and g^{ab}:

$$\begin{aligned} & g^a\left(s, \bar{Z}^a_s - \tilde{\pi}^{a,1}_s \begin{pmatrix} \sigma^S_s S_s \\ 0 \end{pmatrix} - \tilde{\pi}^{a,2}_s \kappa^\theta_s\right) + g^b\left(s, \bar{Z}^b_s - \tilde{\pi}^{b,1}_s \begin{pmatrix} \sigma^S_s S_s \\ 0 \end{pmatrix} - \tilde{\pi}^{b,2}_s \kappa^\theta_s\right) \\ = \; & g^{ab}\left(s, \bar{Z}^a_s + \bar{Z}^b_s - (\tilde{\pi}^{a,1}_s + \tilde{\pi}^{b,1}_s) \begin{pmatrix} \sigma^S_s S_s \\ 0 \end{pmatrix} - (\tilde{\pi}^{a,2}_s + \tilde{\pi}^{b,2}_s)\kappa^\theta_s\right). \end{aligned} \tag{40}$$

(iii) Let us now fix a trading strategy π of the representative agent. In view of the bond price dynamics (9) the associated BSDE can be written as

$$\begin{aligned} \hat{Y}^{ab}_t(\pi) &= - H^a - H^b - H^I + \int_t^T G^{ab}(s, \pi_s, Z^{ab}_s)\, ds - \int_t^T Z^{ab}_s \, dW_s \\ &= - H^a - H^b + \int_t^T G^{ab}(s, \pi_s - (0,1)^t, Z^{ab}_s - \kappa^\theta_s)\, ds - \int_t^T (Z^{ab}_s - \kappa^\theta_s)\, dW_s \end{aligned} \tag{41}$$

with the driver G^{ab} defined by analogy to (18). As a result, the process

$$\left(\hat{Y}^a_t(\tilde{\pi}^a) + \hat{Y}^b_t(\tilde{\pi}^b), \bar{Z}^a_t + \bar{Z}^b_t - \kappa^\theta_t\right) \tag{42}$$

is the solution of the BSDE (41) associated with the trading strategy

$$\bar{\pi} = \tilde{\pi}^a + \tilde{\pi}^b - (0,1).$$

Conversely, any solution of the representative agent's BSDE along with the unique solution of the first agent's BSDE yields, via (40), a solution of the second agent's BSDE. This equation, however, has a unique solution so (42) is in fact the unique solution to (41) given the trading strategy $\bar{\pi}$. This shows that $\bar{\pi}$ is admissible for the representative agent. Since it satisfies the first order conditions

$$g^{ab}_{z_1}\left(s, \bar{Z}^{ab}_s - \kappa^\theta_s - (\tilde{\pi}^{a,1} + \tilde{\pi}^{b,1}) \begin{pmatrix} \sigma^S_s S_s \\ 0 \end{pmatrix} - (\tilde{\pi}^{a,2} + \tilde{\pi}^{b,2} - 1)\kappa^\theta_s\right) = -\theta^S_s,$$

and

$$g^{ab}_{z_2}\left(s, \bar{Z}^{ab}_s - \kappa^\theta_s - (\tilde{\pi}^{a,1} + \tilde{\pi}^{b,1}) \begin{pmatrix} \sigma^S_s S_s \\ 0 \end{pmatrix} - (\tilde{\pi}^{a,2} + \tilde{\pi}^{b,2} - 1)\kappa^\theta_s\right) = -\theta^R_s,$$

we infer from the strict convexity of the function g^{ab} in the space variable that

$$\tilde{\pi}^{ab} = \tilde{\pi}^a + \tilde{\pi}^b - (0,1).$$

In particular

$$0 = \pi^{ab,2}_t = \tilde{\pi}^{a,2}_t + \tilde{\pi}^{b,2}_t - 1 \quad \mathbb{P} \otimes dt\text{-a.s.}$$

□

The preceding theorem yields a formula for a candidate equilibrium market price of external risk: In a first step we solve the first order condition (35) of the representative agent to obtain $\tilde{\pi}_s^{ab,1}$ given that $\tilde{\pi}_s^{ab,2} \equiv 0$; this is possible because (35) is independent of θ^R. A second step consists of defining the candidate process via (36). In a third step we need to verify the admissibility conditions on the candidate market price of risk.

6 An example

In this section we consider an example where the assumptions of Theorem 5.3 can be verified. We restrict ourselves to the more involved case of risk transfer where we need to verify the additional assumption that the equilibrium bond price volatility is strictly positive. In order to guarantee positivity of the volatility process $\kappa^{\theta,R}$ defined in equation (9) we assume that the bond's payoff is increasing in the external risk component.

Assumption 6.1. *The bond's total payoff is increasing in the external risk factor and strictly increasing on a set of positive measure.*

To simplify the notation we assume that the bond's payoff depends only on the terminal state of the forward process, i.e., that $\varphi^a \equiv 0$; the general case follows from similar considerations.

6.1 The microeconomic setup

We consider again the entropic case where

$$g^a(t,z) = \frac{1}{2\gamma_a}\|z\|^2, \quad g^b(t,z) = \frac{1}{2\gamma_b}\|z\|^2,$$

hence

$$g^{ab}(t,z) = \frac{1}{2\gamma_R}\|z\|^2, \quad \gamma_R \triangleq \gamma_a + \gamma_b.$$

In view of Lemma 4.1 the utility optimization problem of an individual agent associated with a market price of risk θ has a unique solution if

$$\mathbb{E}\left[\exp\left(6\int_0^T \|\theta_u\|^2 du\right)\right] < \infty. \tag{43}$$

It turns out from the analysis that follows that under some assumptions on the payoffs the equilibrium market price of external risk θ^{*R} is bounded uniformly and thus for (43) to hold it suffices to ask for

$$\mathbb{E}\left[\exp\left(6\int_0^T |\theta_u^S|^2 du\right)\right] < \infty. \tag{44}$$

For the remainder of this section we work under the preceding condition. In this case, according to Girsanov's Theorem (see Theorem 5.1 p 191 in Karatzas & Shreve 1991) there exists an equivalent measure $\tilde{\mathbb{P}}$ with respect to which

$$\tilde{W}_t^S = W_t^S + \int_0^t \theta_u^S \, du \quad \text{and} \quad \tilde{W}_t^R = W_t^R \tag{45}$$

are independent Brownian motions with respect to the filtration $(\mathcal{F}_t)$. Also notice that the Novikov condition (see Corollary 5.13 p 199 in Karatzas & Shreve 1991) is satisfied under (44).

In order to establish existence and uniqueness of a bounded equilibrium market price of external risk we shall now assume that the market price of financial risk has the following additive decomposition property:

Assumption 6.2. *The market price of financial risk satisfies*

$$(\theta_t^S)^2 = y_t + \Gamma(t, R_t)$$

for some process y_t that is $\tilde{\mathbb{P}}$-independent of W^R, and for a uniformly (in t) bounded non-negative functions $\Gamma(t, \cdot)$, with uniformly bounded first and second derivatives.

The preceding assumption is equivalent to saying that θ^S can be decomposed into two additive components that are independent under $\tilde{\mathbb{P}}$ probability measure. Along with the Novikov condition (44) it is satisfied if, for instance, if θ^S is bounded and independent of the external risk factor as in, e.g., Hu, Imkeller & Müller (2006) and Horst & Müller (2006). But there are also stochastic volatility models where it is satisfied. The following is a modification of a model by Chacko & Viceira (2005).

Example 6.3. *Consider the financial market model with unbounded market price of risk:*

$$\frac{dS_t}{S_t} = \mu^S dt + \sqrt{\frac{1}{y_t}} dW_t^S, \qquad \Gamma(t, \cdot) = 0.$$

Here the volatility process (y_t) is a square root process driven by the Brownian motion W^S:

$$dy_t = k(\zeta - y_t)dt + \sigma\sqrt{y_t}dW_t^S.$$

The condition $2k\zeta > \sigma^2$ guarantees the process y_t remains positive. A recent paper by Wong & Heyde (2006) gives a sufficient condition for which (44) holds true:

$$\frac{k}{\mu^S \sigma} \geq 2\sqrt{3}. \tag{46}$$

Despite the random volatility, the model of the previous example fits into our framework as we may rescale the number of shares of stock. This is possible because the agents' combined demand for the stock does not affect its dynamics.

Our Assumption 6.2 is also satisfied if θ^S depends only on the non-financial risk factor.

Example 6.4. *If the stock price processes of the form*

$$\frac{dS_t}{S_t} = \mu_t^S(R_t)dt + \sigma_t^S(R_t)dW_t^R$$

with bounded functions μ_t^S and σ_t^S and a volatility that is uniformly bounded away from zero, then Assumption 6.2 is satisfied.

The following example can be viewed as mixture of the preceding examples. It considers an unbounded market price of financial risk that depends on both the external risk factor and the randomness driving the stock price.

Example 6.5. *Consider the financial market model where*

$$\frac{dS_t}{S_t} = \mu^S dt + \frac{1}{\sqrt{y_t + \Gamma(t, R_t)}} d\tilde{W}_t^S$$

and

$$dy_t = k(\zeta - y_t)dt + \sigma\sqrt{y_t}d\tilde{W}_t^S. \tag{47}$$

Again, the condition $2k\zeta > \sigma^2$ guarantees the process y_t remains strictly positive. The existence of the process y_t boils down to the existence of a strong solution for the following SDE

$$dy_t = \left[k(\zeta - y_t) + \sigma\mu^S\sqrt{y_t}\sqrt{y_t + \Gamma(t, R_t)} \right] dt + \sigma\sqrt{y_t}dW_t^S, \tag{48}$$

under the original measure $\mathbb{P}$. We prove this in Lemma 6.6. An application of Hölder's inequality shows that the Nokivov condition (44) holds if

$$\frac{k}{\mu^S\sigma} \geq 2\sqrt{6}. \tag{49}$$

Indeed,

$$\mathbb{E}\left[\exp\left(6\int_0^T |\theta_u^S|^2 du\right)\right] = \mathbb{E}\left[\exp\left(6\int_0^T (\mu^S)^2 y_u du\right)\exp\left(6\int_0^T (\mu^S)^2\Gamma(u, R_u)du\right)\right]$$

In light of (49) *it follows that*

$$\mathbb{E}\left[\exp\left(12\int_0^T (\mu^S)^2 y_u du\right)\right] < \infty,$$

according to Wong & Heyde (2006). This combined with the boundedness of Γ, and Hölder's inequality yields (44).

Lemma 6.6. *Under the assumption $2k\zeta > \sigma^2$ there exists a unique strong solution for* (48)*.*

Proof. The existence of a weak solution for SDE (48) it is obvious. Indeed if $2k\zeta > \sigma^2$ then SDE (47) has a unique strong solution given the probability measure $\tilde{\mathbb{P}}$. Then we can construct $\mathbb{P}$ from $\tilde{\mathbb{P}}$ by means of Girsanov's Theorem and this procedure yields a weak solution for SDE (48). By proving pathwise uniqueness for (48) we get uniqueness of the strong solution for (48), due to the Ikeda-Watanabe Theorem (see Corollary 3.23 p. 310 in Karatzas & Shreve 1991). Let us define the stochastic process x_t as the unique strong solution of the SDE

$$dx_t = k(\zeta - x_t)dt + \sigma\sqrt{x_t}dW_t^S, \tag{50}$$

under the measure $\mathbb{P}$. Then, since $2k\zeta > \sigma^2$, it follows that x_t is strictly positive. A standard comparison argument for diffusions shows that y_t (the weak solution of (48)) dominates x_t, i.e., $y_t \geq x_t$ (see Proposition 2.18 p 293 in Karatzas & Shreve 1991) Let us define the following sequence of stopping times

$$\tau_\epsilon \triangleq \inf\{s \leq T, \text{ such that } x_t \leq \epsilon\},$$

for some ϵ small enough (such that $y_0 > \epsilon$). On the interval $[0, \tau_\epsilon]$ it follows that $y_t \geq x_t \geq \epsilon$ hence on this interval SDE (48) admits pathwise uniqueness. Indeed that follows by Theorem 2.5 p 287 in Karatzas & Shreve 1991, since the drift and the volatility are Lipschitz functions. Moreover since x_t is strictly positive it follows that $\tau_\epsilon \uparrow T$ as $\epsilon \to 0$, $\mathbb{P}$ a.s. Therefore (48) admits pathwise uniqueness on $[0, T]$ and this completes the proof. □

For our analysis of equilibrium we are only interested in the market price of financial risk, not the drift nor the volatility coefficients. We may thus with no loss of generality normalize the stock price volatility to one and assume from now on that

$$\frac{dS_t}{S_t} = \theta_t^S dt + dW_t^S = d\tilde{W}_t^S.$$

6.2 The equilibrium BSDE

Under Assumption 6.2 the risk minimization problem for a given market price of risk has a unique solution. In view of the analysis of Section 4.1.2 and Theorem 5.3, a candidate θ^{*R} for an equilibrium market price of external risk is given by z^2, the second component of the integrand part of a solution (Y, Z) of the BSDE

$$Y_t = H^{\text{rep}} - \int_t^T z_s dW_s + \frac{1}{2}\int_t^T [-(z_s^2)^2 + (\theta_s^S)^2 - 2\theta_s^S z_s^1]\, ds. \tag{51}$$

Here $Z = (z^1, z^2)$ and H^{rep} denotes the income of the representative agent weighted by the factors of risk tolerance, i.e.,

$$H^{\text{rep}} \triangleq \frac{H^a + H^b + H^I}{\gamma_R}.$$

Remark 6.7. *When we solved an individual agent's optimization problem for a given market price of risk in an entropic framework, the BSDEs for the optimal risk process turned out to be linear. It is the equilibrium condition that translates the linear BSDE into a quadratic one. We need to replace the process* θ^R *in (27) by* $-\frac{z^2}{\gamma_R}$ *and then rescale the BSDE by a factor* $-\frac{1}{\gamma_R}$.

Notice that under the probability measure $\tilde{\mathbb{P}}$ the BSDE (51) becomes

$$Y_t = H^{\text{rep}} - \int_t^T z_s d\tilde{W}_s + \frac{1}{2}\int_t^T [-(z_s^2)^2 + (\theta_s^S)^2]\, ds. \tag{52}$$

6.2.1 Existence and uniqueness of solutions

If the market price of financial risk θ^S is bounded, the preceding BSDE satisfies the conditions of, e.g., Ankirchner, Imkeller & Dos Reis (2007) and in this case it has a unique solution (Y, Z) and the processes are Malliavin differentiable.[6] In order to do the verification of the sign condition on the equilibrium volatility we will need to go a step further. We need to show that z^2 is of the form $z_t^2 = v(t, S_t, R_t)$ where the dependence on the external risk component is Lipschitz continuous.

The representation of the process z^2 in terms of a Lipschitz function of the forward process is robust with respect to equivalent measure changes, i.e., the function v is deterministic. Hence we may - and will - work under the measure $\tilde{\mathbb{P}}$ defined through (45). More precisely, we are going to establish the following result:

Theorem 6.8. *Under Assumptions 2.1 and 6.2 the following holds:*

(i) There exists a unique solution (Y, Z) *to the equilibrium BSDE (52) such that the second component* z^2 *of the integrand* Z *is bounded in absolute value and* $Z \in L^2(dt \otimes \tilde{\mathbb{P}})$.

(ii) There exists a function $\bar{v}(t, x_1, x_2)$ *that is Lipschitz continuous in the second, third variable such that the process* z^2 *in (i) satisfies*

$$z_t^2 = \bar{v}(t, \ln S_t, R_t).$$

It will be more convenient in the sequel to put $\tilde{H}^{\text{rep}} = H^{\text{rep}} + \int_0^T (\theta_s^S)^2 ds$ and to consider instead the BSDE

$$\tilde{Y}_t = \tilde{H}^{\text{rep}} - \int_t^T \tilde{Z}_s d\tilde{W}_s + \frac{1}{2}\int_t^T -(\tilde{z}_s^2)^2\, ds. \tag{53}$$

This BSDE has a solution, due to Briand & Hu (2007) because the payoff functions are bounded and the market price of financial risk satisfies an exponential integrability condition.

Remark 6.9. *The solutions to (52) and (53) are linked by*

$$(Y_t, Z_t) = \left(\tilde{Y}_t - \int_0^t (\theta_s^S)^2 ds, \tilde{Z}_t\right).$$

[6] We recall the notion of Malliavin differentiability in the appendix.

The following lemma is key to our analysis. It shows that (53) has a solution $(\tilde{Y}, \tilde{Z})$ such that the Malliavin derivative $\tilde{D}_t^2 \tilde{Y}_t$ of the process $\tilde{Y}$ with respect to Brownian motion driving the external risk process exists and that $z_t^2 = \tilde{D}_t^2 \tilde{Y}_t$. Here $\tilde{D}$ is the Malliavin derivative with respect to the $\tilde{\mathbb{P}}$ measure. From this and Assumption 6.2 we deduce that the derivative process $\tilde{D}^2 Y$ exists as well.

Lemma 6.10. *There exists a solution $(\tilde{Y}, \tilde{Z}) \in L^2(dt \otimes \tilde{\mathbb{P}}) \times L^2(dt \otimes \tilde{\mathbb{P}})$ to the BSDE (53) such that the Malliavin derivative $\tilde{D}_u^2 \tilde{Y}_t$ exists for any $u \in [0,T]$ and is uniformly bounded. Furthermore, the process $(\tilde{D}_u^2 \tilde{Y}_t)_{u \leq t \leq T}$ satisfies a linear BSDE.*

Proof. The assertion of the lemma would be covered by standard results from the literature if the terminal value were bounded. To overcome the problem of unbounded terminal values we proceed in three steps.

(i) Notice first that we may with no loss of generality assume that the terminal condition is non-negative because H^{rep} is bounded. Let us now consider a sequence of BSDEs with non-negative terminal conditions

$$\xi^n = H^{\text{rep}} + \int_0^T \Gamma(s, R_s)\, ds + \min\left\{ \int_0^T y_s\, ds, n \right\}.$$

For any $n \in \mathbb{N}$ the "truncated BSDE"

$$\tilde{Y}_t^n = \xi^n - \int_t^T \tilde{Z}_s^n d\tilde{W}_s + \frac{1}{2} \int_t^T -\left(\tilde{z}_s^{2,n}\right)^2 \, ds$$

has a unique solution $(\tilde{Y}^n, \tilde{Z}^n)$ in the space $\mathbb{H}^\infty \times \mathbb{H}^2$. We notice that

$$\tilde{D}_u^2 \xi^n = b H_2^{\text{rep}}(S_T, R_T) + \int_u^T b \Gamma_x(s, R_s)\, ds,$$

with $H_2^{\text{rep}}(x^1, x^2) = \partial_{x_2} H^{\text{rep}}(x_1, x_2)$. Boundedness of $\Gamma(t, \cdot)$ and the derivatives of H^{rep} allows us to argue as in the proof of Theorem 8.4 in Ankirkner, Imkeller & Dos Reis (2007) in order to show that the processes $\tilde{Y}^n$ are Malliavin differentiable with respect to the Brownian motion $\tilde{W}^R$ and that for any $u \in [0,T]$ the derivative processes $\left(\tilde{D}_u^2 \tilde{Y}_t^n\right)_{t \geq u}$ satisfy the linear BSDE

$$\tilde{D}_u^2 \tilde{Y}_t^n = \tilde{D}_u^2 H^{\text{rep}} + \int_u^T b \Gamma_x(s, R_s)\, ds - \int_t^T \tilde{D}_u^2 \tilde{Z}_s^n d\tilde{W}_s - \int_t^T \tilde{z}_s^{2,n} \tilde{D}_u^2 \tilde{z}^{2,n_s} ds. \tag{54}$$

Using the same arguments as in Section 5 of Horst & Müller (2006), and because $\tilde{D}_u^2 H^{\text{rep}}$ does not depend on $n \in \mathbb{N}$ we deduce that the quantities $\tilde{D}_u^2 \tilde{Y}_t^n$ are uniformly bounded in $u, t \in [0,T]$ and $n \in \mathbb{N}$. Furthermore $\left(\tilde{D}_t^2 \tilde{Y}_t^n\right)$ is a version of $\left(\tilde{z}_t^{2,n}\right)$ and hence the family $(\tilde{z}_t^{2,n})_{n \in \mathbb{N}}$ is uniformly bounded.

(ii) Our goal is now to prove that the sequence $\{(\tilde{Y}^n, \tilde{Z}^n)\}_{n\in\mathbb{N}}$ converges along some subsequence in $L^2(dt \otimes \tilde{\mathbb{P}})$ to a process $(\tilde{Y}, \tilde{Z})$ that satisfies the equilibrium BSDE (53).

To this end, we fix $n, m \in \mathbb{N}$. Since the processes $\tilde{z}^{2,n}$ are uniformly bounded a standard comparison result for BSDEs yields a constant $C < \infty$ such that

$$\tilde{\mathbb{E}}\Big[\int_0^T (\tilde{Y}_s^n - \tilde{Y}_s^m)^2 ds + \int_0^T |\tilde{Z}_s^n - \tilde{Z}_s^m|^2 ds\Big] \leq C\,\tilde{\mathbb{E}}\,|\xi^n - \xi^m|^2,$$

where $\tilde{\mathbb{E}}$ is the expectation operator with respect to $\tilde{\mathbb{P}}$. In view of our truncation scheme the difference $\xi^n - \xi^m$ vanishes almost surely as $n, m \to \infty$. Furthermore, the family of random variables $\{\xi^n - \xi^m\}_{n,m}$ is uniformly integrable so we also have convergence in mean square. As a result, the sequence $\{(\tilde{Y}^n, \tilde{Z}^n)\}$ is Cauchy in $L^2(dt \otimes \tilde{\mathbb{P}})$ so it converges to some limit (Y, Z) in L^2.

In particular,

$$\begin{aligned}
\lim_{n\to\infty} \int_0^T \tilde{\mathbb{E}}\left[\int_t^T (\tilde{Z}_s^n - \tilde{Z}_s) dW_s\right]^2 dt &= \lim_{n\to\infty} \int_0^T \tilde{\mathbb{E}}\left[\int_t^T (\tilde{Z}_s^n - \tilde{Z}_s)^2 ds\right] dt \\
&\leq \lim_{n\to\infty} \int_0^T \tilde{\mathbb{E}}\left[\int_0^T (\tilde{Z}_s^n - \tilde{Z}_s)^2 ds\right] dt \\
&\leq \lim_{n\to\infty} T \|\tilde{Z}^n - \tilde{Z}\|_{L^2(dt\otimes\tilde{\mathbb{P}})} \\
&= 0.
\end{aligned}$$

Passing, if necessary, to a subsequence we may in fact assume that

$$\tilde{Z}^n \to \tilde{Z} \quad \text{in} \quad L^2(dt \otimes \tilde{\mathbb{P}}) \quad \text{and} \quad dt \otimes \tilde{\mathbb{P}}\text{-almost surely.}$$

Uniform boundedness of the sequence $\{\tilde{z}^{2,n}\}$ implies that $\tilde{z}^2$ is bounded so dominated convergence yields

$$\lim_{n\to\infty} \left| \int_t^T \tilde{z}_s^{2,n} ds - \int_t^T \tilde{z}_s^2 ds \right| = 0.$$

This shows that the limit process $(\tilde{Y}, \tilde{Z})$ satisfies the equilibrium BSDE.

(iii) In view of step (ii) it follows from Lemma 1.2.3 in Nualart (1995) that the process $(\tilde{Y}, \tilde{Z})$ is Malliavin differentiable and that the derivatives of the approximating sequence converge weakly to $(\tilde{D}.\tilde{Y}., \tilde{D}.\tilde{Z}.)$.

In order to prove strong convergence of the derivative processes and a BSDE representation (54) with $\tilde{z}^{2,n}$ replaced by the limit z^2, we notice that the drivers of (54) converge locally uniformly to the driver of the limit BSDE. Therefore it follows from Theorem 2.8 of Kobylanski (2000) that the sequence $\{(\tilde{D}_u^2 \tilde{Y}_\cdot^n, \tilde{D}_u^2 \tilde{Z}_\cdot^n)\}_{n\in\mathbb{N}}$ $(u \in [0, T])$ converges to the unique solution of the BSDE

$$\tilde{D}_u^2 \tilde{Y}_t = \tilde{D}_u^2 H^{\text{rep}} + \int_u^T b\Gamma_x(s, R_s)\, ds - \int_t^T \tilde{D}_u^2 \tilde{Z}_s d\tilde{W}_s - \int_t^T \tilde{z}_s^2 \tilde{D}_u^2 \tilde{z}_s^2 ds$$

where the integrand converges in $L^2(dt\otimes\tilde{\mathbb{P}})$. Similar arguments as in the preceding step show that the sequence $\{\tilde{D}_u^2 \tilde{Y}_\cdot^n\}_{n\in\mathbb{N}}$ converges in mean square, too. This yields the assertion.

□

We are now ready to prove part (i) of Theorem 6.8

PROOF OF THEOREM 6.8 (I): The proof of the preceding lemma constructs a solution (Y, Z) of the equilibrium BSDE with the desired properties. In order to see that such a solution is unique, let us assume to the contrary that two different solutions $(\hat{Y}, \hat{Z})$ and $(\bar{Y}, \bar{Z})$ exist. In this case

$$\hat{Y}_t - \bar{Y}_t = -\frac{1}{2}\int_t^T \left[(\hat{z}_s^2)^2 - (\bar{z}_s^2)^2\right] ds + \int_t^T \left(\hat{Z}_s - \bar{Z}_s\right) d\tilde{W}_s.$$

There exists a probability measure $\mathbb{Q} \approx \tilde{\mathbb{P}}$ and a $\mathbb{Q}$-Brownian motion $W^{\mathbb{Q}}$ such that

$$\hat{Y}_t - \bar{Y}_t = \int_t^T \left(\hat{Z}_s - \bar{Z}_s\right) dW_s^{\mathbb{Q}}.$$

Taking conditional expectation with respect to $\mathbb{Q}$ shows that $\hat{Y}_t \equiv \bar{Y}_t$ $\mathbb{Q}$-a.s. and hence $\tilde{\mathbb{P}}$-a.s. Thus, $\tilde{\mathbb{P}}$-a.s.

$$\frac{1}{2}\int_t^T \left[(\hat{z}_s^2)^2 - (\bar{z}_s^2)^2\right] ds = \int_t^T \left(\hat{Z}_s - \bar{Z}_s\right) d\tilde{W}_s \quad \text{for all } t \in [0,T].$$

This yields

$$\frac{1}{2}\int_0^t \left[(\hat{z}_s^2)^2 - (\bar{z}_s^2)^2\right] ds = \int_0^t \left(\hat{Z}_s - \bar{Z}_s\right) d\tilde{W}_s \quad \text{for all } t \in [0,T]$$

which implies for the respective quadratic variation processes that

$$0 = \int_0^t \left|\hat{Z}_s - \bar{Z}_s\right|^2 ds \quad \text{for all } t \in [0,T].$$

This proves the assertion. □

In order to finish the proof of Theorem 6.8 it remains to show that the second part of the integrand process can be represented in terms a sufficiently regular function of the forward process. Here, our arguments are based on a decomposition of the process z^2.

6.2.2 A decomposition of the equilibrium process

It follows from the proof of the preceding result that the process (z_t^2) satisfies the following multiplicative decomposition:

$$\begin{aligned} D_0^2\tilde{Y}_t &= \mathbb{E}^{\widehat{\mathbb{P}}_T}\left[b\, H_2^{\text{rep}}(S_T,R_T) + \int_t^T b\,\Gamma_x(s,R_s)\,ds \middle| \mathcal{F}_t \right] + \int_0^t b\,\Gamma_x(s,R_s)\,ds \\ &= D_t^2\tilde{Y}_t + \int_0^t b\,\Gamma_x(s,R_s)\,ds \\ &= z_t^2 + \int_0^t b\,\Gamma_x(s,R_s)\,ds \end{aligned}$$

where the probability measure $\widehat{\mathbb{P}}_T$ is defined by

$$\frac{d\widehat{\mathbb{P}}_s}{d\tilde{\mathbb{P}}} := \mathcal{E}\left(-\int_0^s \begin{pmatrix} 0 \\ z_u^2 \end{pmatrix} d\tilde{W}_u \right),$$

and H_2^{rep} denotes the derivative with the second component of H^{rep}. This decomposition along with the backward equation (54) and the fact that

$$\tilde{D}_0^2 H_T^{\text{rep}} = bH_2^{\text{rep}}(S_T,R_T)$$

shows that the processes $\tilde{D}_0^2\tilde{Y}_t \triangleq \tilde{M}_t$ and $\tilde{D}_0^2\tilde{Z}_t = N_t \triangleq (N_t^1, N_t^2)$ satisfy the BSDE

$$d\tilde{M}_t = \left[\left(\tilde{M}_t - \int_0^t b\Gamma_x(s,R_s)\,ds \right) N_t^2 \right] dt + N_t d\tilde{W}_t \tag{55}$$

with terminal condition

$$\tilde{M}_T = bH_2^{\text{rep}}(S_T,R_T) + \int_0^T b\Gamma_x(s,R_s)\,ds.$$

From this and our additive decomposition we see that the process (z_t^2, N_t) solves the BSDE

$$dM_t = [M_t N_t^2 - b\Gamma_x(t,R_t)]dt + N_t d\tilde{W}_t \tag{56}$$

with terminal condition

$$M_T = bH_2^{\text{rep}}(S_T,R_T).$$

For reasons that will become obvious later on, it will be convenient to denote the forward process from now on by

$$X_t = (X_t^1, X_t^2) = (\ln S_t, R_t).$$

6.2.3 Lipschitz continuity of the equilibrium market price of risk

Using the link between backward stochastic and partial differential equations we can now prove that the candidate for the equilibrium market price of external risk can be represented in terms of a Lipschitz continuous function of the forward process. To this end we first observe that after introducing the forward process $X_t = (X_t^1, X_t^2) = (\ln S_t, R_t)$, the hypothesis $(H4)$ and $(H5)$ of Kobylanski (2000) are satisfied for the BSDE (56) because $\tilde{H}_2^{\text{rep}}$ is bounded. Thus, by her Theorem 3.8 the process $(D_0^2 \tilde{Y}_t^n)$ can be represented as a continuous function $\bar{v}$ of the forward process X. Specifically, $\bar{v}$ is a viscosity solution of the PDE

$$-\bar{v}_t + \mathcal{L}\bar{v} - F(t, x, \bar{v}, \sigma(t, x) \cdot \nabla_x \bar{v}) = 0$$

with boundary condition

$$\bar{v}(T, x) = bH_2^{\text{rep}}(x).$$

Here the operator $\mathcal{L}$ is defined by

$$\mathcal{L}\bar{v} = \frac{1}{2}\bar{v}_{11} + \frac{b^2}{2}\bar{v}_{22} + a_t \bar{v}_2,$$

and

$$F(t, x, y, z) = -yz_2 + \Gamma_x(t, x_2)b.$$

Lemma 6.11. *The function $x \mapsto \bar{v}(t, x)$ is Lipschitz continuous uniformly on compact time intervals.*

Proof. Our proof is based on Theorem 3.3 (b) of Jakobsen & Karlsen (2002). In their notation we have that

$$-tr[A^\theta(t, x, D\bar{v})D^2\bar{v}] = \frac{1}{2}\bar{v}_{11} + \frac{b^2}{2}\bar{v}_{22}$$

and

$$f^\theta(t, x, r, p, X) \quad = \quad rp_2 + a_t p_2 - \Gamma_x(t, x_2)b.$$

Hence conditions (C_1), (C_2), (C_3), (C_6) and (C_8) of Jakobsen & Karlsen (2002) (page 505) are satisfied. For condition (C_2) notice now that

$$f^\theta(t, x, r, p, X) - f^\theta(t, x, s, p, X) \geq \bar{\gamma}_R(r - s),$$

where

$$\bar{\gamma}_R \triangleq p_2 1_{\{p_2 \leq 0\}}.$$

Conditions (C_3) and (C_8) are obviously satisfied and (C_6) follows from a_t being bounded, and from $\Gamma_x(t, x_2)$ being Lipschitz. Thus, our assertion follows from their result since the terminal function is Lipschitz continuous. □

We are now ready to prove the second part of Theorem 6.8.

Proof of Theorem 6.8 (ii): By the preceding lemma we know that

$$z_t^2 = \bar{v}(t, X_t^1, X_t^2) \tag{57}$$

where $\bar{v}(t,\cdot)$ is a Lipschitz continuous uniformly on compact time intervals. □

Remark 6.12. *We establish the existence of a Lipschitz continuous uniformly on compact time intervals function $\bar{v}(t,\cdot,\cdot)$ such that (57) holds. This result was proved by working under the $\tilde{\mathbb{P}}$ probability measure but it holds for every probability measure equivalent to $\tilde{\mathbb{P}}$, in particular the equilibrium pricing measure $\mathbb{P}^{\theta^*}$.*

6.3 A representation of the equilibrium bond price volatility

As a final step toward our equilibrium analysis we are going to verify our Assumption 3.1.

Lemma 6.13. *The volatility process $\{\kappa_t^R\}_{t\in[0,T]}$ is $\mathbb{P}$-a.s. strictly positive.*

Proof. We will use the Clark-Haussmann formula for diffusion forward processes. The forward process is $X_t \triangleq (X_t^1, X_t^2)$. According to Remark 6.12, under $\mathbb{P}^{\theta^*}$ the forward process satisfies

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t^{\theta^*},$$

where with $x = (x_1, x_2)$ we have put

$$f(t,x) = \begin{pmatrix} -\frac{1}{2}t \\ a_t - \bar{v}(t,x) \end{pmatrix} \quad \text{and} \quad g(t,x) = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}.$$

As Bahlali, Mezerdi & Ouknine (2002) have shown, Lipschitz continuity of f and g are enough for the Clark-Haussmann formula to hold for diffusion processes. The payoff H^I was assumed Frechét differentiable with derivative ∂H^I. According to Riesz theorem, there exists a right continuous function $\nu(\cdot, x) = (\nu^1(\cdot, x), \nu^2(\cdot, x))$ of bounded variation, with the second component increasing (because the bond payoff is increasing in the external risk component) such that

$$\partial H^I(x)y = \int_0^T \nu(dt, x)y(t).$$

Let us now denote by $\Phi(s,t)$ the solution of the following first variation equation associated with X_t:

$$d\Phi(t,s) = \left[\frac{\partial f}{\partial x}(t, X(t))\Phi(t,s)\right]dt + \left[\frac{\partial g}{\partial x}(t, X(t))\Phi(t,s)\right]dW_t^{\theta^*} \quad (t > s), \quad \Phi(t,t) = I_2,$$

where $\frac{\partial f}{\partial x}$ and $\frac{\partial g}{\partial x}$ are the generalized derivatives. Thus

$$d\Phi_{22}(t,s) = [-\Phi_{22}(t,s)\bar{v}_2(t,X_t)]dt, \qquad \Phi_{22}(s,s) = 1,$$

so $\Phi_{22}(s,t)$ is positive. According to Clark-Haussmann formula we now have that

$$\kappa_t^R = b\mathbb{E}^{\theta^*}[\lambda_t|\mathcal{F}_t],$$

where

$$\lambda_t = \int_t^T \nu^2(ds,X)\Phi_{22}(s,t).$$

Hence κ_t^R is positive. □

The boundedness of the process $\theta^{*R} = z^2$ together with (44) make (43) hold true. The following theorem summarizes our main result.

Theorem 6.14. *Let the agents have entropic utilities with drivers of the form (23). If Assumptions 2.1, 6.1 and 6.2 are satisfied then the following holds:*

(i) There exists a unique solution (Y,Z), $Z = (z^1,z^2)$, of the BSDE (51) *with bounded z^2.*

(ii) The process z^2 is an equilibrium market price of external risk.

7 A Numerical Example

The BSDE characterization of equilibrium market prices of risk makes our analysis amenable to an efficient numerical analysis. In this section we report some simulations for a benchmark model with two agents. We assume that (R_t) describes the dynamics of a temperature process in California and (S_t) is the price of a share of an energy provider equity.

7.1 Forward dynamics and payoffs

We work in the time interval $[0,T] = [0,1.5]$, where each 0.5 time unit represents a month according to the following setting and assume that

$$dR_t = a_t dt + 2.0\, dW_t^R, \qquad R_0 = r_0$$

describes a temperature process where the deterministic function $a(t) = 4t$ captures seasonal variations in average daytime temperatures for the months of May, June and July[7]. The model is a simple linear approximation of the data presented in the link, centered around 18 degrees Celsius. In our analysis we shall give special attention to $r_0 \in [-4,7]$ corresponding to a variation of 14 to 25 degrees Celsius.

[7]see http://www.wrcc.dri.edu/cgi-bin/cliMONtavt.pl?casjos

7.1.1 Stock price dynamics

Following Example 6.5 and consider a financial market model with unbounded market price of risk on the time interval $[0, 1.5]$:

$$\frac{dS_t}{S_t} = \mu^S dt + \frac{1}{\sqrt{y_t + \Gamma(t, R_t)}}\, dW_t^S, \qquad S_0 = s_0$$
$$dy_t = \left[k(\zeta - y_t) + \mu^S \sigma \sqrt{y_t}\sqrt{y_t + \Gamma(t, R_t)} \right] dt + \sigma \sqrt{y_t} dW_t^S, \qquad y_0 = 1.5.$$

The parameters μ^S, σ, k and ζ are constants satisfying condition (46). We choose:

μ^S	k	ζ	σ	$2k\zeta - \sigma^2$	$\frac{k}{\sigma\mu^S}$
0.08	0.33	24.7718	0.65	15.93	$6.35 > 2\sqrt{6} = 4.9 > 2\sqrt{3}$

The energy provider sells more energy in the summer if temperatures are high as there is more demand for electricity due to a higher use of air-conditioning. Its revenues become the more uncertain - and hence the stock the more volatile - the lower the daytime temperature. The Γ function in the model relates to the impact the temperature has on the value of the stock. An increase of the temperature reflects on the stocks volatility but only so much. This means the variation of Γ when the temperature is above some threshold should be small (and the same when the temperature is too low). In this spirit we model in the following way[8]:

$$\Gamma(t, R_t) = \arctan(-K \vee (-R_t) \wedge K) + 1.6, \qquad K \in \mathbb{R}_+.$$

7.1.2 Payoffs

An institutional investor holding the stock may chose to hedge its financial risk as measured by the stock volatility by issuing a structured derivative that pays yield

$$\varphi^I(t, S_t, R_t) = \exp\left\{ -M * \left(\int_0^t a_s ds - R_t \right)^+ \right\}, \qquad (M > 0) \tag{58}$$

where we interpret $\int_0^t a_s ds$ as the long run average temperature. Along with the institutional investor, two more agents are acting in the market: an orange farmer ("agent a") and a golf resort owner ("agent b"). For the golf resort owner cool season turf grass optimal growth temperature belongs to the interval[9] [15.6 °C, 23.9 °C], for illustration purposes we take 17 °C (i.e. $R^b = -1$) as the golf resort owner's reference temperature. For the orange farmer, low temperatures cause problems in their season yield so his optimal temperature is roughly 22 °C i.e. $R^a = +4$. Finally we assume that the agents' risk preferences are described by entropic utilities with risk tolerance

[8]The truncation of x by K ensures Γ satisfies the conditions of bounded first and second derivatives. In truth, if chosen big enough K will not be really relevant for our simulations.

[9]http://www.nmmastergardeners.org/Manual%20etc/other%20references/turfgrasses.htm

coefficients $\gamma_a = 1.0$ and $\gamma_b = 2.0$ Assuming that both agents also have a position in the stock as their endowment we model their incomes by

$$H^a = c^a S_T + \int_0^T \exp\{-M^a * (R_t - R^a)^2\}dt, \quad H^b = c^b S_T + \int_0^T \exp\{-M^b * (R_t - R^b)^2\}dt$$

for some benchmark temperatures $R^a > R^b$ and positive constants $M^a, M^b \in \mathbb{R}_+$. The constants of our model are chosen as:

γ_a	γ_b	γ_R	M	M^a	M^b	c^a	c^b	R^a	R^b
1.0	2.0	3.0	2.0	0.5	0.5	0.5	0.5	4.0	−1.0

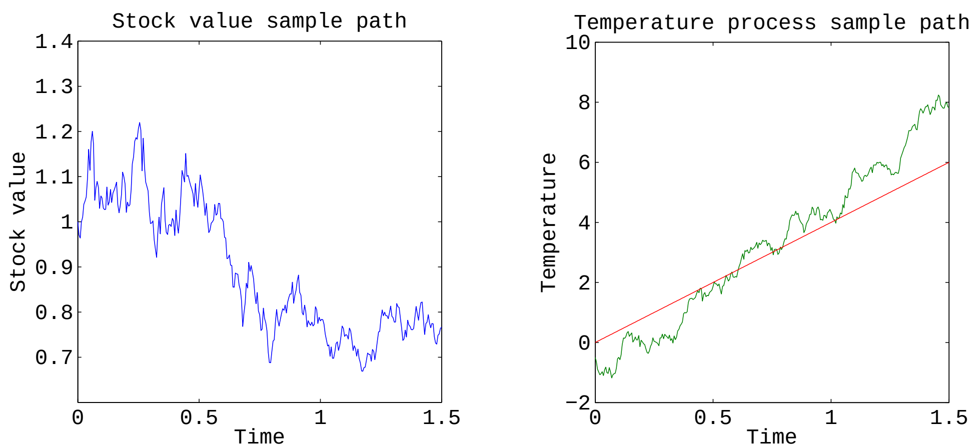

Figure 1: Typical trajectories of the forward processes in the time interval $[0, 1.5]$.

7.2 Solver methodology

In order to simulate our models we first choose the agents' degree of risk tolerance and starting points s_0 and r_0 for the stock and temperature process respectively. After simulating enough trajectories of the forward process (S, R) using the Euler method, we simulate the equilibrium market price of external risk using the representative agent BSDE. In a subsequent step we use the trajectories of θ^{*R} to solve the BSDEs for the bond price processes and agents' residual risk.

The numerical method is a modification of an algorithm by Bender & Denk (2007). It takes advantage of the fact that the driver does not depend on the process Y. For a given partition $\pi = \{0 = t_0, t_1, \ldots, t_M = T\}$ of the time interval $[0, T]$, the corresponding discretization X^π and H^π of the forward process and payoff functions, respectively, we put $(Y^{(0,\pi)}, Z^{(0,\pi)}) = 0$ and define

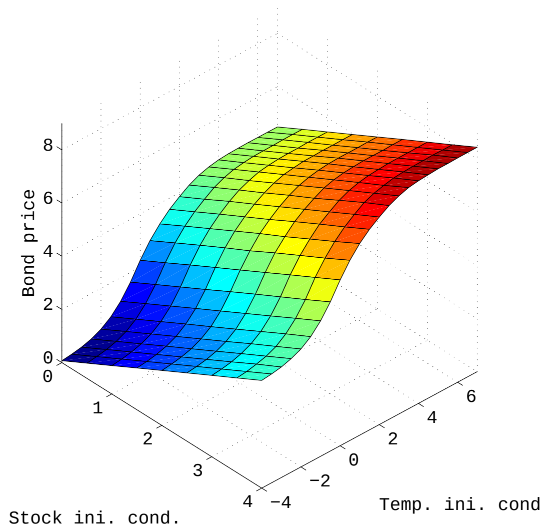

Figure 2: Initial bond prices as a function of the starting point of the forward process.

the processes $Y^{(n,\pi)}$ and $Z^{(n,\pi)}$ recursively by

$$Y_{t_i}^{(n,\pi)} = \mathbb{E}\Big[H^\pi - h\sum_{j=i}^{M-1} f(t_i, X_{t_j}^\pi, Z_{t_j}^{(n-1,\pi)})\Big|\mathcal{F}_{t_i}\Big], \quad (59)$$

$$Z_{t_i}^{(n,\pi)} = \mathbb{E}\Big[\frac{\Delta W_{t_i}}{h}\Big(H^\pi - h\sum_{j=i+1}^{M-1} f(t_i, X_{t_j}^\pi, Z_{t_j}^{(n-1,\pi)})\Big)\Big|\mathcal{F}_{t_i}\Big]. \quad (60)$$

Here $h = t_{i+1} - t_i$ and $\Delta W_i = W_{t_{i+1}} - W_{t_i}$ denotes a Brownian increment. Since only Y_0 is needed we can ignore (59) until the last iteration. The scheme is explicit and converges in $n = M$ iterations.

7.3 Results

In the sequel we display some results. In Figure 2 we display the equilibrium bond price at time $t = 0$ as a function of the starting point of the forward process. The bond price is increasing in the temperature which reflects the fact that the yield curve increases in the external risk component. Figure 3 shows surface plots for the orange farmer's residual risk for the case of risk transfer and risk sharing, respectively. The graphs show that, although the surfaces are qualitatively similar, the case of risk transfer yields a lower risk, i.e., higher utility. A similar result holds for the golf resort owner whose risk exposures for the sharing and transfer case are displayed in Figure 3.

Remark 7.1. *Our simulations suggest that within the framework of our model it is beneficial for both agents to exchange risk exposures through decentralized markets. At first sight this result is*

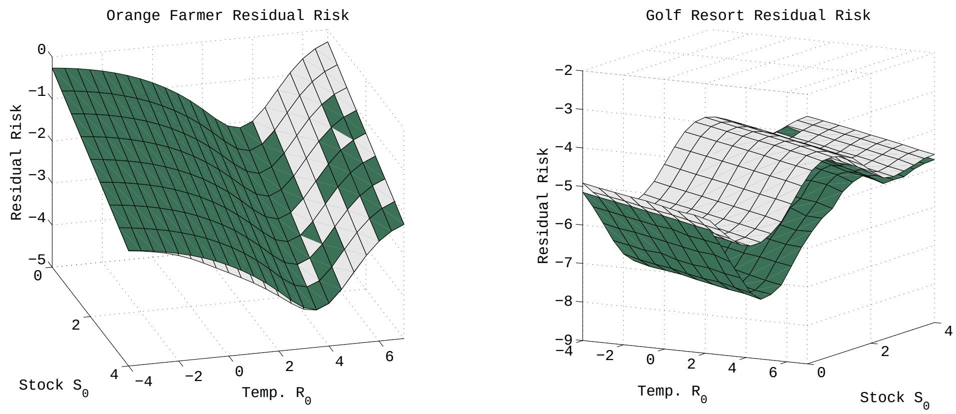

Figure 3: Residual Risk Surfaces for risk sharing (light color) and transfer (dark color): orange farmer (left) and golf resort owner (right)

surprising because in the risk sharing case there is no aggregate cost of hedging whereas in the risk transfer case the agents need to pay for their hedge, i.e, there is an initial outflow of capital. On the other hand, of course, the bond pays dividends whereas the risk sharing case only considers non-dividend paying assets. In our model future dividend payments outweighs the impact of the initial outflow of capital. Notice also that the bond prices depend significantly on both stock price and temperature.

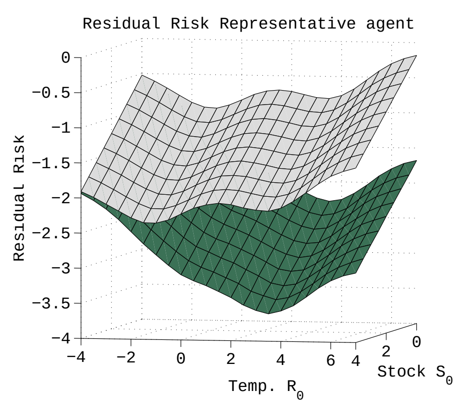

Figure 4: Residual Risk Surface Plot for risk sharing (light color) and transfer (dark color): the represenattive agent

Figure 5 shows the price of one share (unit) of a bond as a function of the number of shares issued and the corresponding revenues. The left plot shows the quantities $\frac{B_0^{(N)}}{N}$ where $B_0^{(N)}$ denotes the equilibrium bond price for the yield curve $N \cdot \varphi^I$. The unit price if decreasing in N, i.e., the price of one share decreases with the number of shares issued. The associated revenues are

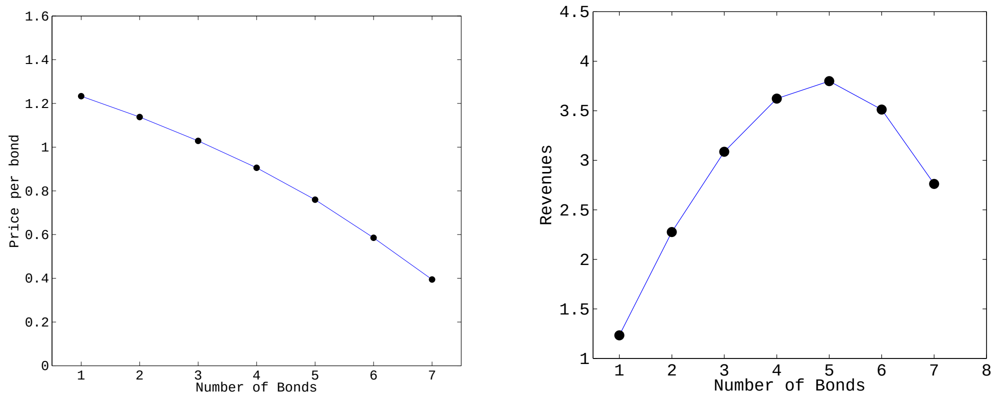

Figure 5: Share prices and revenues as a function of the number of bonds in the market.

displayed in the right plot. We see that for our choice of payoff profiles the revenues are maximal if the issuer sells about 5 units of the bond. A theoretical analysis of the issuer's optimal policy is beyond the scope of this paper and left for future research.

7.4 Some modifications

We close this section with two variations of the previous example. In the first we chose

$$\bar{\varphi}^I(t, S_t, R_t) = \varphi^I(t, S_t, R_t) + (S_t - S_0)^+(-R_t)^+ \wedge K_1 + (S_t - S_0)^+(R_t - 4)^+ \wedge K_2,$$

with big enough constants K_1 and K_2. Of the result we obtained we present only the residual risk plots since all others displayed the same behavior as the ones we presented so far. In Figure 6 we find the residual risk surfaces for both the orange farmer and golf resort owner but now, compared to Figure 3, we see that with this new type of bond the agent's risk is lower. In Figure 7 we present comparatively the risk transfer surfaces for the original example and this variation.

In the second variation we replaced the bond's payoff by

$$\hat{H}^I = \big(\frac{2}{\pi}\arctan(R_T) + 1\big)(S_T - S_0)^+ + \int_0^T \varphi^I(t, S_t, R_t)dt.$$

In Figure 8 we find the residual risk for the markets agents. The representative agent's residual risk surface plot for this case coincided with the surface presented in Figure 4 which suggests that adding such a derivative into the market as a whole plays no impact, although the golf resort owner seems to be much happier.

8 Conclusion

In this paper we provided a general framework within which to price financial securities written on non-tradable risk factors. We assumed that the agents' risk preferences can be described by a

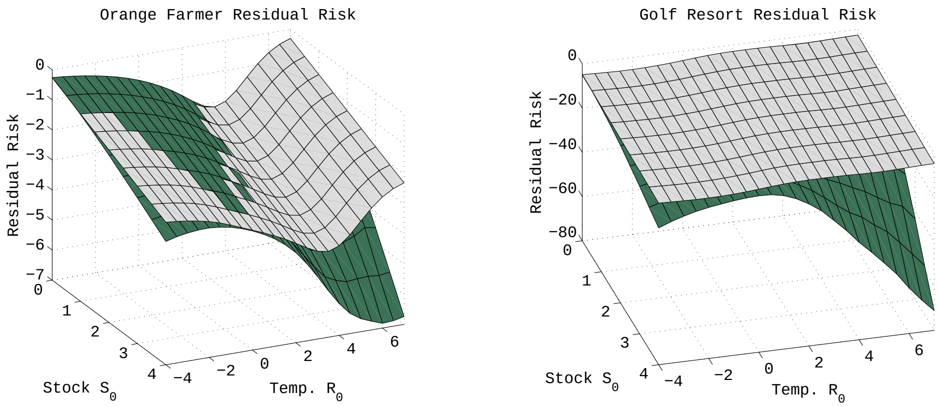

Figure 6: Case $\bar{\varphi}^I(t, S_t, R_t)$: Residual Risk Surfaces for risk sharing (light color) and transfer (dark color): orange producer (left) and golf resort owner (right).

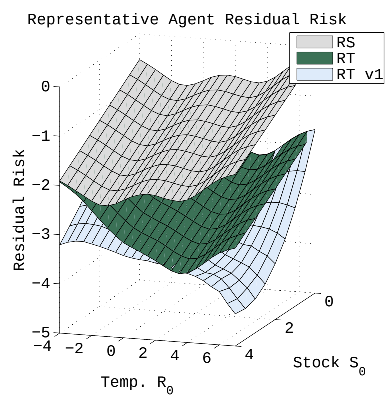

Figure 7: Case $\bar{\varphi}^I(t, S_t, R_t)$: Residual Risk Surface Plot for the represenative agent. From top to bottom: risk sharing case, risk transfer example and risk transfer first variant.

dynamic convex risk measure generated by a backward stochastic differential equation. We solved the individual and representative agents' optimization problems and characterized a certain class of equilibrium markets of risk in terms of a solution to a BSDE. For the specific case of exponential utilities we solve the optimization in closed form. Under the additional assumption that the external risk process follows an Ornstein-Uhlenbeck process and that the market price of financial risk satisfies a modification of the Novikov condition we proved a uniqueness of equilibrium result. The key was a differentiability theorem for quadratic BSDEs with unbounded terminal condition.

Our main goal was to extend the standard representative approach of general equilibrium theory to monetary utility functions and to allow for certain classes of unbounded market prices of financial risk. Many avenues are still open for future research. From personal communication with Freddy Delbaen we believe that any time consistent convex dynamic risk measures can be

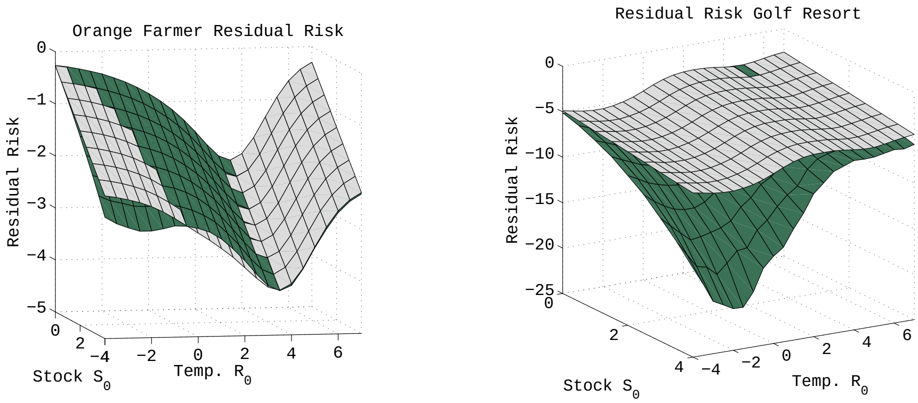

Figure 8: Case $\hat{H}^I$: Residual Risk Surfaces for risk sharing (light color) and transfer (dark color): orange farmer (left) and golf resort owner (right)

represented by a BSDE. The applicability of our results in general and the characterization of equilibrium results in particular are thus limited "only" by the lack of existence, uniqueness and differentiability results for solution of BSDEs beyond the quadratic case. This is a field of active research; a major contribution to this theory was well beyond the scope of this paper. It would also be interesting to consider the problem of optimal derivative design. The simulations of Section 7 can be viewed as a first quantitative step in that direction. Our Figure 5 suggests that for a given payoff profile there exists a revenue maximizing number of shares. Finally, we did not consider models where the assumption of market completion is violated so our method does not cover the incomplete markets.

A Malliavin derivatives

For completeness we review in this appendix the notion of Malliavin derivatives; for details we refer to the textbook of Nualart (1995). We denote by W be an d-dimensional Brownian motion and introduce the space of random variables

$$\mathcal{S} = \left\{ \xi : \xi = F\left(\int_0^T h_d^1 dW_t, \ldots, \int_0^T h_d^d dW_t \right) \right\}$$

where $F : \mathbb{R}^n \to \mathbb{R}^d$ is has bounded partial derivatives of all orders and $h^i \in L^2([0,T];\mathbb{R}^d)$. For any such random variable the d-dimensional operator $D : \mathcal{S} \to L^2(\Omega \times [0,T])^d$ is defined by

$$D_\theta^i \xi = \sum_{j=1}^d \frac{\partial F}{\partial x_{i,j}} \left(\int_0^T h_d^1 dW_t, \ldots, \int_0^T h_d^d dW_t \right) h_\theta^{i,j}$$

and for $p > 1$ the norm $\|\xi\|_{1,p}$ is defined by

$$\|\xi\|_{1,p} = \left(\mathbb{E} \left[|\xi|^p + \left(\int_0^T |D_\theta \xi|^2 \right)^{p/2} \right] \right)^{1/p}.$$

The operator D has a closed extension to the space $\mathbb{D}^{1,p}$, the closure of $\mathcal{S}$ with respect to $\|\cdot\|_{1,p}$.

Lemma A.1. *Let $\{F_n\}$ be a sequence of random variables in $\mathbb{D}^{1,2}$ that converges to F in L^2 and assume that*

$$\sup_n \mathbb{E}\left[\|DF_n\|_{L^2}\right] < \infty.$$

Then $F \in \mathbb{D}^{1,2}$ and the derivatives $\{DF_n\}$ converge to DF in the weak topology of $L^2(\Omega \times [0,T])$.

Let us now consider the diffusion process (M_t) defined by the SDE

$$dM_t = b(t, M_t)dt + \sigma(t, M_t)dW_t \tag{61}$$

where $b(t,\cdot) \in \mathbb{R}^d$ and $\sigma(t,\cdot) = \text{diag}(\sigma^1(t,\cdot),\dots,\sigma^d(t,\cdot))$ is a diagonal matrix. We assume that the diffusion coefficients are differentiable in x with bounded derivatives and fix a differentiable function $g : \mathbb{R}^d \to \mathbb{R}$. With

$$\Delta_{i,t} := \exp\left(\int_0^t \sigma_x^i(s, M_s^i)dW_s^i + \int_0^t \left\{ b_x^i(s, M_s^i) - \frac{1}{2}(\sigma_x^i)^2(s, M_s^i) \right\} ds \right) \tag{62}$$

the Malliavin derivative $D_u^i g(M_t)$ of $g(M_t)$ at time u with respect to the i-th Wiener process is given by

$$D_u^i g(M_t) = \mathbf{1}_{\{t \geq u\}} g_x^i(M_t)\sigma(u, M_u)\Delta_{i,t}\Delta_{i,u}^{-1}.$$

For our external risk process R of Section 6 this means that

$$D_u^2 R_t = \mathbf{1}_{\{t \geq u\}} \exp\left(\int_u^t \mu_x(s, R_s)ds \right).$$

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