Modular representations of general linear groups: An Overview

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To Robert Steinberg on his 90th birthday

The outline of my talk

- Ordinary Representations of GL_n
- Combinatorics of tableaux
- Modular Representations of GL_n
- Blocks
- Decomposition Numbers
- New Methods: Lie Theory

Ordinary representation Theory over K of characteristic 0

Modular Representation Theory over k of characteristic ℓ not dividing q

- a partition of the ordinary characters, or KG-modules, into blocks
- a partition of the Brauer characters, or kG-modules, into blocks
- a partition of the decomposition matrix into blocks

Some main problems of modular representation theory:

- Describe the blocks as sets of characters, or as algebras
- Describe the irreducible modular representations, e.g. their degrees
- Find the decomposition matrix D, the transition matrix between ordinary and Brauer characters.
- Global to local: Describe information on the block B by "local information", i.e. from blocks of subgroups of the form $N_G(P)$, P a p-group

$G = GL(n, q)$ has subgroups:

- Tori, abelian subgroups (e.g. diagonal matrices)
- Levi subgroups, products of subgroups of the form $GL(m, q^d)$
- Borel subgroups, isomorphic to "upper triangular matrices"
- Parabolic subgroups of the form $P = LV$, L a product of subgroups of the form $GL(m, q)$, $V \triangleleft P$

Parabolic subgroup P is of the form

$$\begin{pmatrix} \clubsuit & * & * & \dots & * \\ 0 & \clubsuit & * & \dots * & \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & \clubsuit \end{pmatrix}$$

Then L is of the form

$$\begin{pmatrix} \clubsuit & 0 & 0 & \ldots & 0 \\ 0 & \clubsuit & 0 & \ldots 0 & \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & \clubsuit \end{pmatrix}$$

And V is of the form $\begin{pmatrix} I & * & * & \dots & * \\ 0 & I & * & \dots * & \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & I \end{pmatrix}$

Examples of tori:

- Subgroup of diagonal matrices
- Cyclic torus generated by

$$\begin{pmatrix} \alpha & 0 & . & . & . & 0 \\ 0 & \alpha^q & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \alpha^{q^{n-1}} \end{pmatrix}$$

(α is a primitive $q^n - 1$-th root of unity)

- products of cyclic tori as above

Harish-Chandra theory (applicable to finite reductive groups)

P a parabolic subgroup of G, L a Levi subgroup of P, so that $L \leqslant P \leqslant G$.

Harish-Chandra induction is the following map:

$$R_L^G : K_0(KL) \to K_0(KG).$$

If $\psi \in \mathrm{Irr}(L)$ then $R_L^G(\psi) = \mathrm{Ind}_P^G(\tilde{\psi})$ where $\tilde{\psi}$ is the character of P obtained by inflating ψ to P.

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$\chi \in \mathrm{Irr}(G)$ is cuspidal if $\langle \chi, R_L^G(\psi) \rangle = 0$ for any $L \leqslant P < G$ where P is a proper parabolic subgroup of G. The pair (L, θ) a cuspidal pair if $\theta \in \mathrm{Irr}(L)$ is cuspidal.

THEOREM. (i) Let (L, θ), (L', θ') be cuspidal pairs. Then $\langle R_L^G(\theta), R_{L'}^G(\theta') \rangle = 0$ unless the pairs (L, θ), (L', θ') are G-conjugate.
(ii) If χ is a character of G, then $\langle \chi, R_L^G(\theta) \rangle \neq 0$ for a cuspidal pair (L, θ) which is unique up to G-conjugacy.
$\mathrm{Irr}(G)$ partitioned into Harish-Chandra families: A family is the set of constituents of $R_L^G(\theta)$ where (L, θ) is cuspidal.

Example: If L is the subgroup of diagonal matrices contained in the (Borel) subgroup B of upper triangular matrices, do Harish-CHandra induction, i.e. lift a character of L to B and do ordinary induction.

But if L is a Levi subgroup not in a parabolic subgroup, e.g. a torus of order $q^n - 1$, we must do Deligne-Lusztig induction to obtain generalized characters from characters of L.

Deligne-Lusztig Theory (applicable to finite reductive groups)

Suppose L is a Levi subgroup, not necessarily in a parabolic subgroup P of G.

The Deligne-Lusztig linear operator:

$$R_L^G : K_0(\overline{\mathbf{Q}}_l L) \to K_0(\overline{\mathbf{Q}}_l G).$$

R_L^G takes (ordinary) characters of L to $\mathbf{Z}$-linear combinations of characters of G.

A unipotent character is a constituent of $R_T^G(1)$, T a maximal torus.

Unipotent classes indexed by partitions of n (Jordan form).

unipotent characters of G are constituents of Ind_B^G (not true in general)

Also indexed by partitions of n, denoted by χ_λ, λ a partition of n.

Example: Unipotent characters of $GL(3,q)$, constructed by Harish-Chandra induction, values at unipotent classes [R.Steinberg, Canadian J.Math. 3 (1951)]

$\chi_{[4]}$	1	1	1
$\chi_{[31]}$	q^2+q	1	0
$\chi_{[1^3]}$	q^3	0	0

Recall the blocks of S_n, described by p-cores.

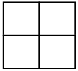
Theorem (Brauer-Robinson) χ_λ, χ_μ are in the same p-block if and only if λ, μ have the same p-core.

ℓ a prime not dividing q, e the order of q mod ℓ

Theorem (Fong-Srinivasan) χ_λ, χ_μ are in the same ℓ-block if and only if λ, μ have the same e-core.

Example: $n = 5$, ℓ divides $q^2 + q + 1$, $e = 3$. Then χ_λ for $5,\ 2^2 1,\ 21^3$ are in a block. Same for S_5, $p = 3$.

Example: $n = 4$, ℓ divides $q^2 + 1$, $e = 4$. Then χ_λ for $4,\ 31,\ 21^2,\ 1^4$ are in a block.


has no 4-hooks.

The ordinary characters in a block can be described via Deligne-Lusztig induction. Brauer meets Lusztig!

Work done on blocks and decomposition matrices for classical groups: Dipper-James, Geck, Gruber, Hiss, Kessar, Malle … …

Describe the unipotent part of the ℓ-modular decomposition matrix of G.

Can write $\chi_\lambda = \sum d_{\lambda\mu}\phi_\mu$ where ϕ_μ are Brauer characters.

Describe $d_{\lambda\mu}$.

M the module induced to G from the trivial character of B, $B=$ Borel=upper triangular matrices

$\mathrm{End}_G(M)$ is isomorphic to The Hecke algebra H_n of type A. Has generators $\{T_1, T_2, \ldots T_{n-1}\}$ and some relations, e.g.
$T_i^2 = (q-1)T_i + q.1$.

When H_n is not semisimple (q a root of unity) we can talk of its modular representations, blocks, decomposition numbers, etc.

Over a field F, define Specht modules S_λ, irreducible modules L_λ, λ partition of n. Then we want the multiplicity $(S_\lambda : L_\mu)$ (as for S_n).

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A new object: The q-Schur algebra $\mathcal{S}_q(n)$ can be defined over any field, as the endomorphism algebra of a H_n-module.

$\mathrm{char}\, k = \ell$. Then $\mathcal{S}_q(n) = \mathrm{End}_{H_n} \oplus M_\lambda$, M_λ are certain permutation H_n-modules.

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$\mathcal{S}_q(n)$ has Weyl modules, simple modules analogous to Specht modules, simple modules for H_n or S_n.

Here $\mathcal{S}_q(n)$ is over k and we can talk of its decomposition numbers, from its Weyl modules and simple modules over k, as $(W_\lambda : L_\lambda)$.

Theorem (Dipper-James) The decomposition matrix of $\mathcal{S}_q(n)$ over a field of characteristic ℓ is the same as the unipotent part of the decomposition matrix of G.

An example of a decomposition matrix D for $n = 4$, $e = 4$:

$$\begin{pmatrix} 4\| & 1 & 0 & 0 & 0 \\ 31\| & 1 & 1 & 0 & 0 \\ 211\| & 0 & 1 & 1 & 0 \\ 1111\| & 0 & 0 & 1 & 1 \end{pmatrix}$$

New modular representation theory connects decomposition numbers for symmetric groups, Hecke algebras, q-Schur algebras, with Lie theory.

The quantized Kac-Moody algebra $\mathcal{U}_v(\widehat{sl_e})$ over $\mathbf{Q}(v)$ is generated by $e_i, f_i, k_i, k_i^{-1}, \ldots, (0 \leqslant i \leqslant e-1)$ with some relations.

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Fock space: $\mathbf{Q}(v)$-vector space with basis s_λ where λ runs over the partitions of $n \geqslant 0$. Can think of the s_λ as indexing

(1) the Weyl modules of $\mathcal{S}_q(n)$ over a field of characteristic 0 with q a primitive e-th root of unity, $n \geqslant 0$

(2) unipotent characters of $GL(n, q)$, $n \geqslant 0$

Then $\mathcal{U}_v(\widehat{sl_e})$ acts on the Fock space! e_i, f_i are functors on the Fock space, called i-induction, i-restriction (as in the case of S_n).

Blocks appear as weight spaces for the subalgebra generated by the k_i.

Work of Ariki, Grojnowski, Vazirani, Lascoux-Leclerc-Thibon, Varagnolo-Vasserot, ...

Decomposition matrix D for the q-Schur algebra $\mathcal{S}_q(n)$ over K with q an e-th root of unity, entries $d_{\lambda\mu}$, can be determined in principle.

This does not give D for G, as the Dipper=James Theorem is for characteristic ℓ.

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Summary

- Known: Decomposition numbers for H_n (also cyclotomic Hecke algebras) over characteristic 0
- Known: Decomposition numbers for $GL_n(q)$, ℓ large
- Not known: Decomposition numbers for S_n, $GL_n(q)$, all ℓ

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