Weyl modules and Schur duality for $GL(n, \mathbb{C}) \times \mathfrak{S}_d$

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Highest Weight Theorem for GL$(n, \mathbb{C})$, $\mathfrak{gl}(n, \mathbb{C})$

Polynomial representation of GL$(n, \mathbb{C})$: group homomorphism

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s.t.: entries of $\pi(g)$ are polynomials in the entries of $g \in GL(n, \mathbb{C})$.

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Example:
$n = 2$, $V = \{A \in M_{2\times 2}(\mathbb{C}) : \mathrm{trace}(A) = 0\}$, $k = \dim V = 3$,

$$\pi(g)(A) = \det(g) g A g^{-1}$$

$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

Every polynomial representation of $\mathrm{GL}(n, \mathbb{C})$ is completely reducible.

A non-polynomial representation:

$$\pi(A) = \begin{pmatrix} 1 & \log\left(|\det A|\right) \\ 0 & 1 \end{pmatrix},$$

it is not completely reducible.

The character of a polynomial representation

The 'usual' character of $\pi : GL(n, \mathbb{C}) \to GL(V)$ is

$$\begin{aligned} \chi_V : GL(n, \mathbb{C}) &\to \mathbb{C} \\ g &\mapsto \operatorname{trace} \pi(g) \end{aligned}$$

Basic properties:

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Basic properties:

- It characterizes π. (U(n) is compact)
- Even the restriction $\chi_V : \mathrm{Diag}(n, \mathbb{C}) \to \mathbb{C}$ characterizes π, this is the *actual character*. (Jordan normal form)
- If $x = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \in \mathrm{Diag}(n, \mathbb{C})$, then $\chi_V(x)$ is a symmetric polynomial in $x_1, \ldots, x_n$.

- $\chi_V(x) = \sum_{\alpha \in \mathbb{N}_0^n} m_\alpha \, x_1^{\alpha_1} \dots x_n^{\alpha_n} \iff V = \bigoplus_{\alpha \in \mathbb{N}_0^n} V_\alpha$

  with $\dim V_\alpha = m_\alpha$ and $\pi(x)|_{V_\alpha} = x_1^{\alpha_1} \dots x_n^{\alpha_n} \, \mathrm{Id}$.

$$P_V = \{\alpha \in \mathbb{N}_0^n : m_\alpha \neq 0 \text{ (in fact } m_\alpha \in \mathbb{N})\}$$

is the *set of weights* of (π, V); V_α is the *weight space* of α; m_α is the *multiplicity* of α.

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$$\pi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

hence, if $x = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$ then

$$\chi_V(x) = x_1^2 + x_1 x_2 + x_2^2 \text{ and } P_V = \{(2,0), (1,1), (0,2)\}.$$

- $\chi_V(x) = \sum_{\alpha \in \mathbb{N}_0^n} m_\alpha\, x_1^{\alpha_1} \dots x_n^{\alpha_n} \iff V = \bigoplus_{\alpha \in \mathbb{N}_0^n} V_\alpha$

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is the *set of weights* of (π, V); V_α is the *weight space* of α;
m_α is the *multiplicity* of α.

Order in P_V: if (π, V) is a polyn repn of $\mathrm{GL}(n, \mathbb{C})$, $\alpha, \beta \in P_V$:

Lexicographic: $\alpha > \beta \iff \alpha_1 = \beta_1, \dots, \alpha_r = \beta_r$ and $\alpha_{r+1} > \beta_{r+1}$.

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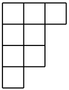
Theorem (Highest Weight Theorem)

If α is the highest weight of an irred polyn repn (π, V) of $GL(n, \mathbb{C})$, then:

- $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$.
- $m_\alpha = 1$, *that is* $\dim V_\alpha = 1$.
- $\pi(g)|_{V_\alpha} = \mathrm{Id}$ *for all* $g = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in GL(n, \mathbb{C})$. *This property characterizes* α.
- α *characterizes* π *and in fact*

$$\left\{ \begin{matrix} \textit{Polyn Irreps} \\ \textit{of } GL(n, \mathbb{C}) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{Partitions } \alpha_1 \geq \cdots \geq \alpha_n \geq 0 \\ \textit{with at most } n \textit{ parts} \end{matrix} \right\}$$

Partitions: if $\alpha = (3, 2, 2, 1)$, we write $\alpha =$  (Young diagram)

and we say that α is a partition of $8 = 3 + 2 + 2 + 1$ and has 4 parts.

$$\mathrm{Irreps}(\mathrm{GL}(n,\mathbb{C})) = \left\{ \begin{matrix} \text{Irred polynomial} \\ \text{repns of } \mathrm{GL}(n,\mathbb{C}) \end{matrix} \right\},$$

$$\mathrm{Part}_{\leq n} = \left\{ \begin{matrix} \text{Partitions } \alpha_1 \geq \cdots \geq \alpha_n \geq 0 \\ \text{with at most } n \text{ parts} \end{matrix} \right\}.$$

Highest Weight Theorem. The following map is a bijection:

$$\begin{aligned} \mathrm{Irreps}(\mathrm{GL}(n,\mathbb{C})) &\longleftrightarrow \mathrm{Part}_{\leq n} \\ (\pi, V) &\longrightarrow \text{the highest weight of } \pi \text{ (or } P_V) \\ \pi_\alpha &\longleftarrow \alpha \qquad \text{(Schur functor)}. \end{aligned}$$

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Examples.

- In the previous example, $\chi_V(x) = x_1^2 + x_1x_2 + x_2^2$ and $\alpha = (2,0)$.
- If $V = \mathbb{C}$ and $\pi : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(V)$, $\pi(g) = 1$ is the trivial representation, $\chi_V(x) = 1$ and $\alpha = (0,0)$.
- If $V = \mathbb{C}^n$ and $\pi : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(V)$, $\pi(g) = g$ is the canonical representation, $\chi_V(x) = x_1 + \cdots + x_n$ and $\alpha = (1,0)$.

- $V = \mathbb{C}^n$ and $\pi : \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(V^{\otimes d})$, $\pi(g) = g^{\otimes d}$ is the *d-tensor power representation* (it is not irred), $\chi_V(x) = (x_1 + \cdots + x_n)^d$.

Thus: $$\mathrm{Irreps}\big(\mathrm{GL}(n, \mathbb{C}), (\mathbb{C}^n)^{\otimes d}\big) \longleftrightarrow \mathrm{Part}_{\leq n}(d)$$
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$$\mathrm{Part}_{\leq n}(d) = \left\{ \begin{matrix} \text{Partitions } \alpha_1 \geq \cdots \geq \alpha_n \geq 0 \\ \text{of } d \text{ with at most } n \text{ parts} \end{matrix} \right\}$$

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Examples. $V = \mathbb{C}^n$.

- $d = 1$, $V^{\otimes 1} = V = \square$
- $d = 2$, $V^{\otimes 2} = V \otimes V = Sym^2(V) \oplus \Lambda^2(V) = \ydiagram{2} \oplus \ydiagram{1,1}$
- $d = 3$, $V^{\otimes 3} = Sym^3(V) \oplus \Lambda^3(V) \oplus W = \ydiagram{3} \oplus \ydiagram{1,1,1} \oplus 2\,\ydiagram{2,1}$

Weyl modules and Schur functor

V : vector space of dimension n ($V = \mathbb{C}^n$). As $\mathrm{GL}(n, \mathbb{C})$-modules

$$V^{\otimes d} = \bigoplus_{\alpha \in \mathrm{Part}_{\leq n}(d)} a_\alpha \, \pi_\alpha, \qquad a_\alpha \in \mathbb{N}$$

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$$V^{\otimes d} = \bigoplus_{\alpha \in \mathrm{Part}_{\leq n}(d)} a_\alpha \, \pi_\alpha, \qquad a_\alpha \in \mathbb{N}$$

$$= Sym^d(\mathbb{C}^n) \oplus \Lambda^d(\mathbb{C}^n) \oplus W$$

$$= \begin{array}{|c|c|c|}\hline & \cdots & \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline \end{array} \oplus \bigoplus_{\substack{\alpha \in \mathrm{Part}_{\leq n}(d) \\ \alpha \neq (d,0,\ldots,0) \\ \alpha \neq (1,1,\ldots,1)}} a_\alpha \, \pi_\alpha$$

$Sym^d(V)$: the subspace of $V^{\otimes d}$ of symmetric tensors,

$\Lambda^d(V)$: the subspace of $V^{\otimes d}$ of skew-symmetric tensors,

$$Sym^d(V) = \left\{ \sum_{\sigma \in \mathfrak{S}_d} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \in V^{\otimes d} \right\}$$

$$\Lambda^d(V) = \left\{ \sum_{\sigma \in \mathfrak{S}_d} (-1)^{|\sigma|} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \in V^{\otimes d} \right\}$$

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Let $\alpha \in \mathrm{Part}_{\leq n}(d)$, $V = \mathbb{C}^n$.

The Weyl module $\mathbb{S}_\alpha(V)$ is a $\mathrm{GL}(n, \mathbb{C})$-submodule of $V^{\otimes d}$.

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– For $\alpha = \square\cdots\square$,

$$\mathbb{S}_\alpha(V) := Sym^d(V) = \left\{ \sum_{\sigma \in \mathfrak{S}_d} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \in V^{\otimes d} \right\}$$

$$:= c_\alpha \cdot V^{\otimes d}, \qquad \text{with } c_\alpha = \sum_{\sigma \in \mathfrak{S}_d} \sigma \in \mathbb{C}[\mathfrak{S}_d]$$

– For $\alpha = \begin{array}{c}\square\\ \vdots \\ \square\end{array}$,

$$\mathbb{S}_\alpha(V) := \Lambda^d(V) = \left\{ \sum_{\sigma \in \mathfrak{S}_d} (-1)^{|\sigma|} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \in V^{\otimes d} \right\}$$

$$:= c_\alpha \cdot V^{\otimes d}, \qquad \text{with } c_\alpha = \sum_{\sigma \in \mathfrak{S}_d} (-1)^{|\sigma|} \sigma \in \mathbb{C}[\mathfrak{S}_d].$$

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$$\text{If } \alpha = \ydiagram{3,2,2,1}\ , \text{ then } T_\alpha = \begin{ytableau} 1 & 2 & 3 \\ 4 & 5 \\ 6 & 7 \\ 8 \end{ytableau} \text{ is the tableau of } \alpha$$

$$p_\alpha = \sum_{\sigma \in P_\alpha} \sigma, \qquad P_\alpha = \{\sigma \in \mathfrak{S}_d : \sigma \text{ preserves each row of } T_\alpha\};$$

$$q_\alpha = \sum_{\sigma \in Q_\alpha} (-1)^{|\sigma|}\sigma, \quad Q_\alpha = \{\sigma \in \mathfrak{S}_d : \sigma \text{ preserves each column of } T_\alpha\};$$

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– For $\alpha = \boxed{}\boxed{\cdots}\boxed{}$, $p_\alpha = \sum_{\sigma \in \mathfrak{S}_d} \sigma$, $q_\alpha = 1$, $c_\alpha = p_\alpha$.

– For $\alpha = \begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline\end{array}$, $p_\alpha = 1$, $q_\alpha = \sum_{\sigma \in \mathfrak{S}_d} (-1)^{|\sigma|}\sigma$, $c_\alpha = q_\alpha$.

– For $\alpha = \ydiagram{2,1}$, $T_\alpha = \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$

$$P_\alpha = \{(1), (12)\}, \qquad p_\alpha = 1 + (12);$$

$$Q_\alpha = \{(1), (13)\}, \qquad q_\alpha = 1 - (13);$$

$$\begin{aligned} c_\alpha &= \big(1 + (12)\big)\big(1 - (13)\big) \\ &= 1 + (12) - (13) - (132) \in \mathbb{C}[\mathfrak{S}_d]. \end{aligned}$$

– For $\alpha = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$, $T_\alpha = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$

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$$= 1 + (12) - (13) - (132) \in \mathbb{C}[\mathfrak{S}_d].$$

$$\mathbb{S}_\alpha(V) = c_\alpha \cdot V^{\otimes 3}$$

$$= \operatorname{span}\{v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 - v_3 \otimes v_1 \otimes v_2\}$$

For $\alpha \in \mathrm{Part}_{\leq n}(d)$, $V = \mathbb{C}^n$

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$$c_\alpha = p_\alpha q_\alpha \text{ in } \mathbb{C}[\mathfrak{S}_d].$$

Then

$$\mathbb{S}_\alpha(V) := c_\alpha \cdot V^{\otimes d}$$

is a $\mathrm{GL}(n, \mathbb{C})$-invariant submodule of $V^{\otimes d}$ and
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By the way,

$$L_\alpha := c_\alpha \cdot \mathbb{C}[\mathfrak{S}_d]$$

is a $\mathfrak{S}_d$-invariant submodule of $\mathbb{C}[\mathfrak{S}_d]$ and it is isomorphic to the irreducible $\mathfrak{S}_d$-module corresponding to α.

Example. $V = \mathbb{C}^n$

Let $\alpha = \ydiagram{2,1}$, $T_\alpha = \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$

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$$(\mathbb{C}^n)^{\otimes 3} = Sym^3(\mathbb{C}^n) \oplus \Lambda^3(\mathbb{C}^n) \oplus 2\,\mathbb{S}_\alpha(V) = \ydiagram{3} \oplus \ydiagram{1,1,1} \oplus 2\ \ydiagram{2,1}$$

There was a free choice in the definition of c_α: for $\alpha \in \mathrm{Part}_{\leq \bullet}(d)$

$$T_\alpha = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array} \rightsquigarrow P_\alpha, Q_\alpha \rightsquigarrow p_\alpha, q_\alpha \rightsquigarrow c_\alpha$$

We could have started with $\tilde{T}_\alpha = \begin{array}{|c|c|c|}\hline 6 & 5 & 2 \\ \hline 4 & 8 \\ \cline{1-2} 1 & 3 \\ \cline{1-2} 7 \\ \cline{1-1}\end{array} \rightsquigarrow \tilde{c}_\alpha.$

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Theorem (Peter and Weyl)

$$\mathbb{C}[\mathfrak{S}_d] = \bigoplus_{\alpha \in Part_{\leq \bullet}(d)} \dim(L_\alpha)\, L_\alpha, \qquad \textit{as } \mathfrak{S}_d\textit{-mod}$$

$$= \bigoplus_{\alpha \in Part_{\leq \bullet}(d)} L_\alpha^* \otimes L_\alpha, \qquad \textit{as } \mathfrak{S}_d \times \mathfrak{S}_d\textit{-mod}$$

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Theorem (Schur Duality for $GL(n) \times \mathfrak{S}_d$)

Let $V = \mathbb{C}^n$, then

$$V^{\otimes d} = \bigoplus_{\alpha \in Part_{\leq n}(d)} \dim(L_\alpha)\, \mathbb{S}_\alpha(V), \qquad \text{as } GL(n)\text{-mod}$$

$$= \bigoplus_{\alpha \in Part_{\leq n}(d)} L_\alpha \otimes \mathbb{S}_\alpha(V), \qquad \text{as } \mathfrak{S}_d \times GL(n)\text{-mod}$$

Hermite's reciprocity law states that

$$\mathrm{Sym}^n\left(\mathrm{Sym}^m(\mathbb{C}^2)\right) \simeq \mathrm{Sym}^m\left(\mathrm{Sym}^n(\mathbb{C}^2)\right)$$

as $\mathsf{GL}(2,\mathbb{C})$-modules. Equivalently

$$\mathbb{S}_{\underbrace{\square\cdots\square}_{n \text{ times}}}(V_m) \;\simeq\; \mathbb{S}_{\underbrace{\square\cdots\square}_{m \text{ times}}}(V_n)\,.$$

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Manivels's extension (2007) if Hermite's reciprocity law:

$$\mathbb{S}_{k\underbrace{\begin{array}{|c|c|c|}\hline & \cdots & \\ \hline \vdots & & \vdots \\ \hline & \cdots & \\ \hline \end{array}}_{n}}(V_m)$$

is symmetric in m, n, k as $\mathsf{SL}(2,\mathbb{C})$-modules.

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Question: find other solutions for the plethysm equation:

$$\mathbb{S}_\alpha\left(V_m\right) \simeq \mathbb{S}_\beta\left(V_n\right), \qquad \text{unknokns: } \alpha, \beta, m, n$$

as $\mathrm{SL}(2,\mathbb{C})$ or $\mathrm{GL}(2,\mathbb{C})$-modules.

We introduce the following notation:

$$\ydiagram{3,2,2,1}_{z} = \mathbb{S}_\alpha\left(V_{z-1}\right), \quad \alpha = (3,2,2,1).$$

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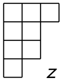
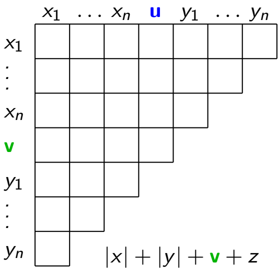


$$= \mathbb{S}_{\alpha}(V_{z-1}), \quad \alpha = (3,2,2,1).$$

Theorem (joint with D. Penazzi)

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be two sequences in $\mathbb{Z}_{\geq 0}$,
set $|x| = \sum x_i$, $|y| = \sum y_i$,
and let $u, v, z \in \mathbb{Z}_{\geq 0}$.

Then the following $SL(2,\mathbb{C})$-isomorphism holds:



$\simeq$