Irreducible Modules Over Virasoro Algebra

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Virasoro Algebra Vir

Let $\mathbb{C}$ be the field of complex numbers, and sometimes we also regard it as a one dimensional Lie algebra. All vector spaces and algebras are over $\mathbb{C}$. For any Lie algebra $\mathcal{L}$, we denote its universal enveloping algebra by $U(\mathcal{L})$.

Let $\mathbb{C}[t^{\pm 1}]$ be the Laurent polynomial algebra in t. The Lie algebra of all derivations $\mathrm{Der}(\mathbb{C}[t^{\pm 1}]) = \{f(t)\frac{\partial}{\partial t} \mid f \in \mathbb{C}[t^{\pm 1}]\}$ is the **Witt algebra,** denoted by W.

The universal central extension of Witt algebra is the **Virasoro algebra**, denote by Vir. As a vector space, $\mathrm{Vir} = W \oplus \mathbb{C}z_1$, where $\mathbb{C}z_1$ is the center of Vir.

Virasoro Algebra Vir

If we denote $d_i = t^{i+1}\frac{\partial}{\partial t}$, then Vir has a standard basis $\{d_i, z_1 \mid i \in \mathbb{Z}\}$, and the Lie brackets are given by

$$[d_m, d_n] = (n-m)d_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}z_1, \qquad \forall\ m,n \in \mathbb{Z},$$

$$[d_m, z_1] = 0, \qquad \forall\ m \in \mathbb{Z}.$$

We have the triangular decomposition

$$\mathrm{Vir} = \mathrm{Vir}_+ \oplus \mathrm{Vir}_0 \oplus \mathrm{Vir}_-,$$

where $\mathrm{Vir}_\pm = \sum_{i>0}\mathbb{C}d_{\pm i}$ and $\mathrm{Vir}_0 = \mathbb{C}d_0 + \mathbb{C}z_1$, which acts semi-simply on the algebra via the adjoint representation.

Weight Modules over Vir

A Vir-module V is called a **weight module** if

$$V = \bigoplus_{\lambda \in \mathrm{Vir}_0^*} V_\lambda,$$

where $\mathrm{Vir}_0^* = \mathrm{Hom}_{\mathbb{C}}(\mathrm{Vir}_0, \mathbb{C})$ is the dual space and

$$V_\lambda = \{v \in V \mid xv = \lambda(x)v, \forall x \in \mathrm{Vir}_0\}$$

is called the weight space.

The **support** of a weight module V is defined as

$$\text{supp}(V) = \{\lambda \in \mathrm{Vir}_0^* \mid V_\lambda \neq 0.\}$$

Weight Modules over Vir

A weight Vir-module V is called

highest/lowest weight if $V = U(\mathrm{Vir})v$ for some nonzero v with $\mathrm{Vir}_{\pm}v = 0$ respectively;

Harish-Chandra if all its weight spaces are finitely dimensional;

a module of the intermediate series if it is indecomposable and all its weight spaces are no larger than 1-dimensional.

Irreducible Weight Modules over Vir

Theorem (1992, Mathieu [1]). Any irreducible Harish-Chandra module over Vir is either a highest weight module, or a lowest weight module or a module of intermediate series.

Theorem (2007, Mazorchuk and Zhao [2]). Let V be an irreducible weight Vir-module with $\mathrm{supp}(V) \subseteq \lambda + \mathbb{Z}$ for some $\lambda \in \mathrm{Vir}_0^*$, then

$$\exists\, n_0 \in \mathbb{Z},\ \dim V_{\lambda+n_0} < +\infty \implies \dim V_{\lambda+n} < +\infty,\ \forall\, n \in \mathbb{Z}.$$

Remark. For any irreducible weight Vir-module V, we have $\mathrm{supp}(V) \subseteq \lambda + \mathbb{Z}$ for some $\lambda \in \mathrm{Vir}_0^*$.

Irreducible Weight Modules over Vir

In 1997, H. Zhang [3] constructed a class of weight Vir-modules with infinitely dimensional weight spaces. Among them are many irreducible ones.

In 2001, C. Conley and C. Martin [4] constructed another class of irreducible weight Vir-modules with infinitely dimensional weight spaces, indexed by 4 parameters.

Classical Whittaker Modules over Vir

Let $\varphi : \ \mathrm{Vir}_+ \longrightarrow \mathbb{C}$ be a nonzero Lie algebra homomorphism and $c \in \mathbb{C}$.

We can define a 1-dimensional $\mathrm{Vir}_+ \oplus \mathbb{C}z_1$-module $\mathbb{C}v$ by

$$d_i v = \varphi(d_i) v, \quad z_1 v = cv, \ \forall \ i \in \mathbb{N}.$$

Then we have the induced Vir-module

$$L_{\varphi,c} = \mathrm{Ind}_{\mathrm{Vir}_+ \oplus \mathbb{C}z_1}^{\mathrm{Vir}} \mathbb{C}v = U(\mathrm{Vir}) \bigotimes_{U(\mathrm{Vir}_+ \oplus \mathbb{C}z_1)} \mathbb{C}v.$$

$L_{\varphi,c}$ is called the **universal Whittaker module** of type φ.

Classical Whittaker Modules over Vir

Theorem (2009, M. Ondrus and E. Wiesner[5]). The module $L_{\varphi,c}$ is irreducible if $\varphi(d_1) \neq 0$ and $\varphi(d_2) \neq 0$.

In 2011, S. Yanagida [6] proved that $L_{\varphi,c}$ is irreducible if $\varphi(d_1) \neq 0$ and $\varphi(d_2) = 0$.

We will generalize the above results to a much more general case and with less restrictions.

New Irreducible Whittaker Modules over Vir

For any $m \in \mathbb{N}$, we have a subalgebra $\mathrm{Vir}^{(m)} = \bigoplus_{i \geq m} d_i$. Let $\varphi_m : \mathrm{Vir}^{(m)} \to \mathbb{C}$ be any nonzero Lie algebra homomorphism and c be any complex number.

We can define a 1-dimensional $\mathrm{Vir}^{(m)} \oplus \mathbb{C}z_1$-module $\mathbb{C}v$ by

$$d_i v = \varphi(d_i)v \quad z_1 v = cv, \ \forall\, i \geq m.$$

Then we have the induced Vir-module

$$L_{\varphi_m,c} = \mathrm{Ind}_{\mathrm{Vir}^{(m)} \oplus \mathbb{C}z_1}^{\mathrm{Vir}} \mathbb{C}v = U(\mathrm{Vir}) \bigotimes_{U(\mathrm{Vir}^{(m)} \oplus \mathbb{C}z_1)} \mathbb{C}v.$$

$L_{\varphi_m,c}$ is a Whittaker module in the sense of Batra and Mazorchuk [7].

The Heisenberg-Virasoro Algebra

Heisenberg-Virasoro algebra HVir is a Lie algebra over $\mathbb{C}$ with the basis

$$\{d_n, t^n, z_1, z_2, z_3 \mid n \in \mathbb{Z}\}$$

and the Lie bracket given by

$$[d_n, d_m] = (m-n)d_{n+m} + \delta_{n,-m}\frac{n^3-n}{12}z_1, \tag{1}$$

$$[d_n, t^m] = mt^{m+n} + \delta_{n,-m}(n^2+n)z_2, \tag{2}$$

$$[t^n, t^m] = n\delta_{n,-m}z_3, \tag{3}$$

$$[\mathrm{HVir}, z_1] = [\mathrm{HVir}, z_2] = [\mathrm{HVir}, z_3] = 0. \tag{4}$$

Weight Modules over HVir

Remark. The subalgebra of HVir spanned by $\{d_i, z_1 \mid i \in \mathbb{Z}\}$ is just the Virasoro algebra.

We have the triangular decomposition

$$\mathrm{HVir} = \mathrm{HVir}_+ \oplus \mathrm{HVir}_0 \oplus \mathrm{HVir}_-,$$

where $\mathrm{HVir}_0 = \mathbb{C}d_0 + \mathbb{C}t^0 + \mathbb{C}z_1 + \mathbb{C}z_2 + \mathbb{C}z_3$ and $\mathrm{HVir}_\pm = \sum_{i>0}(\mathbb{C}d_{\pm i} \oplus \mathbb{C}t^{\pm i})$.

We can define the concepts of weight modules and other special modules for HVir as we do for Vir.

Automorphisms of HVir

For any $\alpha = \sum_{i\in\mathbb{Z}} a_i t^i \in \mathbb{C}[t^{\pm 1}]$ and $b \in \mathbb{C}$, we have an automorphism $\sigma = \sigma_{\alpha,b} \in$ Aut(HVir) defined as

$$\begin{aligned}\sigma(d_n) =& d_n + t^n(\alpha + nb) - (n+1)a_{-n}z_2 \\ & - (\frac{\sum_{i\in\mathbb{Z}} a_i a_{-n-i}}{2} + a_{-n}nb)z_3 + \delta_{n,0}b(z_2 + \frac{b}{2}z_3),\end{aligned}$$

$$\sigma(t^n) = t^n + \delta_{n,0}bz_3 - a_{-n}z_3,$$

$$\sigma(z_1) = z_1 - 24bz_2 - 12b^2z_3 \quad \sigma(z_2) = z_2 + bz_3 \quad \sigma(z_3) = z_3.$$

Twisted modules from Automorphism

For any $\sigma \in$ Aut(HVir) and weight HVir-module V, we can give V a different HVir-module structure by defining:

$$x \circ v = \sigma(x)v, \ \forall\, x \in \mathrm{HVir}, \ v \in V.$$

We denote the resulted new module by V^σ.

Then by restricting the action to the Virasoro subalgebra, we can view the new module as a Vir-module, which we still denote by V^σ.

Oscillator Representation of HVir

Let $B = \mathbb{C}[x_1, x_2 \cdots, x_n, \cdots]$ be the Fock space. Make B into an HVir-module by the following actions (See [8]):

$$t^n = \frac{\partial}{\partial x_n}, \quad t^{-n} = n x_n, \; \forall\, n \in \mathbb{N}$$

$$t^0 = 0, \quad z_1 = 1, \quad z_2 = 0, \quad z_3 = 1$$

$$d_n = -\frac{1}{2} \sum_{i \in \mathbb{Z}} : t^{-i} \cdot t^{i+n} :, \; \forall\, n \in \mathbb{Z}.$$

Remark. $(t^i \cdot t^j)v = t^i(t^j v)$ and $: t^i \cdot t^j := \begin{cases} t^i \cdot t^j & \text{if } i \leq j \\ t^j \cdot t^i & \text{if } i > j \end{cases}.$

The action of d_n is well defined.

New Irreducible Whittaker Modules over Vir

For $\sigma_{\alpha,b} \in$ Aut$(\mathrm{HVir}), \alpha \in \mathbb{C}[t^{\pm 1}], b \in \mathbb{C}$, we can construct the Vir-module $B^{\sigma_{\alpha,b}}$, which we denote by $B_{\alpha,b}$ for convenience.

Theorem. For any $\alpha \in \mathbb{C}[t^{\pm 1}] \setminus \mathbb{C}[t]$ and $b \in \mathbb{C}$, the module $B_{\alpha,b}$ is irreducible over Vir.

New Irreducible Whittaker Modules over Vir

Theorem. For any $\alpha = \sum_{i=-m}^{m'} a_i t^i$ with $a_{-m} \neq 0, m \in \mathbb{N}$ and $b \in \mathbb{C}$, we have

$$B_{\alpha,b} \cong L_{\varphi_m,c},$$

where $\varphi_m(d_n) = -(\frac{\sum_{i\in\mathbb{Z}} a_i a_{-n-i}}{2} + nba_{-n})$ for all $n \geq m$ and $c = 1 - 12b^2$.

Remark. For any Lie algebra homomorphism $\varphi_m : \ \mathrm{Vir}^{(m)} \to \mathbb{C}$ with $\varphi_m(d_{2m}) \neq 0$ and any $c \in \mathbb{C}$, we can find some $\alpha \in \mathbb{C}[t^{\pm 1}] \setminus \mathbb{C}[t]$ and some $b \in \mathbb{C}$ such that

$$B_{\alpha,b} \cong L_{\varphi_m,c}.$$

Thus we have $L_{\varphi_m,c}$ is irreducible if $\varphi_m(d_{2m}) \neq 0$.

New Whittaker modules over Vir

Use the theorem and remark on the last page, we can deduce the following more general result:

Theorem. For any Lie algebra homomorphism $\varphi_m : \mathrm{Vir}^{(m)} \to \mathbb{C}$ and any $c \in \mathbb{C}$, the Whittaker module $L_{\varphi_m,c}$ is irreducible if and only if $\psi_m(d_{2m}) \neq 0$ or $\psi_m(d_{2m-1}) \neq 0$.

Remark. Recently, E. Felinska, Z. Jaskolski and M. Kosztolowicz [9] reobtained the above result. Their method is based on some technique computations, similar to that of M. Ondrus and E. Wiesner [5].

Intermediate Series Modules over HVir

For any $\lambda, \mu, F \in \mathbb{C}$, we have the modules of intermediate series $V(\lambda, \mu, F)$ for HVir. As a vector space

$$V(\lambda, \mu, F) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} v_n,$$

the action of HVir on $V(\lambda, \mu, F)$ is

$$d_m v_n = (\lambda + n + m\mu) v_{m+n}, \quad t^m v_n = F v_{m+n}, \ \forall\ m, n \in \mathbb{Z},$$

$$z_1 v_n = z_2 v_n = z_3 v_n = 0, \ \forall\ n \in \mathbb{Z}.$$

$V(\lambda, \mu, F)$ is reducible if and only if $\lambda \in \mathbb{Z}, \mu = 0, 1$ and $F = 0$.

New Irreducible Modules over Vir

For convenience, we denote $A = V(0,0,1)$.

Let $\alpha \in \mathbb{C}[t^{\pm 1}], b \in \mathbb{C}$ and $\sigma_{\alpha,b} \in$ Aut(HVir) be an automorphism. We obtain the Vir-module $A_{\alpha,b} = A^{\sigma_{\alpha,b}}$.

Remark. One can construct Vir-modules using the general HVir-modules of intermediate series to obtain $V(\lambda,\mu,F)^{\sigma_{\alpha,b}}$. However, we can not get essentially different irreducible modules from them other than from $A = V(0,0,1)$.

New Irreducible Modules over Vir

Irreducibility of $A_{\alpha,b}$:

Theorem. Let $\alpha \in \mathbb{C}[t^{\pm 1}]$ and $b \in \mathbb{C}$.

1. If $b \notin \{0, 1\}$, then $A_{\alpha,b}$ is irreducible;
2. $A_{\alpha,0}$ is irreducible if and only if $\alpha \notin \mathbb{Z}$;
3. $A_{\alpha,1}$ is irreducible if and only if $\alpha \in \mathbb{C} \setminus \mathbb{Z}$;
4. If $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, then $A_{\alpha,1} \cong A_{\alpha,0}$;
5. If $\alpha \notin \mathbb{C}$, then $A_{\alpha,1}$ contains a unique nonzero proper submodule and this submodule is isomorphic to $A_{\alpha,0}$.

New Irreducible Modules over Vir

Isomorphisms between $A_{\alpha,b}$:

Theorem. Let $\alpha_1, \alpha_2 \in \mathbb{C}[t^{\pm 1}]$ and $b_1, b_2 \in \mathbb{C}$.

1. If $b_1 \notin \{0, 1\}$, then $A_{\alpha_1,b_1} \cong A_{\alpha_2,b_2}$ if and only if $\alpha_1 - \alpha_2 \in \mathbb{Z}$ and $b_1 = b_2$;
2. If $b_1 \in \{0, 1\}$, then $A_{\alpha_1,b_1} \cong A_{\alpha_2,b_1}$ if and only if $\alpha_1 - \alpha_2 \in \mathbb{Z}$;
3. $A_{\alpha_1,0} \cong A_{\alpha_2,1}$ if and only if $\alpha_1 \in \mathbb{C} \setminus \mathbb{Z}$ and $\alpha_1 - \alpha_2 \in \mathbb{Z}$.

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THANKS!