Knörr lattices for elementary abelian $p$-groups


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Obtain: Blocks of irreducible characters of $G$.

- What do irreducibles in same block have in common?
- How many irreducibles appear in a given block?
- In what terms can we expect answers to questions like this?

# Defect Groups

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If $e$ is a block idempotent, $D$ is called a *defect group* of $e$ if $D$ is minimal such that

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Fact: $D$ is determined up to conjugacy by $e$.

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Brauer's Height Zero Conjecture:

$h = 0$ for all $\chi$ in $e$ if and only if $D$ is abelian.

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- Can choose $V \subseteq D$.

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## Theorem

*Let $\chi$ be irreducible, and $M$, $V$, $D$, and $S$ as above. Then*

$$h = \textit{height of } \chi = \nu|D : V| + \nu(\textit{rank}(S)).$$

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*With the same assumptions*

$$C_D(V) \subseteq V.$$

*So if $D$ is abelian, then $V = D$.*

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Question:

- Can an abelian $p$-group have a Knörr lattice of rank divisible by $p$?

# Knörr 1988, example

Let $R = \mathbb{Z}_2[2^{1/3}]$ and $D = C_2 \times C_2 \times C_2 \times C_2$.

THEN: there is a rank 6 Knörr $RD$-lattice (gives four explicit $6 \times 6$ matrices).

# But...

The group $\mathrm{Gal}(\mathbb{Q}_2(2^{1/3}, \zeta_3)/\mathbb{Q}_2)$ is not abelian, yet $\chi$ can be afforded by a $KG$ module in which $K/\mathbb{Q}_2$ is cyclotomic!

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Specifically, if a positive height irreducible character of some finite group in a 2-block with elementary abelian 16 defect group exists, THEN a Knörr lattice of even rank *defined over a cyclotomic extension* of $\mathbb{Z}_2$ exists. And Knörr does not display such a lattice.

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Facts:

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- There is a $K'$ which is unramified over $Z(\Delta) \cong \mathbb{Q}_p(\chi)$.

Also (Green/Brauer)

- If $g \in G$ and $g_p$ is not conjugate to an element of $D$, then $\chi(g) = 0$.

# A better splitting field

**Lemma**

*Let*

$$K = \mathbb{Q}_p(\textit{all } p' \textit{ roots of unity}, \zeta_{p^m})$$

*where $p^m$ is the exponent of $D$. Let $R$ be the valuation ring of $K$. Then $\chi$ is afforded by an $RG$-lattice.*

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The point:

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The point:

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The hope:

- Knörr lattices (with rank divisible by $p$) for abelian $p$-groups with above uniformizer might be understandable.

# First results

**Theorem**

*Let $D$ be elementary abelian of order $2^d$ with $d \geq 2$. Let $M$ be a Knörr RD-lattice. Then*

$$\nu(\mathit{rank}(M)) \leq d - 2.$$

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**Theorem**

*Let $D$ be elementary abelian of order 8, and assume $R/\mathbb{Z}_2$ is unramified. Then an even rank Knörr RD-lattice must have rank at least 14.*

# Idea for a rank 6 lattice with $D \cong C_2 \times C_2 \times C_2$

- Let $H$ be a maximal subgroup of $D$. If $M_H$ has an odd-rank summand, consider the transfer of the projection to an $RD$-endomorphism of $M$. We conclude that $M_H$ has no odd-rank summands.

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- If $M_H$ has a projective summand, then $M_H \cong P \oplus 2$. This forces $\chi_M(g) = 0$ for some $g \in H$.
- If $M$ has no projective summands, then a result of MCR Butler shows that (using fact that $R/(2) = k$)

$$2M \subseteq (M_H)_{ss} \subseteq M.$$

  Now $(M_H)_{ss}$ is an $RD$-lattice with a rank-1 summand. Consider multiplication by 2 followed by projection.

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## Lemma

*Let $M$ be a cyclic $RD$-lattice with $\mathrm{rank}(M)_p = p^h$. Then $M$ is a Knörr lattice if and only if*

$$\chi_M(g) \equiv \chi_M(1) \pmod{\pi p^h}$$

*for all $g \in D$.*

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## Lemma

*Let $g$ have order $p$, and let $\chi$ be a character of $D$. Then*

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*if and only if* $[\chi_{\langle g \rangle}, \lambda] \equiv 0 \pmod{p^h}$ *for all* $\lambda \in \hat{\langle g \rangle}$.

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If $D$ is an elementary abelian 2-group, and $h = 1$, this is equivalent to saying that

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For example

$$\chi = 1_D + \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_1\lambda_2\lambda_3\lambda_4$$

is the character afforded by a *cyclic* rank-6 lattice for the e.a. group of order 16. Here $\hat{D} = \langle \lambda_i \mid 1 \leq i \leq 4 \rangle$.

# Subsets of $\mathbb{F}_p^d$

Assume now that $D$ is elementary abelian. If $\chi$ is multiplicity-free, then $\chi$ can be thought of as a subset of the $\mathbb{F}_p$ vector space $\hat{D}$.

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*Let $\chi$ be multiplicity-free character of the e.a. $p$-group $D$. Say $\chi(1)_p = p^h$. Then $\chi$ is the character of a cyclic Knörr RD-lattice if and only if*

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*for every codimension 1 hyperplane $W$ in $\hat{D}$.*

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### Theorem

*If $|D| = p^d$, then $h \leq \frac{d}{2}$. This bound is tight.*

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- Powers of radical.
- Stabilizer of source in $N_G(D)$?