

Global crystal bases and q -Schur algebras

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The quantum enveloping algebra $U_q(\mathfrak{gl}_n)$

- $U_q(\mathfrak{gl}_n)$ is the associative algebra over $\mathbb{C}(q)$ with generators

$$e_i, f_i, \ 1 \leq i < n, \ K_j, K_j^{-1}, \ 1 \leq i \leq n$$

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- $U_q(\mathfrak{sl}_n)$ is the subalgebra generated by $e_i, f_i, K_i K_{i+1}^{-1}$ and $K_i^{-1} K_{i+1}$.

Comultiplication on $U_q(\mathfrak{gl}_n)$

- The comultiplication $\Delta : U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n) \rightarrow U_q(\mathfrak{gl}_n)$ gives an action on tensor products of $U_q(\mathfrak{gl}_n)$ -modules:

$$\Delta(e_i) = e_i \otimes 1 + K_i^{-1} K_{i+1} \otimes e_i, \quad \Delta(f_i) = f_i \otimes K_i K_{i+1}^{-1} + 1 \otimes f_i,$$

$$\Delta(K_i) = K_i \otimes K_i$$

- A sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is a **partition** of r if $\lambda_1 \geq \dots \geq \lambda_n \geq 0$ and $\lambda_1 + \dots + \lambda_n = r$.

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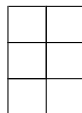
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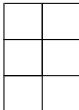


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- A λ -**tableau** is obtained by filling the boxes of the Young diagram of shape λ with numbers from the set $\{1, 2, \dots, n\}$.

Semistandard Young tableaux

$$T = \begin{array}{|c|c|c|} \hline 2 & 3 & 4 \\ \hline 6 & 6 & \\ \hline \end{array}, \quad S = \begin{array}{|c|c|c|} \hline 2 & 1 & 5 \\ \hline 4 & 5 & \\ \hline \end{array}$$

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- T is *semistandard* since the entries in its rows are weakly increasing and entries in its columns are strictly increasing. S is column increasing but not semistandard.

Highest weight modules

- A $U_q(\mathfrak{gl}_n)$ -module V is a *highest weight module* if it contains a highest weight vector v , where $e_i v = 0$ for $1 \leq i < n$, such that $U_q(\mathfrak{gl}_n)v = V$.

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- For each partition λ with at most n nonzero parts there is a unique highest weight finite-dimensional irreducible $U_q(\mathfrak{gl}_n)$ -module $V(\lambda)$.
- For $\chi = (\chi_1, \dots, \chi_n)$, an n -tuple of nonnegative integers, the subspace $V(\lambda)^\chi = \{v \in V(\lambda) \mid K_i v = q^{\chi_i} v, i = 1, \dots, n\}$ is a weight space and

$$V(\lambda) = \bigoplus_{\chi} V(\lambda)^\chi.$$

- Let $\Lambda_k = (1, \dots, 1, 0, \dots, 0) \vdash k \leq n$; can write $\lambda = \sum_{i=1}^n a_i \Lambda_i$.

Fundamental modules

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- The $U_q(\mathfrak{gl}_n)$ -modules $V(\Lambda_k)$ are called *fundamental modules*.
- $V(\Lambda_k)$ is an $\binom{n}{k}$ -dimensional vector space with basis $\{[T] \mid T \text{ column increasing}\}$ labeled by one-column Young tableau of shape $\Lambda_k = (1)^k$ with entries from $\{1, 2, \dots, n\}$.

$$K_i[T] = \begin{cases} [T] & \text{if } i \notin T \\ q[T] & \text{otherwise} \end{cases}$$

$$f_i[T] = \begin{cases} 0 & \text{if } i+1 \in T \text{ or } i \notin T \\ [T'] & \text{otherwise, where } i \text{ is replaced with } i+1 \end{cases}$$

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Example. $n = 3$, $V(\Lambda_2)$ has basis

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}; \quad f_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad f_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 0.$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ is a highest weight vector.}$$

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- $W(\lambda) = V(\Lambda_n)^{\otimes a_n} \otimes V(\Lambda_{n-1})^{\otimes a_{n-1}} \otimes \cdots \otimes V(\Lambda_1)^{\otimes a_1}$ is a $U_q(\mathfrak{gl}_n)$ -module.

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- $V(\lambda) = U_q(\mathfrak{gl}_n)v_\lambda$

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$$B = \left\{ \left[\begin{array}{c} \boxed{1} \\ \boxed{2} \end{array} \right] \otimes \boxed{1}, \left[\begin{array}{c} \boxed{1} \\ \boxed{3} \end{array} \right] \otimes \boxed{1}, \left[\begin{array}{c} \boxed{2} \\ \boxed{3} \end{array} \right] \otimes \boxed{1}, \dots \right\}$$

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$$v_\lambda = \left[\begin{array}{c} \boxed{1} \\ \boxed{2} \end{array} \right] \otimes \boxed{1}, \quad V(\lambda) = U_q(\mathfrak{gl}_3)v_\lambda.$$

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- Basis for $W(\lambda)$ is indexed by column-increasing λ -tableaux;
 $B = \{ \omega(T) \mid T \text{ column-increasing} \}$

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- Carter-Lusztig basis in terms of q -Schur algebra (G. Cliff, AS, 2010).

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- Define an involution $- : U_q(\mathfrak{gl}_n) \rightarrow U_q(\mathfrak{gl}_n)$ by

$$\overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad \overline{K_j} = K_j^{-1}, \quad \overline{q} = q^{-1}, \quad 1 \leq i < n, \quad 1 \leq j \leq n.$$

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- For $w = uv_\lambda \in V(\lambda)$, define $\overline{w} = \overline{u}v_\lambda$, $u \in U_q(\mathfrak{gl}_n)$.

Global crystal basis for $V(\lambda)$

Theorem. (Kashiwara) There exists a unique $\mathbb{Q}[q, q^{-1}]$ -basis $\{G(T) \mid T \text{ semistandard}\}$ of $V(\lambda)$ with the properties that

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- In general, difficult to find.
- Leclerc, Toffin provided an intermediate basis for $V(\lambda)$, related to global crystal basis by an upper triangular matrix. Yields an algorithm for producing the global crystal basis.

(Quantum) Carter-Lusztig basis

- Let $f_{i,i+1} = f_i$ and for $j > i + 1$, define

$$f_{ij} = f_{i+1,j}f_i - q^{-1}f_if_{i+1,j}.$$

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- Define $f_i^{(k)} = \frac{f_i^k}{[k]!}$, where $[k] = \frac{q^m - q^{-m}}{q - q^{-1}}$ and $[k]! = [k][k-1] \dots [1]$.

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- For T a semistandard λ -tableau, define

$$F_T = \prod_{1 \leq i < k, i < j \leq n} f_{ij}^{(\gamma_{ij})} = f_{12}^{(\gamma_{12})} f_{13}^{(\gamma_{13})} \dots f_{1k}^{(\gamma_{1k})} f_{23}^{(\gamma_{23})} \dots f_{2k}^{(\gamma_{2k})} \dots f_{k-1,k}^{(\gamma_{k-1,k})},$$

where γ_{ij} is number of entries equal to j in row i of T .

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- Example.** Let $T_1 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$ and $T_2 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$. Then $F_{T_1} = f_{12}f_{23}$ and $F_{T_2} = f_{13} = f_{23}f_{12} - q^{-1}f_{12}f_{23}$.

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$V(\lambda)$ has one 2-dimensional weight space; Any basis for the weight space is indexed by tableaux $T_1 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}$ and $T_2 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$.

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CL-basis is $\{F_{T_1} v_\lambda, F_{T_2} v_\lambda\} = \{f_{13} v_\lambda = (f_2 f_1 - q^{-1} f_1 f_2) v_\lambda, f_1 f_2 v_\lambda\}$ which is not the same as the global basis.

Example cont'd

Replace $F_{T_1} v_\lambda$ with $F_{T_1} v_\lambda + q^{-1} F_{T_2} v_\lambda = f_2 f_1 v_\lambda$. Then $\overline{f_2 f_1} v_\lambda = f_2 f_1 v_\lambda$.

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$$f_2 f_1 v_\lambda = \left[\begin{array}{c|c} 1 & \\ \hline 2 & \end{array} \right] \otimes \left[\begin{array}{c|c} & \\ \hline & 3 \end{array} \right] + q \left[\begin{array}{c|c} 1 & \\ \hline 3 & \end{array} \right] \otimes \left[\begin{array}{c|c} & \\ \hline & 2 \end{array} \right],$$

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$$f_2 f_1 v_\lambda = \left[\begin{array}{c} 1 \\ 2 \end{array} \right] \otimes \left[\begin{array}{c} 3 \end{array} \right] + q \left[\begin{array}{c} 1 \\ 3 \end{array} \right] \otimes \left[\begin{array}{c} 2 \end{array} \right],$$

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$\{f_2 f_1 v_\lambda, f_1 f_2 v_\lambda\}$ is the global crystal basis.

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- Define $\mathbb{C}_q[x_{ij} \mid 1 \leq i, j \leq n]$ to be the associative \mathbb{C} -algebra generated by x_{ij} , $1 \leq i, j \leq n$ subject to the relations:

$$x_{il}x_{ik} = qx_{ik}x_{il} \quad 1 \leq k < l \leq n$$

$$x_{jk}x_{ik} = qx_{ik}x_{jk} \quad 1 \leq i < j \leq n$$

$$x_{il}x_{jk} = x_{jk}x_{il} \quad 1 \leq i < j \leq n,$$

$$1 \leq k < l \leq n$$

$$x_{ik}x_{jl} - x_{jl}x_{ik} = (q^{-1} - q)x_{il}x_{jk} \quad 1 \leq i < j \leq n,$$

$$1 \leq k < l \leq n.$$

- $A_q(n, r)$ is the subspace of $\mathbb{C}_q[x_{ij}]$ consisting of homogeneous polynomials of degree r , $r \geq 0$.

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Example. $x_{11}^2 + x_{22}x_{13} \in A_q(3, 2)$

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- Also...algorithm for producing global basis elements in terms of the q -Schur algebra.

The end

Thank you!