Counting characters in blocks of solvable groups with abelian defect group

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June 4, 2012

Joint work with James P. Cossey
2012 CMS Summer Meeting - Regina
Special Session on Representations of Groups, Lie Algebras, and Hopf Algebras

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Let α^o denote the restriction of the complex valued class function α to the p-regular elements of G.

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It has long been known that if $\chi \in \mathrm{Irr}(G)$, we have

$$\chi^o = \sum_{\varphi \in \mathrm{IBr}_p(G)} d_{\chi\varphi}\varphi,$$

where the $\{d_{\chi\varphi}\}$ are uniquely defined non-negative integers, called the decomposition numbers of χ.

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The Fong-Swan theorem says that if $\varphi \in \mathrm{IBr}_p(G)$, then there is a character $\chi \in \mathrm{Irr}(G)$ such that $\chi^o = \varphi$.

In this case, we say that χ is a lift of φ.

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We examine necessary and sufficient conditions for a character $\varphi \in \mathrm{IBr}_p(B)$ with abelian defect group to attain the upper bound of $|L_\varphi| = |D|$.

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In particular this means that all of the results in this paper also hold in the setting of Isaacs' "π-partial" characters of π-separable groups and the proofs would be the same.

Theorems

Given a Brauer character φ of G, we let $\mathrm{Irr}(G : \varphi) = \{\chi \in \mathrm{Irr}(G) \mid d_{\chi\varphi} \neq 0\}$.

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Theorem

Let G be a p-solvable group and B a block of G with abelian defect group D and let $\varphi \in \mathrm{IBr}_p(B)$. Then the following conditions are equivalent:

1. $|L_\varphi| = |D|$.
2. $\mathrm{Irr}(G : \varphi) = L_\varphi$.
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It is interesting to have a condition about lifts be equivalent to a "global" condition that does not involve lifts.

It turns out that these global conditions are fairly easy to prove. It is perhaps more interesting to look at a local condition.

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Given any block B of a solvable group G with defect group D, we define (up to conjugacy) a certain canonical subgroup V_B containing (a conjugate of) D as a Sylow p-subgroup, called the Fong subgroup of B.

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In our Main Theorem, we show that condition that $|L_\varphi| = |D|$ is equivalent to a condition about V_B.

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Theorem

Let G be a p-solvable group and B a block of G with abelian defect group, let V_B be a Fong subgroup for B, and let D be a defect group for B contained in V_B. Suppose $\varphi \in \mathrm{IBr}_p(B)$. Then $|L_\varphi| = |D|$ if and only if $D \leq Z(N_{V_B}(D))$.

The subgroup V_B

Let B be a block of a p-solvable group G. We will define the Fong pair (V_B, B_V) and discuss some of the basic properties of this pair.

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Thus we develop a purely "character theoretic" approach.

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If α is not invariant in G, then we let T be the stabilizer of α in G.

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In this case we define the Fong pair (V_B, B_V) for B to be a Fong pair for B_1.

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One could have chosen a different character α of M covered by B, but a different choice would have yielded a conjugate Fong pair.

The subgroup V_B

Induction preserves decomposition numbers. In other words, with the above notation, if $\chi \in \mathrm{Irr}(B)$ and $\varphi \in \mathsf{IBr}_p(B)$ are induced by $\chi_1 \in \mathrm{Irr}(B_1)$ and $\varphi_1 \in \mathsf{IBr}_p(B_1)$, respectively, then $d_{\chi\varphi} = d_{\chi_1\varphi_1}$.

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Lemma

Let G be a p-solvable group and B a block of G with Fong pair (V_B, B_V). Let $\varphi \in \mathrm{IBr}_p(B)$ and let $\widehat{\varphi}$ denote the corresponding character of V_B. Then induction is a bijection from $L_{\widehat{\varphi}}$ to L_φ and from $\mathrm{Irr}(V_B : \widehat{\varphi})$ to $\mathrm{Irr}(G : \varphi)$.

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We note that this works whether or not D is abelian.

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We then use a result of Dade (whose proof was finally published by Turull) to obtain the following (this result uses the fact that D is abelian):

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Lemma

Let G be a p-solvable group and B a block of G with Fong pair (V_B, B_V) and abelian defect group D contained in V_B. Then there exists a Brauer character $\widetilde{\varphi} \in \mathrm{IBr}_p(N_{V_B}(D))$ that satisfies $|\mathrm{Irr}(N_{V_B}(D) : \widetilde{\varphi})| = |\mathrm{Irr}(G : \varphi)|$ and $|L_\varphi| = |L_{\widetilde{\varphi}}|$.

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This gives a way of translating the hypotheses between G and $N_{V_B}(D)$.

The Local condition

We now give some idea of the proofs.

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We start by assuming that $D \leq Z(N_{V_B}(G))$. This next lemma can be viewed as having $G = N_{V_B}(G)$, and we prove the result in this case.

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Lemma

Let p be a prime, let G be a p-solvable group, let $N = O_{p'}(G)$, and let $\varphi \in \mathrm{IBr}_p(G)$ be contained in the block B with defect group D. Suppose $\alpha \in \mathrm{Irr}(N)$ is an irreducible constituent of φ_N and is G-invariant. If $D \leq Z(G)$, then $|L_\varphi| = |D|$. Moreover, φ is the unique Brauer character of B.

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$\varphi_H = \theta \in \mathrm{Irr}(H)$, and the set of lifts of φ is the set $\mathrm{Irr}(G \mid \theta)$.

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$\varphi_H = \theta \in \mathrm{Irr}(H)$, and the set of lifts of φ is the set $\mathrm{Irr}(G \mid \theta)$.

Since $|\mathrm{Irr}(G \mid \theta)| = |D|$, this gives $|L_\varphi| = |D|$, as desired.

Finally, since G is the direct product of a p-group and a p'-group, then φ is the unique Brauer character in B.

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Theorem

Let G be a p-solvable group and let B be a block of G with abelian defect group D, and suppose $\varphi \in \mathrm{IBr}_p(B)$. Let V_B be a Fong subgroup associated to B. If D is central in $N_{V_B}(D)$, then $|L_\varphi| = |D|$. Moreover, in this case φ is the unique Brauer character in B.

$|\mathrm{Irr}(G : \varphi)| = |D|$

We next consider $|\mathrm{Irr}(G : \varphi)| = |D|$ when D is a normal, Sylow subgroup.

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Lemma

Let G be a p-solvable group, let D be a normal, abelian Sylow p-subgroup, and let $\varphi \in \mathrm{IBr}_p(G)$. Then the following are true:

1. $|\mathrm{Irr}(G : \varphi)| \le |D|$.
2. *If* $|\mathrm{Irr}(G : \varphi)| = |D|$, *then* $\mathrm{Irr}(G : \varphi) = L_\varphi$.

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2. *If* $|\mathrm{Irr}(G : \varphi)| = |D|$, *then* $\mathrm{Irr}(G : \varphi) = L_\varphi$.

We note that the first conclusion is a consequence of the $k(B)$-conjecture, but we have a simple proof.

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We use the correspondence to reduce this next theorem to the situation of the previous lemma, and applying that lemma, we obtain:

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Theorem

Let G be a p-solvable group, and let $\varphi \in \mathrm{IBr}_p(G)$ lie in a block B with abelian defect group D. If $|\mathrm{Irr}(G : \varphi)| = |D|$, then $\mathrm{Irr}(G : \varphi) = L_\varphi$. Moreover, $\mathrm{IBr}_p(B) = \{\varphi\}$ and $\mathrm{Irr}(B) = L_\varphi$.

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For a character δ of the normal subgroup D of G, we will let G_δ denote the inertia subgroup of δ.

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For a character δ of the normal subgroup D of G, we will let G_δ denote the inertia subgroup of δ.

Lemma

Let G be a p-solvable group, and let $\varphi \in \mathrm{IBr}_p(G)$ be in a block with a normal, abelian defect group D. Let $\delta \in \mathrm{Irr}(D)$. If $|L_\varphi| = |D|$, then $|L_\varphi \cap \mathrm{Irr}(G \mid \delta)| = |G : G_\delta|$.

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Theorem

Let G be a p-solvable group, let D be a normal, abelian Sylow p-subgroup, and let $\varphi \in \mathrm{IBr}_p(G)$. Assume that $\varphi_{\mathbf{O}_{p'}(G)}$ is homogeneous. If $|L_\varphi| = |D|$, then $D \leq Z(G)$.

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In particular, this uses the following theorem of Dolfi:

Theorem (Dolfi)

Let G be a solvable group acting faithfully and coprimely on a group K. Then there exist $x, y \in K$ so that $C_G(x) \cap C_G(y) = 1$.

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Corollary

Let G be a p-solvable group, and let $\varphi \in \mathrm{IBr}_p(G)$ lie in a block B with abelian defect group D. Let V be a Fong subgroup for B containing D. If $|L_\varphi| = |D|$, then $D \le Z(N_V(D))$.

$\mathrm{Irr}(G : \varphi) = L_\varphi$

Interestingly, in the next lemma we conclude that the defect group is abelian if the defect group is a normal Sylow p-subgroup and every character in $\mathrm{Irr}(G : \varphi)$ is a lift of φ.

$\mathrm{Irr}(G : \varphi) = L_\varphi$

Interestingly, in the next lemma we conclude that the defect group is abelian if the defect group is a normal Sylow p-subgroup and every character in $\mathrm{Irr}(G : \varphi)$ is a lift of φ.

Lemma

Let G be a p-solvable group, let D be a normal Sylow p-subgroup, and let $\varphi \in \mathrm{IBr}_p(G)$. If $\mathrm{Irr}(G : \varphi) = L_\varphi$, then $|L_\varphi| = |D|$ and D is abelian.

Example

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Take $p = 7$.

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Let G be the semi-direct product of a group of order 3 acting on a group of order 91 as a fixed-point free automorphism, so G is a Frobenius group of order 273.

Take $p = 7$.

Let D be the subgroup of order 7, and notice that D is normal in G but not central.

Example

The group G has a Brauer character $\varphi \in \mathrm{IBr}_7(G)$ whose defect group is D where $|L_\varphi| = 7 = |D|$.