Fundamental invariants for the action of $SL_3(\mathbb{C}) \times SL_3(\mathbb{C}) \times SL_3(\mathbb{C})$ on $3 \times 3 \times 3$ arrays

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Introduction

- Classical invariant theory: the determinant of an $n \times n$ matrix $X = (x_{ij})$ over the field $\mathbb{C}$ of complex numbers freely generates the algebra of invariant polynomials in the matrix entries x_{ij} under the action $X \mapsto AXB$ of $SL_n(\mathbb{C}) \times SL_n(\mathbb{C})$.
- For cubic $n \times n \times n$ arrays $X = (x_{ijk})$: consider homogeneous polynomials invariant under $SL_n(\mathbb{C}) \times SL_n(\mathbb{C}) \times SL_n(\mathbb{C})$ acting by changes of basis along the three directions.
- The cubic invariants are freely generated only for $n \leq 3$.
- $n = 2$: one generator, Cayley's hyperdeterminant (degree 4).
- $n = 3$: Vinberg (1976), three generators (degrees 6, 9 and 12).
- We calculate explicitly these three generating invariants; they have respectively 1152, 9216 and 209061 terms.
- We express these polynomials in terms of the orbits of the symmetry group $(S_3 \times S_3 \times S_3) \rtimes S_3$ acting on monomials of weight zero for the Lie algebra $\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C})$.

Preliminaries

- $X = (x_{ijk})$ is a $3 \times 3 \times 3$ array, $x_{ijk} \in \mathbb{C}$ for $1 \le i, j, k \le 3$.
- $\mathbb{C}^{333} = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$: every element is a finite sum of simple tensors $u \otimes v \otimes w$ where $u, v, w \in \mathbb{C}^3$.
- e_1, e_2, e_3 are the standard basis vectors of $\mathbb{C}^3$.
- $\mathbb{C}^{333}$ has basis $e_{ijk} = e_i \otimes e_j \otimes e_k$ for $1 \le i, j, k \le 3$.
- Identify $\mathbb{C}^3$ with $(\mathbb{C}^3)^*$, and $X = (x_{ijk})$ with the element

$$X = \sum_{i,j,k=1}^{3} x_{ijk} e_{ijk}.$$

- Polynomial ring $P = \mathbb{C}[x_{ijk} \,|\, 1 \le i, j, k \le 3]$; monomial basis

$$M(F) = \prod_{i,j,k=1}^{3} x_{ijk}^{f_{ijk}},$$

where $F = (f_{ijk})$ is exponent array of non-negative integers.

- Homogeneous subspace P_N of degree N has monomial basis:

$$\{ M(F) \mid \sum_{i,j,k=1}^{3} f_{ijk} = N \}.$$

- Action of a triple of invertible matrices (A, B, C) on x_{ijk}:

$$(A, B, C) \cdot x_{ijk} = \sum_{p,q,r=1}^{3} a_{pi} b_{qj} c_{rk} x_{pqr},$$

- Action extends to polynomials in the obvious way:

$$(A, B, C) \cdot p(\dots, x_{ijk}, \dots) = p(\dots, (A, B, C) \cdot x_{ijk}, \dots).$$

- Homogeneous polynomial p is called an invariant if

$$\det(A) = \det(B) = \det(C) = 1 \implies (A, B, C) \cdot p = p.$$

Representations of Lie algebras

- Finite-dimensional representations of the Lie group $SL_n(\mathbb{C})$ can be studied in terms of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$.
- $\mathfrak{sl}_n(\mathbb{C})$ has this standard basis:
  - matrix units $U_{i,j}$ for $1 \leq i \neq j \leq n$,
  - diagonal matrices $H_i = U_{i,i} - U_{i+1,i+1}$ for $1 \leq i \leq n{-}1$.
- Simple root vectors: $T_i = U_{i,i+1}$ for $1 \leq i \leq n{-}1$.
- Natural irreducible representation of $\mathfrak{sl}_n(\mathbb{C})$ on $\mathbb{C}^n$:

$$H_i \cdot e_j = \begin{cases} e_j & \text{if } j = i \\ -e_j & \text{if } j = i{+}1 \\ 0 & \text{otherwise,} \end{cases} \qquad T_i \cdot e_j = \begin{cases} e_{j-1} & \text{if } j = i{+}1 \\ 0 & \text{otherwise.} \end{cases}$$

- Semisimple Lie algebra $\mathfrak{sl}_{333}(\mathbb{C}) = \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C})$ acts irreducibly on $\mathbb{C}^{333}$.
- Superscript (ℓ) for $\ell = 1, 2, 3$ indicates ℓ-th $\mathfrak{sl}_3(\mathbb{C})$ summand.

Lemma

Action of Cartan subalgebra (diagonal matrices $H_m^{(\ell)}$) and simple root vectors (raising operators $T_m^{(\ell)}$) on the entries of $X = (x_{ijk})$:

$$H_m^{(1)} \cdot x_{ijk} = \begin{cases} x_{ijk} & \text{if } i = m \\ -x_{ijk} & \text{if } i = m+1 \\ 0 & \text{otherwise} \end{cases} \qquad T_m^{(1)} \cdot x_{ijk} = \begin{cases} x_{i-1,j,k} & \text{if } i = m+1 \\ 0 & \text{otherwise} \end{cases}$$

$$H_m^{(2)} \cdot x_{ijk} = \begin{cases} x_{ijk} & \text{if } j = m \\ -x_{ijk} & \text{if } j = m+1 \\ 0 & \text{otherwise} \end{cases} \qquad T_m^{(2)} \cdot x_{ijk} = \begin{cases} x_{i,j-1,k} & \text{if } j = m+1 \\ 0 & \text{otherwise} \end{cases}$$

$$H_m^{(3)} \cdot x_{ijk} = \begin{cases} x_{ijk} & \text{if } k = m \\ -x_{ijk} & \text{if } k = m+1 \\ 0 & \text{otherwise} \end{cases} \qquad T_m^{(3)} \cdot x_{ijk} = \begin{cases} x_{i,j,k-1} & \text{if } k = m+1 \\ 0 & \text{otherwise} \end{cases}$$

- Action of a Lie algebra on tensor product $V \otimes W$ of modules is defined on simple tensors and extended linearly:

$$A \cdot (v \otimes w) = (A \cdot v) \otimes w + v \otimes (A \cdot w).$$

- Homogeneous polynomials of degree N on basis $v_1, \ldots, v_p$ of L-module V coincide with N-th symmetric power $S^N V$.
- Action of L on $S^N V$:

$$\begin{aligned} A \cdot (v_1^{f_1} v_2^{f_2} \cdots v_p^{f_p}) &= \sum_{i=1}^{p} v_1^{f_1} \cdots (A \cdot v_i^{f_i}) \cdots v_p^{f_p} \\ &= \sum_{i=1}^{p} v_1^{f_1} \cdots \left(f_i v_i^{f_i - 1} (A \cdot v_i) \right) \cdots v_p^{f_p} \\ &= \sum_{i=1}^{p} f_i \, v_1^{f_1} \cdots v_i^{f_i - 1} \cdots v_p^{f_p} (A \cdot v_i). \end{aligned}$$

Lemma

The Cartan subalgebra of $\mathfrak{sl}_{333}(\mathbb{C})$ *acts on monomials as follows:*

$$H_m^{(1)} \cdot \prod_{i,j,k} x_{ijk}^{f_{ijk}} = \sum_{j,k} (f_{m,j,k} - f_{m+1,j,k}) \prod_{i,j,k} x_{ijk}^{f_{ijk}},$$
$$H_m^{(2)} \cdot \prod_{i,j,k} x_{ijk}^{f_{ijk}} = \sum_{i,k} (f_{i,m,k} - f_{i,m+1,k}) \prod_{i,j,k} x_{ijk}^{f_{ijk}},$$
$$H_m^{(3)} \cdot \prod_{i,j,k} x_{ijk}^{f_{ijk}} = \sum_{i,j} (f_{i,j,m} - f_{i,j,m+1}) \prod_{i,j,k} x_{ijk}^{f_{ijk}}.$$

- Eigenvalue of monomial $\prod_{i,j,k} x_{ijk}^{f_{ijk}}$ for $H_m^{(\ell)}$ is denoted $\omega_{\ell,m}$.
- Weight of monomial $\prod_{i,j,k} x_{ijk}^{f_{ijk}}$ is the 6-tuple

$$(\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22}, \omega_{31}, \omega_{32}).$$

- A monomial has weight zero if $\omega_{\ell m} = 0$ for all ℓ and m.
- $W(N \mid \omega_{11}, \omega_{12}, \omega_{21}, \omega_{22}, \omega_{31}, \omega_{32})$ is subspace of P_N spanned by monomials of given degree and given weight.

Lemma

The raising operators in $\mathfrak{sl}_{333}(\mathbb{C})$ act on monomials as follows:

$$T_m^{(1)} \cdot \prod_{i,j,k} x_{ijk}^{f_{ijk}} = \sum_{j,k} f_{m+1,j,k}\, x_{111}^{f_{111}} \cdots x_{m,j,k}^{f_{m,j,k}+1} \cdots x_{m+1,j,k}^{f_{m+1,j,k}-1} \cdots x_{333}^{f_{333}}$$

$$T_m^{(2)} \cdot \prod_{i,j,k} x_{ijk}^{f_{ijk}} = \sum_{i,k} f_{i,m+1,k}\, x_{111}^{f_{111}} \cdots x_{i,m,k}^{f_{i,m,k}+1} \cdots x_{i,m+1,k}^{f_{i,m+1,k}-1} \cdots x_{333}^{f_{333}}$$

$$T_m^{(3)} \cdot \prod_{i,j,k} x_{ijk}^{f_{ijk}} = \sum_{i,j} f_{i,j,m+1}\, x_{111}^{f_{111}} \cdots x_{i,j,m}^{f_{i,j,m}+1} \cdots x_{i,j,m+1}^{f_{i,j,m+1}-1} \cdots x_{333}^{f_{333}}$$

- The raising operators $T_m^{(\ell)}$ induce linear maps:

$$
\begin{aligned}
T_1^{(1)} &\colon W(N \mid 0,0,0,0,0,0) \longrightarrow W(N \mid 2,-1,0,0,0,0), \\
T_2^{(1)} &\colon W(N \mid 0,0,0,0,0,0) \longrightarrow W(N \mid -1,2,0,0,0,0), \\
T_1^{(2)} &\colon W(N \mid 0,0,0,0,0,0) \longrightarrow W(N \mid 0,0,2,-1,0,0), \\
T_2^{(2)} &\colon W(N \mid 0,0,0,0,0,0) \longrightarrow W(N \mid 0,0,-1,2,0,0), \\
T_1^{(3)} &\colon W(N \mid 0,0,0,0,0,0) \longrightarrow W(N \mid 0,0,0,0,2,-1), \\
T_2^{(3)} &\colon W(N \mid 0,0,0,0,0,0) \longrightarrow W(N \mid 0,0,0,0,-1,2).
\end{aligned}
$$

- $\Omega_{\ell m}$ is the nonzero weight in the image of $T_m^{(\ell)}$.
- $\Lambda_N = (T_1^{(1)}, \ldots, T_2^{(3)})$ is the direct sum of these maps:

$$
\Lambda_N \colon W(N \mid 0,0,0,0,0,0) \longrightarrow \bigoplus_{\ell=1}^{3} \bigoplus_{m=1}^{2} W(N \mid \Omega_{\ell m}).
$$

Lemma

The invariant polynomials in degree N for the action of the Lie algebra $\mathfrak{sl}_{333}(\mathbb{C}) = \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C})$ on the vector space $\mathbb{C}^{333} = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ of $3 \times 3 \times 3$ arrays are the (nonzero) elements of the kernel of the linear map Λ_N.

Theorem

The algebra of invariants for $\mathfrak{sl}_{333}(\mathbb{C})$ acting on $\mathbb{C}^{333}$ is freely generated by three fundamental invariants in degrees 6, 9 and 12.

Proof.

Vinberg (1976). $\square$

Corollary

For $N = 6, 9, 12$ the kernel of Λ_N has dimension 1, 1, 2 (resp.).

Combinatorics of monomials

- $F = (f_{ijk})$ is exponent array for monomial $M(F) = \prod_{i,j,k} x_{ijk}^{f_{ijk}}$.
- $\mathrm{flatten}(F) = [f_{111}, f_{112}, f_{113}, f_{121}, f_{122}, f_{123}, \ldots, f_{331}, f_{332}, f_{333}]$
- Total order on exponent arrays is lex order on flattenings.
- Matrix form: the third index distinguishes the frontal slices:

$$\left[\begin{array}{ccc|ccc|ccc} f_{111} & f_{121} & f_{131} & f_{112} & f_{122} & f_{132} & f_{113} & f_{123} & f_{133} \\ f_{211} & f_{221} & f_{231} & f_{212} & f_{222} & f_{232} & f_{213} & f_{223} & f_{233} \\ f_{311} & f_{321} & f_{331} & f_{212} & f_{222} & f_{232} & f_{213} & f_{223} & f_{233} \end{array}\right].$$

- A slice is any 3×3 submatrix obtained by fixing one subscript (giving horizontal, vertical, and frontal slices).

Lemma

A basis for $W(N \mid 0,0,0,0,0,0)$ consists of the monomials $M(F)$ for which every slice of F has sum $N/3$. There are no monomials of weight zero, and hence no invariants, unless N is a multiple of 3.

- Semidirect product $G = (S_3 \times S_3 \times S_3) \rtimes S_3$ acts on weight zero monomials in each degree.
- Three factors in $S_3 \times S_3 \times S_3$ permute parallel slices in three directions; last S_3 permutes directions.
- More precisely, (α, β, γ) and δ act on $F = (f_{ijk})$ by:

$$\big((\alpha, \beta, \gamma) \cdot F\big)_{i,j,k} = f_{\alpha(i),\beta(j),\gamma(k)}, \qquad \big(\delta \cdot F\big)_{i,j,k} = f_{i^\delta,j^\delta,k^\delta}.$$

- Symmetric and alternating orbit sums for monomial $M(F)$:

$$\mathcal{O}^+(M) = \frac{|\mathcal{O}(M)|}{|G|} \sum_{g \in G} M(g \cdot F),$$
$$\mathcal{O}^-(M) = \frac{|\mathcal{O}(M)|}{|G|} \sum_{g \in G} \epsilon(g)\, M(g \cdot F),$$

- $\epsilon(g)$ is product of signs of components of $g = (\alpha, \beta, \gamma, \delta) \in G$.

Degree 6

Lemma

In degree 6, there are 1152 monomials of weight zero, and 792 monomials of each higher weight $\Omega_{\ell m}$ for $\ell = 1, 2, 3$ and $m = 1, 2$.

Lemma

- *The nullspace of the matrix representing Λ_6 has dimension 1 using modular arithmetic with $p = 101$.*
- *Using symmetric representatives of the congruence classes, the canonical basis vector of the nullspace has coefficients*

$$-10,\ -4,\ -2,\ 1,\ 2,\ 4,\ 8.$$

- *If we interpret these coefficients as integers then the corresponding polynomial is an invariant for $\mathfrak{sl}_{333}(\mathbb{C})$.*

Proof.

- Use Maple's `LinearAlgebra[Modular]` to create matrix B with upper and lower blocks 1152×1152 and 792×1152.
- For $\ell = 1, 2, 3$ and $m = 1, 2$ store matrix representing $T_m^{(\ell)}$ in lower block; then compute the row canonical form (RCF).
- At termination, B has rank 1151.
- From RCF, extract a basis vector for the nullspace.
- Perform another computation using integer arithmetic to verify that corresponding polynomial is invariant over $\mathbb{C}$.

$\square$

- Coefficients of canonical basis vector for nullspace of Λ_6 are constant on orbits for the action of $(S_3 \times S_3 \times S_3) \rtimes S_3$.
- For each orbit, the coefficient, the matrix form of the minimal representative, and the orbit size, are in the following table:

I_6	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
-10	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & 1 & . \\ . & . & 1 & . & . & . & 1 & . & . \\ . & 1 & . & 1 & . & . & . & . & . \end{array}\right]$	36	1
-4	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & . & 1 \\ . & 1 & . & . & . & . & 1 & . & . \\ 1 & . & . & . & 1 & . & . & . & . \end{array}\right]$	324	2
-2	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & . & . & . & 2 \\ . & 1 & . & . & 1 & . & . & . & . \\ 1 & . & . & 1 & . & . & . & . & . \end{array}\right]$	162	3
1	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & . & . & . & 2 \\ . & . & . & . & 2 & . & . & . & . \\ 2 & . & . & . & . & . & . & . & . \end{array}\right]$	36	4
2	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & . & 1 \\ . & 1 & . & . & . & . & . & 1 & . \\ 1 & . & . & 1 & . & . & . & . & . \end{array}\right]$	108	5
2	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & . & 1 \\ . & . & . & . & 1 & . & 1 & . & . \\ 1 & 1 & . & . & . & . & . & . & . \end{array}\right]$	324	6
4	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & . & . & . & 2 \\ . & 1 & . & 1 & . & . & . & . & . \\ 1 & . & . & . & 1 & . & . & . & . \end{array}\right]$	54	7
8	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & 1 & . \\ . & . & 1 & . & . & . & 1 & . & . \\ 1 & . & . & . & 1 & . & . & . & . \end{array}\right]$	108	8

Table: The fundamental invariant in degree 6

Theorem

Every invariant in degree 6 for $SL_3(\mathbb{C}) \times SL_3(\mathbb{C}) \times SL_3(\mathbb{C})$ acting on $3 \times 3 \times 3$ arrays $X = (x_{ijk})$ is a scalar multiple of the following linear combination of symmetric orbits sums:

$$
\begin{aligned}
&-10\,\mathcal{O}^+\big(x_{123}x_{132}x_{213}x_{231}x_{312}x_{321}\big)\\
&-4\,\mathcal{O}^+\big(x_{132}x_{133}x_{213}x_{221}x_{311}x_{322}\big)\\
&-2\,\mathcal{O}^+\big(x_{133}^2x_{221}x_{222}x_{311}x_{312}\big)\\
&+\,\mathcal{O}^+\big(x_{133}^2x_{222}^2x_{311}^2\big)\\
&+2\,\mathcal{O}^+\big(x_{132}x_{133}x_{221}x_{223}x_{311}x_{312}\big)\\
&+2\,\mathcal{O}^+\big(x_{132}x_{133}x_{213}x_{222}x_{311}x_{321}\big)\\
&+4\,\mathcal{O}^+\big(x_{133}^2x_{212}x_{221}x_{311}x_{322}\big)\\
&+8\,\mathcal{O}^+\big(x_{123}x_{132}x_{213}x_{231}x_{311}x_{322}\big).
\end{aligned}
$$

Degree 9

Lemma

- *In degree 9, there are 22620 weight zero monomials, and 17802 monomials of each weight $\Omega_{\ell m}$ ($\ell = 1, 2, 3$; $m = 1, 2$).*
- *The kernel of Λ_9 has dimension 1 modulo $p = 101$.*
- *Using symmetric representatives, the canonical basis vector has coefficients $\{-1, 0, 1\}$, and ± 1 each occur 4608 times.*
- *If we interpret these coefficients as integers then the corresponding polynomial is an invariant for $\mathfrak{sl}_{333}(\mathbb{C})$.*
- *The canonical basis vector for the kernel of Λ_9 is a linear combination of alternating orbit sums for the action of $(S_3 \times S_3 \times S_3) \rtimes S_3$ on weight zero monomials.*

For each orbit, the coefficient, the matrix form of the minimal representative, and the orbit size, are in the following tables:

I_9	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
1	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & 1 & 1 \\ . & 1 & . & . & . & 1 & 1 & . & . \\ 2 & . & . & . & 1 & . & . & . & . \end{array}\right]$	648	1
1	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & 1 & 1 \\ . & 1 & 1 & . & . & . & 1 & . & . \\ . & 1 & . & 2 & . & . & . & . & . \end{array}\right]$	648	2
1	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & 1 & 1 \\ . & 1 & . & . & 1 & . & 1 & . & . \\ 1 & . & 1 & 1 & . & . & . & . & . \end{array}\right]$	1296	3
1	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & 1 & 1 \\ . & 2 & . & . & . & . & 1 & . & . \\ 1 & . & . & 1 & . & 1 & . & . & . \end{array}\right]$	648	4
1	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & 1 & 1 \\ . & . & 1 & 1 & . & . & . & 1 & . \\ 1 & 1 & . & 1 & . & . & . & . & . \end{array}\right]$	648	5
1	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & 1 & 1 \\ . & . & 1 & 1 & 1 & . & . & . & . \\ 2 & . & . & . & . & . & . & 1 & . \end{array}\right]$	1296	6
1	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & 1 & 1 \\ . & 1 & . & 1 & . & 1 & . & . & . \\ 1 & 1 & . & . & . & . & 1 & . & . \end{array}\right]$	648	7

Table: The fundamental invariant in degree 9 – part 1

I_9	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
1	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & 1 & 1 \\ . & 1 & 1 & 1 & . & . & . & . & . \\ 1 & . & . & . & 1 & . & 1 & . & . \end{array}\right]$	1296	8
1	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & 1 & 1 \\ . & 1 & . & 1 & . & . & 1 & . & . \\ . & 1 & 1 & 1 & . & . & . & . & . \end{array}\right]$	216	9
1	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & 1 & 1 \\ . & . & 1 & 2 & . & . & . & . & . \\ . & 2 & . & . & . & . & 1 & . & . \end{array}\right]$	648	10
1	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 1 & . & 1 & 1 \\ 1 & . & . & 1 & . & 1 & . & . & . \\ 1 & 1 & . & . & . & . & . & 1 & . \end{array}\right]$	72	11
1	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 2 & . & 1 & . \\ . & 1 & 1 & . & . & . & 1 & . & . \\ 1 & . & . & 1 & . & . & . & 1 & . \end{array}\right]$	432	12
1	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 2 & . & 1 & . \\ . & 1 & . & . & 1 & . & 1 & . & . \\ 2 & . & . & . & . & . & . & . & 1 \end{array}\right]$	648	13
1	$\left[\begin{array}{ccc\|ccc\|ccc} . & . & . & . & . & 2 & . & 1 & . \\ . & . & 1 & . & . & . & 2 & . & . \\ . & 2 & . & 1 & . & . & . & . & . \end{array}\right]$	72	14

Table: The fundamental invariant in degree 9 – part 2

Theorem

Every invariant in degree 9 is a scalar multiple of the following linear combination of alternating orbit sums:

$$
\begin{aligned}
&\mathcal{O}^-\big(x_{123}x_{132}x_{133}x_{213}x_{221}x_{232}x_{311}^2x_{322}\big)\\
+&\mathcal{O}^-\big(x_{123}x_{132}x_{133}x_{213}x_{221}x_{231}x_{312}^2x_{321}\big)\\
+&\mathcal{O}^-\big(x_{123}x_{132}x_{133}x_{213}x_{221}x_{222}x_{311}x_{312}x_{331}\big)\\
+&\mathcal{O}^-\big(x_{123}x_{132}x_{133}x_{213}x_{221}^2x_{311}x_{312}x_{332}\big)\\
+&\mathcal{O}^-\big(x_{123}x_{132}x_{133}x_{212}x_{223}x_{231}x_{311}x_{312}x_{321}\big)\\
+&\mathcal{O}^-\big(x_{123}x_{132}x_{133}x_{212}x_{222}x_{231}x_{311}^2x_{323}\big)\\
+&\mathcal{O}^-\big(x_{123}x_{132}x_{133}x_{212}x_{221}x_{232}x_{311}x_{313}x_{321}\big)\\
+&\mathcal{O}^-\big(x_{123}x_{132}x_{133}x_{212}x_{221}x_{231}x_{311}x_{313}x_{322}\big)\\
+&\mathcal{O}^-\big(x_{123}x_{132}x_{133}x_{212}x_{213}x_{221}x_{312}x_{321}x_{331}\big)\\
+&\mathcal{O}^-\big(x_{123}x_{132}x_{133}x_{212}^2x_{231}x_{313}x_{321}^2\big)\\
+&\mathcal{O}^-\big(x_{123}x_{132}x_{133}x_{211}x_{212}x_{232}x_{311}x_{321}x_{323}\big)\\
+&\mathcal{O}^-\big(x_{123}x_{132}^2x_{213}x_{221}x_{231}x_{311}x_{312}x_{323}\big)\\
+&\mathcal{O}^-\big(x_{123}x_{132}^2x_{213}x_{221}x_{222}x_{311}^2x_{333}\big)\\
+&\mathcal{O}^-\big(x_{123}x_{132}^2x_{213}^2x_{231}x_{312}x_{321}^2\big).
\end{aligned}
$$

Degree 12

- We find explicit forms of two linearly independent invariants, and verify that the simpler one can be taken as the generator.

Lemma

In degree 12, there are 302274 weight zero monomials, and 254961 monomials of each higher weight $\Omega_{\ell m}$ ($\ell = 1, 2, 3$; $m = 1, 2$).

- Even with modular arithmetic, Maple cannot process a matrix with 302274 columns and $302274 + 254961 = 557235$ rows.
- We make the problem much smaller by looking for invariants that are linear combinations of symmetric orbit sums.

Lemma

In degree 12, there are 359 orbits for the action of the symmetry group $(S_3 \times S_3 \times S_3) \rtimes S_3$ on weight zero monomials.

Lemma

The nullspace of the matrix representing the restriction of Λ_{12} *to the span of the symmetric orbit sums has dimension* 2 *modulo* 101.

Proof.

- We use modular arithmetic on a matrix M with an upper block of size 359×359 and a lower block of size 254961×359.
- For $\ell = 1, 2, 3$ and $m = 1, 2$ we consider the restriction of $T_m^{(\ell)}$ to the subspace spanned by the symmetric orbit sums.
- We store the corresponding matrix in the lower block, and compute the row canonical form.
- After the first iteration ($\ell = m = 1$) the rank is 357, and does not increase for the remaining five iterations.
- This agrees with the corollary to Vinberg's theorem (nullity 2).
- A basis of the kernel consists of I_6^2 and a new generator I_{12}.

$\square$

- Useful fact relating rank of an integer matrix over $\mathbb{Q}$ and $\mathbb{F}_p$:

Lemma

Let A be a matrix with integer entries, and let p be a prime.

- *If r_0 is the rank of A over $\mathbb{Q}$ and r_p is its rank over the field $\mathbb{F}_p$, then $r_p \leq r_0$.*
- *Hence the dimension of the nullspace of A over $\mathbb{Q}$ is no larger than the dimension of the nullspace over $\mathbb{F}_p$.*

Proof.

The rank is r if and only if all $(r+1)$-minors are zero, and at least one r-minor is not zero. Since the minors are polynomials in the matrix entries with integer coefficients, the claim follows. $\square$

Outline of the computation for invariants in degree 12:

- For each $\ell = 1, 2, 3$ and $m = 1, 2$ we totally order the set of 254961 higher weight monomials, and partition this set into 399 groups of 639 consecutive monomials.
- For $j = 1, 2, \ldots 359$ we compute the action of $T_m^{(\ell)}$ on the j-th symmetric orbit sum, and separate the terms by their groups.
- We create an integer matrix M with a 359×359 upper block and a 639×359 lower block, initialized to zero.
- For $\ell = 1, 2, 3$, $m = 1, 2$, $j = 1, 2, \ldots, 359$, $k = 1, 2, \ldots, 399$ we consider the terms in group k obtained by applying $T_m^{(\ell)}$ to the j-th symmetric orbit sum.
- We store the integer coefficients of these higher weight monomials in the lower block, and compute the Hermite normal form (HNF); this is the analogue of the row canonical form over a PID (in this case the ring of integers).

- After the first iteration the rank is 357, and does not increase for the remaining iterations.
- At termination, we have an integer matrix of size 357×359, also called M, whose integer nullspace consists of the coefficient vectors of the invariant polynomials of degree 12.
- To find a lattice basis for this integer nullspace, we compute the HNF of the transpose M^t, obtaining integer matrices U (invertible) and H, of sizes 359×359 and 359×357, for which $UM^t = H$.
- The last two rows of U form a lattice basis for the integer nullspace of M.
- We apply the algorithm of Gauss-Lagrange to these last two rows to find a short basis of the integer nullspace.
- Since the lattice is 2-dimensional, the output consists of a shortest nonzero lattice vector I_{12} and a shortest lattice vector I'_{12} which is not a multiple of the first vector.

- Finally, we perform an independent check using integer arithmetic that I_{12} and I'_{12} are annihilated by all the $T_m^{(\ell)}$.

Theorem

In degree 12, the polynomials I_{12} and I'_{12} form a reduced basis for the lattice of invariants with integer coefficients.

Lemma

We have $I'_{12} = I_6^2 + 21\, I_{12}$. we take I_{12} is the fundamental invariant; it involves 235 of the 359 orbits, with a total of 209061 monomials.

The components of these reduced basis vectors I_{12} and I'_{12} are given on the following twelve pages!

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
0	1	0 0 0 0 0 0 0 0 4 0 0 0 0 4 0 0 0 0 4 0 0 0 0 0 0 0 0	36	1
0	−4	0 0 0 0 0 0 0 0 4 0 0 0 1 3 0 0 0 0 3 1 0 0 0 0 0 0 0	324	2
0	6	0 0 0 0 0 0 0 0 4 0 0 0 2 2 0 0 0 0 2 2 0 0 0 0 0 0 0	162	3
0	4	0 0 0 0 0 0 0 0 4 0 1 0 1 2 0 0 0 0 2 1 0 1 0 0 0 0 0	324	4
0	8	0 0 0 0 0 0 0 0 4 0 1 0 1 2 0 0 0 0 3 0 0 0 1 0 0 0 0	216	5
0	−16	0 0 0 0 0 0 0 0 4 0 1 0 2 1 0 0 0 0 2 1 0 0 1 0 0 0 0	324	6
0	16	0 0 0 0 0 0 0 0 4 0 2 0 2 0 0 0 0 0 2 0 0 0 2 0 0 0 0	54	7
0	56	0 0 0 0 0 0 0 0 4 1 1 0 1 1 0 0 0 0 1 1 0 1 1 0 0 0 0	27	8
0	12	0 0 0 0 0 0 0 1 3 0 0 0 1 2 1 0 0 0 3 1 0 0 0 0 0 0 0	324	9
0	4	0 0 0 0 0 0 0 1 3 0 0 0 1 3 0 0 0 0 3 0 1 0 0 0 0 0 0	216	10
0	−12	0 0 0 0 0 0 0 1 3 0 0 0 2 1 1 0 0 0 2 2 0 0 0 0 0 0 0	648	11
0	4	0 0 0 0 0 0 0 1 3 0 0 1 0 3 0 0 0 0 3 0 0 1 0 0 0 0 0	648	12
0	−4	0 0 0 0 0 0 0 1 3 0 0 1 1 2 0 0 0 0 2 1 0 1 0 0 0 0 0	1296	13
0	−8	0 0 0 0 0 0 0 1 3 0 0 1 1 2 0 0 0 0 3 0 0 0 1 0 0 0 0	1296	14
0	16	0 0 0 0 0 0 0 1 3 0 0 1 2 1 0 0 0 0 2 1 0 0 1 0 0 0 0	648	15
0	12	0 0 0 0 0 0 0 1 3 0 1 0 0 2 1 0 0 0 3 0 0 1 0 0 0 0 0	648	16
0	−8	0 0 0 0 0 0 0 1 3 0 1 0 1 1 1 0 0 0 2 1 0 1 0 0 0 0 0	1296	17
0	−16	0 0 0 0 0 0 0 1 3 0 1 0 1 1 1 0 0 0 3 0 0 0 1 0 0 0 0	648	18
0	−4	0 0 0 0 0 0 0 1 3 0 1 0 1 2 0 0 0 0 2 0 1 1 0 0 0 0 0	1296	19
0	16	0 0 0 0 0 0 0 1 3 0 1 0 2 0 1 0 0 0 2 1 0 0 1 0 0 0 0	648	20
0	−12	0 0 0 0 0 0 0 1 3 0 1 1 0 2 0 0 0 0 2 0 0 2 0 0 0 0 0	1296	21
0	−8	0 0 0 0 0 0 0 1 3 0 1 1 1 1 0 0 0 0 1 1 0 2 0 0 0 0 0	648	22
0	32	0 0 0 0 0 0 0 1 3 0 1 1 1 1 0 0 0 0 2 0 0 1 1 0 0 0 0	1296	23
0	−32	0 0 0 0 0 0 0 1 3 0 1 1 2 0 0 0 0 0 2 0 0 0 2 0 0 0 0	648	24
0	−4	0 0 0 0 0 0 0 1 3 0 2 0 1 0 1 0 0 0 1 1 0 2 0 0 0 0 0	1296	25
0	16	0 0 0 0 0 0 0 1 3 0 2 0 1 0 1 0 0 0 2 0 0 1 1 0 0 0 0	1296	26
0	16	0 0 0 0 0 0 0 1 3 0 2 0 1 1 0 0 0 0 2 0 0 1 0 1 0 0 0	1296	27
0	−24	0 0 0 0 0 0 0 1 3 0 2 1 1 0 0 0 0 0 1 0 0 2 1 0 0 0 0	648	28
0	−8	0 0 0 0 0 0 0 1 3 0 3 0 1 0 0 0 0 0 1 0 0 2 0 1 0 0 0	648	29
0	32	0 0 0 0 0 0 0 1 3 1 0 0 1 1 1 0 0 0 1 2 0 1 0 0 0 0 0	648	30

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
0	-56	0 0 0 0 0 0 0 1 3 1 0 1 1 1 0 0 0 0 1 1 0 1 1 0 0 0 0	648	31
0	6	0 0 0 0 0 0 0 2 2 0 0 0 2 0 2 0 0 0 2 2 0 0 0 0 0 0 0	108	32
0	24	0 0 0 0 0 0 0 2 2 0 0 0 2 1 1 0 0 0 2 1 1 0 0 0 0 0 0	162	33
0	8	0 0 0 0 0 0 0 2 2 0 0 1 1 1 1 0 0 0 2 1 0 1 0 0 0 0 0	1296	34
0	16	0 0 0 0 0 0 0 2 2 0 0 1 1 1 1 0 0 0 3 0 0 0 1 0 0 0 0	648	35
0	4	0 0 0 0 0 0 0 2 2 0 0 1 1 2 0 0 0 0 2 0 1 1 0 0 0 0 0	1296	36
0	8	0 0 0 0 0 0 0 2 2 0 0 1 1 2 0 0 0 0 3 0 0 0 0 1 0 0 0	648	37
0	-16	0 0 0 0 0 0 0 2 2 0 0 1 2 0 1 0 0 0 2 1 0 0 1 0 0 0 0	648	38
0	-16	0 0 0 0 0 0 0 2 2 0 0 1 2 1 0 0 0 0 2 1 0 0 0 1 0 0 0	324	39
0	6	0 0 0 0 0 0 0 2 2 0 0 2 0 2 0 0 0 0 2 0 0 2 0 0 0 0 0	324	40
0	4	0 0 0 0 0 0 0 2 2 0 0 2 1 1 0 0 0 0 1 1 0 2 0 0 0 0 0	648	41
0	-16	0 0 0 0 0 0 0 2 2 0 0 2 1 1 0 0 0 0 2 0 0 1 1 0 0 0 0	1296	42
0	16	0 0 0 0 0 0 0 2 2 0 0 2 2 0 0 0 0 0 2 0 0 0 2 0 0 0 0	324	43
0	24	0 0 0 0 0 0 0 2 2 0 1 1 0 1 1 0 0 0 2 0 0 2 0 0 0 0 0	162	44
0	8	0 0 0 0 0 0 0 2 2 0 1 1 1 0 1 0 0 0 1 1 0 2 0 0 0 0 0	648	45
0	-32	0 0 0 0 0 0 0 2 2 0 1 1 1 0 1 0 0 0 2 0 0 1 1 0 0 0 0	1296	46
0	64	0 0 0 0 0 0 0 2 2 0 1 1 2 0 0 0 0 0 2 0 0 0 1 1 0 0 0	162	47
0	24	0 0 0 0 0 0 0 2 2 0 1 2 1 0 0 0 0 0 1 0 0 2 1 0 0 0 0	648	48
0	-16	0 0 0 0 0 0 0 2 2 1 0 0 1 0 2 0 0 0 1 2 0 1 0 0 0 0 0	324	49
0	-64	0 0 0 0 0 0 0 2 2 1 0 0 1 1 1 0 0 0 1 1 1 1 0 0 0 0 0	162	50
0	56	0 0 0 0 0 0 0 2 2 1 0 1 1 0 1 0 0 0 1 1 0 1 1 0 0 0 0	324	51
0	56	0 0 0 0 0 0 0 2 2 1 0 1 1 1 0 0 0 0 1 1 0 1 0 1 0 0 0	162	52
0	-12	0 0 0 0 0 0 1 1 2 0 0 0 1 2 1 0 0 0 2 1 1 0 0 0 0 0 0	108	53
0	28	0 0 0 0 0 0 1 1 2 0 0 1 0 2 1 0 0 0 2 1 0 1 0 0 0 0 0	1296	54
0	-4	0 0 0 0 0 0 1 1 2 0 0 1 0 2 1 0 0 0 3 0 0 0 1 0 0 0 0	1296	55
0	-8	0 0 0 0 0 0 1 1 2 0 0 1 0 3 0 0 0 0 3 0 0 0 0 1 0 0 0	324	56
0	-24	0 0 0 0 0 0 1 1 2 0 0 1 1 1 1 0 0 0 1 2 0 1 0 0 0 0 0	1296	57
0	8	0 0 0 0 0 0 1 1 2 0 0 1 1 2 0 0 0 0 2 1 0 0 0 1 0 0 0	324	58
0	-16	0 0 0 0 0 0 1 1 2 0 0 2 0 2 0 0 0 0 1 1 0 2 0 0 0 0 0	1296	59
0	4	0 0 0 0 0 0 1 1 2 0 0 2 0 2 0 0 0 0 2 0 0 1 1 0 0 0 0	648	60

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
0	56	0 0 0 0 0 0 1 1 2 0 0 2 1 1 0 0 0 0 1 1 0 1 1 0 0 0 0	324	61
0	8	0 0 0 0 0 0 1 1 2 0 1 0 0 1 2 0 0 0 2 1 0 1 0 0 0 0 0	1296	62
0	16	0 0 0 0 0 0 1 1 2 0 1 0 0 1 2 0 0 0 3 0 0 0 1 0 0 0 0	648	63
0	−32	0 0 0 0 0 0 1 1 2 0 1 0 0 2 1 0 0 0 2 0 1 1 0 0 0 0 0	648	64
0	−32	0 0 0 0 0 0 1 1 2 0 1 0 1 0 2 0 0 0 2 1 0 0 1 0 0 0 0	648	65
0	32	0 0 0 0 0 0 1 1 2 0 1 0 1 1 1 0 0 0 1 1 1 1 0 0 0 0 0	324	66
0	16	0 0 0 0 0 0 1 1 2 0 1 0 1 1 1 0 0 0 2 0 1 0 1 0 0 0 0	648	67
0	−24	0 0 0 0 0 0 1 1 2 0 1 1 0 1 1 0 0 0 1 1 0 2 0 0 0 0 0	1296	68
0	8	0 0 0 0 0 0 1 1 2 0 1 1 0 2 0 0 0 0 1 0 1 2 0 0 0 0 0	648	69
0	28	0 0 0 0 0 0 1 1 2 0 1 1 0 2 0 0 0 0 2 0 0 1 0 1 0 0 0	648	70
0	−8	0 0 0 0 0 0 1 1 2 0 1 1 1 0 1 0 0 0 1 1 0 1 1 0 0 0 0	648	71
0	48	0 0 0 0 0 0 1 1 2 0 1 1 1 0 1 0 0 0 2 0 0 0 2 0 0 0 0	648	72
0	72	0 0 0 0 0 0 1 1 2 0 1 1 1 1 0 0 0 0 1 1 0 1 0 1 0 0 0	324	73
0	−64	0 0 0 0 0 0 1 1 2 0 1 1 1 1 0 0 0 0 2 0 0 0 1 1 0 0 0	648	74
0	−12	0 0 0 0 0 1 0 1 2 0 0 1 0 2 0 1 0 0 2 1 0 1 0 0 0 0 0	324	75
0	16	0 0 0 0 0 1 0 1 2 0 0 1 0 2 0 1 0 0 3 0 0 0 1 0 0 0 0	648	76
1	13	0 0 0 0 0 1 0 1 2 0 0 1 1 1 0 0 1 0 2 1 0 1 0 0 0 0 0	648	77
−1	−5	0 0 0 0 0 1 0 1 2 0 0 1 1 1 0 0 1 0 3 0 0 0 1 0 0 0 0	1296	78
−1	19	0 0 0 0 0 1 0 1 2 0 0 1 1 1 0 1 0 0 1 2 0 1 0 0 0 0 0	1296	79
1	−27	0 0 0 0 0 1 0 1 2 0 0 1 1 1 0 1 0 0 2 1 0 0 1 0 0 0 0	1296	80
−1	11	0 0 0 0 0 1 0 1 2 0 0 1 1 2 0 0 0 0 2 1 0 0 0 0 1 0 0	1296	81
1	13	0 0 0 0 0 1 0 1 2 0 0 1 1 2 0 0 0 0 3 0 0 0 0 0 0 1 0	1296	82
−1	−9	0 0 0 0 0 1 0 1 2 0 0 1 2 0 0 0 1 0 1 2 0 1 0 0 0 0 0	1296	83
1	5	0 0 0 0 0 1 0 1 2 0 0 1 2 0 0 0 1 0 2 1 0 0 1 0 0 0 0	1296	84
1	−7	0 0 0 0 0 1 0 1 2 0 0 1 2 0 0 1 0 0 0 3 0 1 0 0 0 0 0	648	85
2	2	0 0 0 0 0 1 0 1 2 0 0 1 2 1 0 0 0 0 1 2 0 0 0 0 1 0 0	1296	86
−2	−26	0 0 0 0 0 1 0 1 2 0 0 1 2 1 0 0 0 0 2 1 0 0 0 0 0 1 0	1296	87
−1	−5	0 0 0 0 0 1 0 1 2 0 0 1 3 0 0 0 0 0 0 3 0 0 0 0 1 0 0	648	88
1	13	0 0 0 0 0 1 0 1 2 0 0 1 3 0 0 0 0 0 1 2 0 0 0 0 0 1 0	648	89
−2	−18	0 0 0 0 0 1 0 1 2 0 1 0 1 0 1 0 1 0 2 1 0 1 0 0 0 0 0	324	90

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
2	26	000001012010101010300010000	648	91
1	13	000001012010101100120100000	1296	92
−1	−21	000001012010101100210010000	1296	93
0	16	000001012010110001210100000	648	94
0	32	000001012010110001300010000	216	95
1	13	000001012010110010201100000	1296	96
−1	−5	000001012010110010300001000	1296	97
0	−8	000001012010110100111100000	1296	98
1	−27	000001012010110100201010000	1296	99
−1	−21	000001012010110100210001000	1296	100
0	−12	000001012010120000201000100	216	101
0	16	000001012010120000300000001	648	102
0	8	000001012010200001120100000	1296	103
0	−32	000001012010200001210010000	648	104
1	13	000001012010200010111100000	1296	105
−2	−26	000001012010200010201010000	1296	106
1	5	000001012010200010210001000	1296	107
−1	−29	000001012010200100021100000	216	108
1	53	000001012010200100111010000	1296	109
0	8	000001012010200100120001000	1296	110
−1	−9	000001012010201000120000100	1296	111
1	5	000001012010201000210000010	1296	112
−1	19	000001012010210000111000100	1296	113
1	5	000001012010210000201000010	1296	114
0	−32	000001012010210000210000001	648	115
−1	−5	000001012010300000111000010	1296	116
−2	14	000001012011100100020200000	648	117
4	−60	000001012011100100110110000	648	118
−2	54	000001012011100100200020000	648	119
2	22	000001012011200000020100100	648	120

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
−3	−15	0 0 0 0 0 1 0 1 2 0 1 1 2 0 0 0 0 0 1 1 0 0 1 0 1 0 0	1296	121
−1	−21	0 0 0 0 0 1 0 1 2 0 1 1 2 0 0 0 0 0 1 1 0 1 0 0 0 1 0	1296	122
2	42	0 0 0 0 0 1 0 1 2 0 1 1 2 0 0 0 0 0 2 0 0 0 1 0 0 1 0	1296	123
2	−38	0 0 0 0 0 1 0 1 2 0 2 0 2 0 0 0 0 0 1 0 1 0 1 0 1 0 0	648	124
1	5	0 0 0 0 0 1 0 1 2 0 2 0 2 0 0 0 0 0 1 0 1 1 0 0 0 1 0	1296	125
−2	−10	0 0 0 0 0 1 0 1 2 0 2 0 2 0 0 0 0 0 2 0 0 0 0 1 0 1 0	648	126
0	64	0 0 0 0 0 1 0 1 2 0 2 0 2 0 0 0 0 0 2 0 0 0 1 0 0 0 1	216	127
−1	−21	0 0 0 0 0 1 0 1 2 1 0 0 0 1 1 1 0 0 1 2 0 1 0 0 0 0 0	1296	128
1	13	0 0 0 0 0 1 0 1 2 1 0 0 0 1 1 1 0 0 2 1 0 0 1 0 0 0 0	1296	129
0	48	0 0 0 0 0 1 0 1 2 1 0 0 0 2 0 0 0 1 2 1 0 1 0 0 0 0 0	108	130
1	−27	0 0 0 0 0 1 0 1 2 1 0 0 0 2 0 1 0 0 1 1 1 1 0 0 0 0 0	1296	131
−1	−9	0 0 0 0 0 1 0 1 2 1 0 0 0 2 0 1 0 0 2 1 0 0 0 1 0 0 0	648	132
2	10	0 0 0 0 0 1 0 1 2 1 0 0 1 0 1 0 1 0 1 2 0 1 0 0 0 0 0	1296	133
−2	−18	0 0 0 0 0 1 0 1 2 1 0 0 1 0 1 0 1 0 2 1 0 0 1 0 0 0 0	648	134
0	−64	0 0 0 0 0 1 0 1 2 1 0 0 1 1 0 0 0 1 1 2 0 1 0 0 0 0 0	648	135
−2	−42	0 0 0 0 0 1 0 1 2 1 0 0 1 1 0 0 1 0 1 1 1 1 0 0 0 0 0	1296	136
−2	−2	0 0 0 0 0 1 0 1 2 1 0 0 1 1 0 1 0 0 1 1 1 0 1 0 0 0 0	648	137
1	13	0 0 0 0 0 1 0 1 2 1 0 0 1 1 0 1 0 0 1 2 0 0 0 1 0 0 0	1296	138
−1	−21	0 0 0 0 0 1 0 1 2 1 0 0 1 1 1 0 0 0 1 2 0 0 0 0 1 0 0	1296	139
1	13	0 0 0 0 0 1 0 1 2 1 0 0 1 1 1 0 0 0 2 1 0 0 0 0 0 1 0	648	140
1	−27	0 0 0 0 0 1 0 1 2 1 0 0 1 2 0 0 0 0 1 1 1 0 0 0 1 0 0	1296	141
−1	−9	0 0 0 0 0 1 0 1 2 1 0 0 2 0 0 0 1 0 1 2 0 0 0 1 0 0 0	648	142
−1	−9	0 0 0 0 0 1 0 1 2 1 0 0 2 0 1 0 0 0 1 2 0 0 0 0 0 1 0	216	143
−2	78	0 0 0 0 0 1 0 1 2 1 0 1 0 1 0 1 0 0 1 1 0 1 1 0 0 0 0	648	144
0	−32	0 0 0 0 0 1 0 1 2 1 0 1 0 1 0 1 0 0 2 0 0 0 2 0 0 0 0	648	145
−2	−2	0 0 0 0 0 1 0 1 2 1 0 1 1 0 0 0 1 0 1 1 0 1 1 0 0 0 0	1296	146
1	5	0 0 0 0 0 1 0 1 2 1 0 1 1 0 0 0 1 0 2 0 0 0 2 0 0 0 0	1296	147
0	32	0 0 0 0 0 1 0 1 2 1 0 1 1 1 0 0 0 0 1 1 0 0 1 0 1 0 0	648	148
4	60	0 0 0 0 0 1 0 1 2 1 0 1 1 1 0 0 0 0 1 1 0 1 0 0 0 1 0	1296	149
−1	−21	0 0 0 0 0 1 0 1 2 1 0 1 1 1 0 0 0 0 2 0 0 0 1 0 0 1 0	1296	150

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
0	112	000001012110100001110110000	216	151
−2	−2	000001012110100010110101000	648	152
−1	−21	000001012110100010200011000	1296	153
2	10	000001012110110000200001010	648	154
1	−11	000001021001011100300010000	1296	155
−1	−21	000001021001020100300001000	1296	156
0	−24	000001021001101100120100000	648	157
0	16	000001021001101100210010000	1296	158
1	−11	000001021001110001300010000	1296	159
2	−14	000001021001110100111100000	1296	160
−2	22	000001021001110100201010000	648	161
0	16	000001021001110100210001000	1296	162
−1	−21	000001021001120000300000001	1296	163
1	5	000001021001200001120100000	1296	164
−1	11	000001021001200001210010000	1296	165
1	−27	000001021001200100111010000	1296	166
1	5	000001021001200100120001000	1296	167
−1	11	000001021001201000120000100	1296	168
−3	−15	000001021001210000111000100	1296	169
2	42	000001021001210000210000001	648	170
2	10	000001021001300000021000100	648	171
1	−15	000001021002010100110200000	648	172
−1	11	000001021002010100200110000	1296	173
2	−2	000001021002100100020200000	648	174
−4	28	000001021002100100110110000	1296	175
2	−22	000001021002100100200020000	1296	176
0	16	000001021002110000110100100	1296	177
−1	11	000001021002110000200010100	1296	178
−2	−10	000001021002200000020100100	648	179
3	−1	000001021002200000110010100	1296	180

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
1	13	0 0 0 0 0 1 0 2 1 0 1 0 0 0 2 1 0 0 3 0 0 0 1 0 0 0 0	648	181
−1	−5	0 0 0 0 0 1 0 2 1 0 1 0 0 1 1 1 0 0 3 0 0 0 0 1 0 0 0	1296	182
−2	−42	0 0 0 0 0 1 0 2 1 0 1 0 1 0 1 0 0 1 3 0 0 0 1 0 0 0 0	648	183
−2	−2	0 0 0 0 0 1 0 2 1 0 1 0 1 0 1 1 0 0 1 1 1 1 0 0 0 0 0	1296	184
0	16	0 0 0 0 0 1 0 2 1 0 1 0 1 0 1 1 0 0 2 0 1 0 1 0 0 0 0	1296	185
2	10	0 0 0 0 0 1 0 2 1 0 1 0 1 0 1 1 0 0 2 1 0 0 0 1 0 0 0	1296	186
0	16	0 0 0 0 0 1 0 2 1 0 1 0 1 1 0 1 0 0 2 0 1 0 0 1 0 0 0	648	187
1	−11	0 0 0 0 0 1 0 2 1 0 1 0 1 1 1 0 0 0 3 0 0 0 0 0 0 0 1	1296	188
−1	−21	0 0 0 0 0 1 0 2 1 0 1 0 2 0 0 0 0 1 1 1 1 1 0 0 0 0 0	1296	189
2	42	0 0 0 0 0 1 0 2 1 0 1 0 2 0 0 0 0 1 2 0 1 0 1 0 0 0 0	648	190
−1	−29	0 0 0 0 0 1 0 2 1 0 1 0 2 0 0 1 0 0 1 0 2 0 1 0 0 0 0	1296	191
−1	−21	0 0 0 0 0 1 0 2 1 0 1 0 2 0 0 1 0 0 1 1 1 0 0 1 0 0 0	1296	192
3	7	0 0 0 0 0 1 0 2 1 0 1 0 2 0 1 0 0 0 1 1 1 0 0 0 1 0 0	648	193
1	−15	0 0 0 0 0 1 0 2 1 0 1 0 2 1 0 0 0 0 1 0 2 0 0 0 1 0 0	432	194
−2	2	0 0 0 0 0 1 0 2 1 0 1 0 3 0 0 0 0 0 0 1 2 0 0 0 1 0 0	648	195
3	7	0 0 0 0 0 1 0 2 1 0 1 1 0 0 1 1 0 0 1 1 0 2 0 0 0 0 0	648	196
−3	−15	0 0 0 0 0 1 0 2 1 0 1 1 0 0 1 1 0 0 2 0 0 1 1 0 0 0 0	1296	197
3	47	0 0 0 0 0 1 0 2 1 0 1 1 0 1 0 1 0 0 2 0 0 1 0 1 0 0 0	648	198
1	53	0 0 0 0 0 1 0 2 1 0 1 1 1 0 0 0 0 1 2 0 0 1 1 0 0 0 0	1296	199
0	−24	0 0 0 0 0 1 0 2 1 0 1 1 1 0 0 1 0 0 0 1 1 2 0 0 0 0 0	648	200
2	26	0 0 0 0 0 1 0 2 1 0 1 1 1 0 0 1 0 0 1 0 1 1 1 0 0 0 0	1296	201
−2	38	0 0 0 0 0 1 0 2 1 0 1 1 1 0 0 1 0 0 1 1 0 1 0 1 0 0 0	1296	202
0	−64	0 0 0 0 0 1 0 2 1 0 1 1 1 0 0 1 0 0 2 0 0 0 1 1 0 0 0	1296	203
−2	−2	0 0 0 0 0 1 0 2 1 0 1 1 1 0 1 0 0 0 1 1 0 1 0 0 1 0 0	1296	204
3	47	0 0 0 0 0 1 0 2 1 0 1 1 1 0 1 0 0 0 2 0 0 0 1 0 1 0 0	1296	205
2	−14	0 0 0 0 0 1 0 2 1 0 1 1 1 1 0 0 0 0 1 0 1 1 0 0 1 0 0	1296	206
−3	−15	0 0 0 0 0 1 0 2 1 0 1 1 1 1 0 0 0 0 2 0 0 0 0 1 1 0 0	1296	207
1	53	0 0 0 0 0 1 0 2 1 0 1 1 1 1 0 0 0 0 2 0 0 1 0 0 0 0 1	1296	208
−1	59	0 0 0 0 0 1 0 2 1 0 1 1 2 0 0 0 0 0 1 0 1 0 1 0 1 0 0	1296	209
1	−27	0 0 0 0 0 1 0 2 1 0 1 1 2 0 0 0 0 0 1 1 0 0 0 1 1 0 0	1296	210

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
-2	-106	0 0 0 0 0 1 0 2 1 0 1 1 2 0 0 0 0 0 2 0 0 0 1 0 0 0 1	648	211
-2	-58	0 0 0 0 0 1 0 2 1 0 1 2 1 0 0 0 0 0 1 0 0 1 1 0 1 0 0	1296	212
1	5	0 0 0 0 0 1 0 2 1 0 2 0 0 0 1 1 0 0 2 0 0 1 0 1 0 0 0	648	213
-1	11	0 0 0 0 0 1 0 2 1 0 2 0 1 0 0 0 0 1 2 0 0 1 0 1 0 0 0	648	214
4	4	0 0 0 0 0 1 0 2 1 0 2 0 1 0 0 1 0 0 1 0 1 1 0 1 0 0 0	1296	215
-2	-10	0 0 0 0 0 1 0 2 1 0 2 0 1 0 0 1 0 0 2 0 0 0 0 2 0 0 0	648	216
0	-24	0 0 0 0 0 1 0 2 1 0 2 0 1 0 1 0 0 0 1 0 1 1 0 0 1 0 0	1296	217
1	5	0 0 0 0 0 1 0 2 1 0 2 0 1 0 1 0 0 0 2 0 0 0 0 1 1 0 0	1296	218
-1	11	0 0 0 0 0 1 0 2 1 0 2 0 1 0 1 0 0 0 2 0 0 1 0 0 0 0 1	1296	219
-3	17	0 0 0 0 0 1 0 2 1 0 2 0 2 0 0 0 0 0 1 0 1 0 0 1 1 0 0	1296	220
2	-22	0 0 0 0 0 1 0 2 1 0 2 0 2 0 0 0 0 0 2 0 0 0 0 1 0 0 1	648	221
2	10	0 0 0 0 0 1 0 2 1 0 2 1 1 0 0 0 0 0 1 0 0 1 0 1 1 0 0	1296	222
0	32	0 0 0 0 0 1 0 2 1 1 0 0 0 1 1 1 0 0 1 1 1 1 0 0 0 0 0	1296	223
-2	22	0 0 0 0 0 1 0 2 1 1 0 0 1 0 1 0 0 1 1 2 0 1 0 0 0 0 0	324	224
0	-8	0 0 0 0 0 1 0 2 1 1 0 0 1 0 1 1 0 0 1 1 1 0 1 0 0 0 0	1296	225
-2	-18	0 0 0 0 0 1 0 2 1 1 0 0 1 0 1 1 0 0 1 2 0 0 0 1 0 0 0	648	226
2	106	0 0 0 0 0 1 0 2 1 1 0 0 1 1 0 0 0 1 1 1 1 1 0 0 0 0 0	648	227
0	-8	0 0 0 0 0 1 0 2 1 1 0 0 1 1 0 1 0 0 1 1 1 0 0 1 0 0 0	1296	228
0	32	0 0 0 0 0 1 0 2 1 1 0 0 1 1 1 0 0 0 1 1 1 0 0 0 1 0 0	1296	229
4	-20	0 0 0 0 0 1 0 2 1 1 0 1 0 0 1 1 0 0 1 1 0 1 1 0 0 0 0	1296	230
1	53	0 0 0 0 0 1 0 2 1 1 0 1 0 1 0 0 0 1 1 1 0 2 0 0 0 0 0	648	231
2	-54	0 0 0 0 0 1 0 2 1 1 0 1 0 1 0 1 0 0 1 0 1 1 1 0 0 0 0	1296	232
-2	-82	0 0 0 0 0 1 0 2 1 1 0 1 0 1 0 1 0 0 1 1 0 1 0 1 0 0 0	1296	233
0	32	0 0 0 0 0 1 0 2 1 1 0 1 0 1 1 0 0 0 1 1 0 1 0 0 1 0 0	1296	234
-2	22	0 0 0 0 0 1 0 2 1 1 0 1 0 2 0 0 0 0 1 0 1 1 0 0 1 0 0	1296	235
2	-54	0 0 0 0 0 1 0 2 1 1 0 1 1 0 0 0 0 1 1 1 0 1 1 0 0 0 0	1296	236
0	32	0 0 0 0 0 1 0 2 1 1 0 1 1 0 0 1 0 0 1 1 0 0 1 1 0 0 0	1296	237
-4	-36	0 0 0 0 0 1 0 2 1 1 0 1 1 0 1 0 0 0 1 1 0 0 1 0 1 0 0	1296	238
2	26	0 0 0 0 0 1 0 2 1 1 0 1 1 1 0 0 0 0 1 1 0 0 0 1 1 0 0	1296	239
-4	-116	0 0 0 0 0 1 0 2 1 1 0 1 1 1 0 0 0 0 1 1 0 1 0 0 0 0 1	648	240

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
−2	22	000001021110001001110200000	648	241
−2	−2	000001021110001100110101000	1296	242
2	10	000001021110002000110100100	648	243
1	21	000001030001011100300001000	648	244
0	24	000001030001101100111100000	648	245
−1	−5	000001030001101100210001000	1296	246
−2	−10	000001030001110001300001000	108	247
2	10	000001030001200001210001000	648	248
3	−17	000001030002100100110101000	432	249
−2	22	000001030002100100200011000	648	250
2	42	000001030002200000200010001	216	251
1	1	000001030003000100010300000	72	252
−4	−4	000001030011100100101101000	648	253
2	−22	000001030100101001111100000	648	254
2	18	000001030100101100111001000	216	255
2	58	000001030101010100101101000	648	256
−4	4	000001111001020100111100000	324	257
3	47	000001111001020100210001000	1296	258
0	24	000001111001110010111100000	648	259
−2	−42	000001111001110010210001000	1296	260
4	20	000001111001120000111000100	648	261
−1	−21	000001111001120000210000001	648	262
−3	17	000001111002010100020200000	648	263
4	−20	000001111002010100110110000	1296	264
−1	11	000001111002010100200020000	1296	265
0	16	000001111002020000110100100	648	266
4	4	000001111002110000020100100	1296	267
−4	−36	000001111002110000110010100	1296	268
0	32	000001111010011100201010000	1296	269
2	18	000001111010101010111100000	648	270

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
2	−22	000001111010101100111010000	648	271
−2	−42	000001111010110001201010000	648	272
−4	−84	000001111010111000111000100	216	273
−2	−42	000001111010200001111010000	324	274
−2	−2	000001111010200010111001000	324	275
−4	36	000001111011001100110110000	1296	276
2	58	000001111011010100101110000	648	277
−2	−50	000001111011010100110101000	1296	278
2	18	000001111011011000110100100	648	279
0	32	000001111011011000200100010	1296	280
−2	−10	000001111011100001110110000	1296	281
0	−64	000001111011100001200020000	1296	282
0	−56	000001111011100010110101000	1296	283
2	106	000001111011100010200011000	648	284
0	−48	000001111011100100101020000	1296	285
2	58	000001111011100100110011000	1296	286
6	46	000001111011101000110100010	648	287
−4	−36	000001111011101000200010010	1296	288
0	−96	000001111011110000110100001	648	289
2	106	000001111011110000200010001	1296	290
4	−108	000001111110001001110110000	324	291
2	10	000001120001110001210001000	324	292
−4	−20	000001120001200001120001000	324	293
1	21	000001120002020000200100001	1296	294
−1	−13	000001120002100001020200000	432	295
0	16	000001120002100001110110000	1296	296
4	84	000001120002110000110100001	648	297
−3	−63	000001120002110000200010001	1296	298
2	10	000001120010101001201010000	648	299
−2	−26	000001120010200001102010000	324	300

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
0	−48	000001120011100001101110000	1296	301
2	26	000001120011100001110101000	1296	302
−2	22	000001120011100001200011000	648	303
−4	−36	000001120011101000110100001	1296	304
1	53	000001120011101000200010001	1296	305
3	−33	000001120011110000200001001	1296	306
−2	−58	000001120020100001101101000	1296	307
4	4	000001120020101000101100001	648	308
−2	22	000001120021100000100101001	324	309
2	26	000001120101010001101110000	1296	310
0	−48	000001120101010001110101000	648	311
−4	44	000001120101110000110001001	648	312
−4	18	000002020002000200020200000	36	313
6	−34	000002020002000200110110000	324	314
−2	22	000002020002000200200020000	108	315
−5	23	000002020002100100110010100	648	316
−12	92	000002020011100100011100100	162	317
4	−76	000002020011100100101010100	648	318
0	128	000002020011100100200010001	324	319
−8	40	000002020101000101110110000	324	320
−4	108	000002020101010100101010100	324	321
4	164	000002020101010100110100001	324	322
2	10	000002020101100001110010100	648	323
8	−72	000002110011000101110110000	1296	324
−6	50	000002110011010100101100010	648	325
−2	38	000002110011010100110100001	1296	326
5	41	000002110011010100200010001	648	327
0	112	000002110011100001110010100	1296	328
−6	−30	000002110011100001110100010	432	329
−2	158	000002110011100010101010100	324	330

I_{12}	I'_{12}	MINIMAL REPRESENTATIVE	ORBIT SIZE	#
−1	−181	000002110011100010200010001	648	331
−4	−116	000002110011100100110010001	1296	332
2	−54	000002110011110000110000101	648	333
−6	34	000002110011110000200000011	324	334
4	20	000002110110010001110100001	324	335
24	−88	000110110111001000111001 00	27	336
−8	8	000110110111001001010101 00	648	337
10	−6	000110111010001011101100 00	108	338
−8	48	000110111010001101011100 00	324	339
−2	70	000110111010001101101010 00	648	340
4	172	000110111010101001011000 10	324	341
−2	−130	000110111010101001101000 01	324	342
−12	60	000111010110001101011100 00	108	343
−4	76	000111010110001101101010 00	324	344
2	−22	000111010110011001101000 10	1296	345
4	−108	000111010110101001011000 10	648	346
0	−136	000111010111000011100101 00	216	347
6	46	000111010111000101011000 10	648	348
4	92	000111010111000101100011 00	648	349
0	−16	000111010111000101101000 01	1296	350
8	88	001001110001020100110100001	216	351
−4	−164	001001110001110010110100001	648	352
4	180	001001110010101010200010001	324	353
0	144	001001110010101100110010001	648	354
−4	236	001001110010110001110100001	108	355
0	−128	001001110010110001200010001	216	356
−2	−82	001010101010110001200001010	324	357
6	6	001010101100011010110100001	72	358
2	222	001010200020100001100002010	36	359

Theorem

The three fundamental invariants for $3 \times 3 \times 3$ arrays are:

- I_6, *which is a linear combination of the* 8 *symmetric orbits in degree* 6*; this invariant has* 1152 *terms and coefficients* $\{-10, -4, -2, 1, 2, 4, 8\}$;
- I_9, *which is the sum of* 14 *alternating orbits in degree* 9*; this invariant has* 9216 *terms and coefficients* $\{-1, 0, 1\}$;
- I_{12}, *which is a linear combination of* 235 *symmetric orbits in degree* 12*; this invariant has* 209061 *terms and coefficients* $\{-12, -8, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 8, 10, 24\}$.

Open Problem

The $3 \times 3 \times 3$ *hyperdeterminant (GKZ) has degree 36, and hence equals* $a\, I_6^6 + b\, I_6^4 I_{12} + c\, I_6^3 I_9^2 + d\, I_6^2 I_{12}^2 + e\, I_6 I_9^2 I_{12} + f\, I_9^4 + g\, I_{12}^3$, *for some* $a, b, c, d, e, f, g \in \mathbb{C}$. *Determine these coefficients.*

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