Semisimple Symplectic Characters of Finite Unitary Groups

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Frobenius-Schur Indicators

- Let G be a finite group, and (π, V) an irreducible complex representation of G with character χ.
- The *Frobenius-Schur indicator* of π, or of χ, is $\varepsilon(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$.
- Then $\varepsilon(\chi) = 1, -1$, or 0 when π is an $\mathbb{R}$-representation, χ is real-valued but π is not an $\mathbb{R}$-representation, or χ is not real-valued, respectively.
- If $\varepsilon(\chi) = -1$, χ is *symplectic*. If $\varepsilon(\chi) = 1$, χ is *orthogonal*.

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Basic Goal

- Given a finite group of Lie type G, determine the Frobenius-Schur indicator $\varepsilon(\chi)$ for each complex irreducible character χ of G. This determines the classification of the irreducible representations of G over $\mathbb{R}$.
- Much work has been done on this question (along with the related questions regarding rationality) by Geck, Gow, Janusz, Lusztig, Malle, Marberg, Ohmori, Prasad, Przygocki, Tiep, Turull, Zalesski,...

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However, this goal is not yet completely accomplished for the finite unitary group, $G = \mathrm{U}(n, \mathbb{F}_q)$.
Here, $\mathrm{U}(n, \mathbb{F}_q)$ is defined as $\boldsymbol{G}^F$, where $\boldsymbol{G} = \mathrm{GL}(n, \bar{\mathbb{F}}_q)$ and

$$F((a_{ij})) = {}^T(a_{ij}^q)^{-1}.$$

Note that on $\mathrm{GL}(1, \bar{\mathbb{F}}_q)$, which we may identify with $\bar{\mathbb{F}}_q^\times$, F acts as $F(\alpha) = \alpha^{-q}$.

Semisimple characters

Let $\boldsymbol{G}$ be a connected reductive group with connected center, defined over $\mathbb{F}_q$ by Frobenius F, and let $G = \boldsymbol{G}^F$. Suppose $p = \mathrm{char}(\mathbb{F}_q)$ is a *good* prime for $\boldsymbol{G}$. Then $\chi \in \mathrm{Irr}(G)$ is a *semisimple* character of G when $\chi(1)$ is prime to p.

Main Result

Theorem (Srinivasan-V)

Let $G = \mathrm{U}(2m, \mathbb{F}_q)$, with q odd. Then the number of semisimple symplectic characters of G is q^{m-1}, and these are in one-to-one correspondence with self-dual polynomials in $\mathbb{F}_q[t]$ of degree $2m$ with constant term -1.

A polynomial $f(t) \in \mathbb{F}_q[t]$ is self-dual when it is monic, and $\alpha \in \bar{\mathbb{F}}_q$ is a root of $f(t)$ with multiplicity k if and only if α^{-1} is.

Proposition (Ohmori, V)

Let $G = \mathrm{U}(n, \mathbb{F}_q)$, and $\chi \in \mathrm{Irr}(G)$ a semisimple real-valued character. Then:

1. *If n is odd or q is even, then $\varepsilon(\chi) = 1$.*
2. *If n is even and q is odd, then $\varepsilon(\chi) = \omega_\chi(z)$, where ω_χ is the central character of the representation affording χ, and z is a generator for $Z(G)$.*

That is, G can only have semisimple symplectic characters if $n = 2m$ and q is odd.

Remark. Ohmori calculated $\varepsilon(\chi)$ when χ is a unipotent character of $G = \mathrm{U}(n, \mathbb{F}_q)$, q odd. The result implies there can be no $t \in Z(G)$ such that $\varepsilon(\chi) = \omega_\chi(t)$ for all $\chi \in \mathrm{Irr}(G)$.

Real-valued semisimple characters

- If $\boldsymbol{G}$ is a connected reductive group with connected center, defined over $\mathbb{F}_q$ by Frobenius F, let $\boldsymbol{G}^*$ be a group dual to $\boldsymbol{G}$, with dual Frobenius F^*. Let $G = \boldsymbol{G}^F$ and $G^* = {\boldsymbol{G}^*}^{F^*}$.
- The Deligne-Lusztig theory of characters for $\boldsymbol{G}^F$ gives a bijection $(s) \leftrightarrow \chi_{(s)}$ between semisimple classes of G^* and semisimple characters of G.
- The conjugacy class (s) is *real* if s is conjugate to s^{-1}.

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Real-valued semisimple characters

Lemma

In the bijection $(s) \leftrightarrow \chi_{(s)}$, (s) is a real semisimple class if and only if $\chi_{(s)}$ is a real-valued character.

Navarro-Tiep: One may replace "real" with "rational" in the above, for a more general result.

Take $\boldsymbol{G}$ in duality to $\boldsymbol{G}^*$ as before, with respect to the F-stable maximal torus $\boldsymbol{T}$ of $\boldsymbol{G}$, in an F-stable Borel, and an F^*-stable maximal torus $\boldsymbol{T}^*$ of $\boldsymbol{G}^*$ in an F^*-stable Borel of $\boldsymbol{G}^*$. There is an isomorphism

$$(\boldsymbol{T}^*)^{F^*} \cong \mathrm{Irr}(\boldsymbol{T}^F),$$

which gives rise to a homomorphism $z \mapsto \hat{z}$ from $Z(\boldsymbol{G}^*)^{F^*}$ to linear characters of $\boldsymbol{G}^F$.

This map, in turn, allows us to define a map $s \mapsto \hat{s}$ from semisimple elements of G^* (an F^*-stable element of the center of $C_{\boldsymbol{G}^*}(s)$) to linear characters of $C_{\boldsymbol{G}}(s)^F$.

Lemma (Fong-Srinivasan)

Let $\boldsymbol{G}^F = \mathrm{U}(n, \mathbb{F}_q)$, $\chi_{(s)}$ be a semisimple character of $\boldsymbol{G}^F$, and $z \in Z(\boldsymbol{G}^F)$. Then $\omega_{\chi_{(s)}}(z) = \hat{s}(z)$.

Lemma

$\hat{s}(z) = \hat{z}(s)$.

So, for $G = \mathrm{U}(2m, \mathbb{F}_q)$, q odd, we have $\varepsilon(\chi_{(s)}) = \omega_{\chi_{(s)}}(z) = \hat{z}(s)$ for real-valued $\chi_{(s)}$, where z is a generator for $Z(G)$.

Lemma

When $G = \mathrm{U}(n, \mathbb{F}_q)$, q is odd, and z is a generator of $Z(G)$, then for any semisimple s of G, $\hat{z}(s) = (-1)^k$, where k is the multiplicity of -1 as an eigenvalue of s.

To obtain the main result, we now analyze the real semisimple classes of $\mathrm{U}(n, \mathbb{F}_q)$.

- When $\boldsymbol{G}^F = \mathrm{U}(n, \mathbb{F}_q)$, $F(\alpha) = \alpha^{-q}$ for $\alpha \in \bar{\mathbb{F}}_q^\times$.
- A semisimple class of $\boldsymbol{G}^F$ consists of a choice of F-orbits $[\alpha]_F$ with multiplicity, with total size n.
- The semisimple class is real whenever the orbit $[\alpha]_F$ always has the same multiplicity as $[\alpha^{-1}]_F$.
- If $\tilde{F}(\alpha) = \alpha^q$, then $[\alpha]_{\tilde{F}} \cup [\alpha^{-1}]_{\tilde{F}} = [\alpha]_F \cup [\alpha^{-1}]_F$.

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It follows that real semisimple classes of $\mathrm{U}(n, \mathbb{F}_q)$ correspond to self-dual polynomials in $\mathbb{F}_q[t]$ of degree n.
When $n = 2m$, q odd, the constant term of the polynomial corresponding to real (s) is exactly $(-1)^k$, where k is the multiplicity of -1 as an eigenvalue of (s).

The main result follows, since, for (s) a real semisimple class, $\chi_{(s)}$ is real-valued, and

$$\varepsilon(\chi_{(s)}) = \hat{z}(s) = (-1)^k,$$

while $(-1)^k$ is precisely the constant term of the self-dual polynomial of degree $2m$ in $\mathbb{F}_q[t]$ corresponding to (s).
One can quickly enumerate the number of such self-dual polynomials with k odd to be q^{m-1}, corresponding to semisimple symplectic characters, and those with k even to be q^m, corresponding to semisimple orthogonal characters.

- As mentioned, when $G = \mathrm{U}(n, \mathbb{F}_q)$, q odd, Ohmori determined $\varepsilon(\psi)$ for ψ unipotent. We also know $\varepsilon(\chi_{(s)})$ when $\chi_{(s)}$ is semisimple.
- If one can understand the behavior of the F-S indicator under Deligne-Lusztig induction R_L^G, we could take advantage of the Jordan decomposition of characters, and then the basic goal of computing $\varepsilon(\chi)$ would be reduced to that of unipotent characters.

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- More generally, if $G = \boldsymbol{G}^F$ is a finite reductive group, and χ is a complex irreducible character of G with Jordan decomposition (s, ψ), so s is a semisimple element of G^*, and ψ is a unipotent character of $C_{G^*}(s)$, precisely what conditions of (s, ψ) are equivalent to χ being real-valued? It seems we need s is a real element of G^*, and if $wsw^{-1} = s^{-1}$, then we also need ${}^w\psi = \bar{\psi}$.
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