On Tensor Products with the Steinberg Character of the Unitary Group

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Outline

The Unitary Group

For the purposes of this talk, we will restrict our attention to the unitary group, $U(n,q)$.

Note here that we consider what is sometimes called the *general* unitary group.

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Let $\mathbb{F}_{q^2}$ denote the finite field of order q^2.

The unitary group $U(n,q)$ is the group of isometries of a non-degenerate hermitian form defined on $V \times V$, where V is a vector space of dimension n over $\mathbb{F}_{q^2}$.

Properties of the Unitary Group

The order of the unitary group is

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The first is the Steinberg representation, which is irreducible of degree $q^{n(n-1)/2}$. It can be constructed by using the theory of a group with a B,N pair.

The Steinberg representation is defined over the rational numbers and Z_n is contained in its kernel.

The Steinberg and Weil Characters

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Given an element g of $U(n,q)$, we have

$$\phi_n(g) = (-1)^n(-q)^{c(g)}$$

where $c(g) = \dim \ker (g - I)$.

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Let z denote a generator of Z_n and let ϵ denote a complex root of unity of order $q+1$.

Let λ_i denote the linear character of Z_n given by

$$\lambda_i(z^j) = \epsilon^{ij}$$

for $0 \leq i, j \leq q$.

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We have the degree formulae

$$\begin{aligned} \phi_n^0(1) &= \frac{q^n - (-1)^n}{q+1} + (-1)^n \\ \phi_n^i(1) &= \frac{q^n - (-1)^n}{q+1} \text{ for } 1 \le i \le q. \end{aligned}$$

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These two characters are in fact rational-valued.

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$\nu(\chi) = 1$ if χ is both real-valued and the character of a representation defined over $\mathbb{R}$.

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In this talk, we are are looking at a special case in $U(n, q)$.

The Indicators of Constituents of the Weil Character

Theorem

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- In all other cases, the real-valued irreducible constituents of the Weil character have indicator equal to 1.

The proof of this theorem is by induction on n, making use of knowledge of the restriction of ϕ_n to smaller unitary subgroups.

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We now apply a theorem of Hiss, Zalesski and Brunat, *The Weil-Steinberg character of finite classical groups*. With an appendix by Olivier Brunat. Represent. Theory **13** (2009), 427–459.

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We now prove a result about the Schur index of each constituent of $\phi_n St_n$.

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Corollary

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St_{n+1} is the character of a representation of $U(n+1, q)$ defined over $\mathbb{Q}$.

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St_{n+1} is the character of a representation of $U(n+1, q)$ defined over $\mathbb{Q}$.

It follows that the restriction of St_{n+1} to $U(n, q)$ is the character of a representation of $U(n, q)$ defined over $\mathbb{Q}$.

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It follows that the restriction of St_{n+1} to $U(n, q)$ is the character of a representation of $U(n, q)$ defined over $\mathbb{Q}$.

Since each irreducible constituent of this restriction occurs with multiplicity 1, the result is clear, as we know now that $\phi_n St_n$ is the character of this restriction.

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However, we know that it must also be the case that each irreducible constituent of the products $\phi_n^i St_n$, for $0 \leq i \leq q$, has Schur index 1 over $\mathbb{Q}$, since $\phi_n^i St_n$ is a subcharacter of $\phi_n St_n$.

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This has interesting consequences when we choose ϕ_n^i to be a character with $\nu(\phi_n^i) = -1$.

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Given this, and the fact that all irreducible constituents of $\phi_n^i St_n$ have Schur index 1, these constituents cannot be real-valued.

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where the parentheses denote the usual inner product of characters.

Proof Concluded

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As far as we know, St^2 does contain every other irreducible character χ of $U(n,q)$ as a constituent, provided Z_n is contained in the kernel of χ.

Indeed, it may be the case that the square of the Steinberg character of a simple group of Lie type contains all irreducible characters of the group, except for this special case.

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This is not necessarily true when q is odd.