The universal associative envelope of the anti-Jordan triple system of $n \times n$ matrices

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Part of the speaker's doctoral thesis

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June 1, 2012

PLAN OF THE TALK:

1. Anti-Jordan Triple Systems,
2. Noncommutative Gröbner bases in free associative algebras,
3. The universal associative envelope of the anti-Jordan triple system of $n \times n$ matrices,
4. Structure constants of the universal associative envelope,
5. The center of the universal associative envelope,
6. Explicit decomposition of the universal associative envelope into matrix algebras,
7. Conjecture.

Anti-Jordan triple systems were introduced by Faulkner and Ferrar in (1980). The classification of finite-dimensional simple anti-Jordan triple systems over an algebraically closed field of characteristic 0 was given by Bashir (2008).

Definition. A vector space V over a field F endowed with a trilinear operation $V \times V \times V \to V$, $(a, b, c) \to \langle a, b, c \rangle$ is said to be an *anti-Jordan triple system* if the following conditions are satisfied for all $a, b, c, d, e \in V$:

$$\langle a, b, a \rangle = 0,$$
$$\langle a, b, \langle c, d, e \rangle \rangle = \langle \langle a, b, c \rangle, d, e \rangle + \langle c, \langle b, a, d \rangle, e \rangle + \langle c, d, \langle a, b, e \rangle \rangle.$$

Remark. If A is an associative algebra, A defines an anti-Jordan triple system A_- relative to the product $\langle a, b, c \rangle = abc - cba$.

Anti-Jordan triple systems

Definition.

- A *representation* of an anti-Jordan triple system $\mathfrak{J}$ is a homomorphism $\rho\colon \mathfrak{J} \to (\mathrm{End}\, V)_-$ from $\mathfrak{J}$ to the anti-Jordan triple system of endomorphisms of a vector space V. In other words, ρ is a linear mapping that satisfies

$$\rho(\langle a,b,c\rangle) = \rho(a)\rho(b)\rho(c) - \rho(c)\rho(b)\rho(a),$$

  for all $a,b,c \in \mathfrak{J}$.
- Two representations ρ_1 and ρ_2 of an anti-Jordan triple system $\mathfrak{J}$ on the same vector space V are *equivalent* if there exists an invertible endomorphism T such that $\rho_2(a) = T^{-1}\rho_1(a)T$ for all $a \in \mathfrak{J}$.

Examples. Let F be an algebraically closed field of characteristic 0. The following are simple finite-dimensional anti-Jordan triple systems over F:

(i) $T_1 = M_{nn}(F)$, together with the trilinear operation $\langle x, y, z\rangle_1 = xyz - zyx$.

(ii) $T_2 = M_{mn}(F)$, together with the trilinear operation $\langle x, y, z\rangle_2 = xy^t az - zy^t ax$, where $a = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$, and $2r = m$.

(iii) $T_3 = M_{mn}(F)$, together with the trilinear operation $\langle x, y, z\rangle_3 = xby^t z - zby^t x$, where $b = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$, and $2r = n$.

Noncommutative Gröbner bases in free associative algebras

Definition. Let $X = \{x_1, \ldots, x_n\}$ be a set of symbols with the total order $x_i < x_j$ if and only if $i < j$.

- The *free monoid* generated by X is the set X^* of all (possibly empty) words $w = x_{i_1} \cdots x_{i_k}$ ($\deg(w) = k \geq 0$) with the (associative) operation of concatenation.
- The *degree-lexicographical* (*deglex*) order $<$ on X^* is defined as follows: $u < v$ if and only if either (*i*) $\deg(u) < \deg(v)$ or (*ii*) $\deg(u) = \deg(v)$ and $u = wx_iu', v = wx_jv'$ where $x_i < x_j$ $(w, u', v' \in X^*)$.
- The *free (unital) associative algebra* generated by X is the vector space $F\langle X\rangle$ with basis X^* and multiplication extended bilinearly from concatenation in X^*.

Definition.

- The *leading monomial* of $f \in F\langle X \rangle$, denoted $\mathrm{LM}(f)$, is the highest element of f with respect to deglex order.
- If I is any ideal of $F\langle X \rangle$ then the set of *normal words* modulo I is defined by $N(I) = \{u \in X^* \mid u \neq \mathrm{LM}(f) \text{ for any } f \in I\}$.
- We write $C(I)$ for the subspace of $F\langle X \rangle$ spanned by $N(I)$.

Proposition. If $I \subset F\langle X \rangle$ is an ideal then $F\langle X \rangle = C(I) \oplus I$.

Definition. Let $G \subset F\langle X \rangle$ be a subset generating an ideal $I \subset F\langle X \rangle$. A noncommutative polynomial $f \in F\langle X \rangle$ is in *normal form modulo* G if no monomial occurring in f has a factor of the form $\mathrm{LM}(g)$ for any $g \in G$.

Noncommutative Gröbner bases

Definition. A subset $G \subset I$ is a *Gröbner basis* of I if for all $f \in I$ there is a $g \in G$ such that $\mathrm{LM}(g)$ is a factor of $\mathrm{LM}(f)$. A subset $G \subset F\langle X\rangle$ is *self-reduced* if every $g \in G$ is in normal form modulo $G \setminus \{g\}$ and every $g \in G$ is *monic*: the coefficient of $\mathrm{LM}(g)$ is 1.

Definition. Let $g, h \in F\langle X\rangle$ be two monic noncommutative polynomials. Assume that $\mathrm{LM}(g)$ is not a factor of $\mathrm{LM}(h)$ and that $\mathrm{LM}(h)$ is not a factor of $\mathrm{LM}(g)$. Let $u, v \in X^*$ be such that (i) $\mathrm{LM}(g)\,u = v\,\mathrm{LM}(h)$, (ii) u is a proper right factor of $\mathrm{LM}(h)$, and v is a proper left factor of $\mathrm{LM}(g)$. In this case the element $gu - vh \in F\langle X\rangle$ is called a *composition* of g and h.

Lemma. (Diamond Lemma, (Bergman, 1978)) If $I \subset F\langle X\rangle$ is an ideal generated by a self-reduced set G, then G is a Gröbner basis of I if and only if for all compositions f of the elements of G the normal form of f modulo G is zero.

Given an ideal I in $F\langle X\rangle$, we want a basis for the quotient algebra $F\langle X\rangle/I$. If G is a Gröbner basis for I then we can easily determine a basis for $F\langle X\rangle/I$: it is the set of all $w \in X^*$ such that $\mathrm{LM}(g)$ is not a subword of w for all $g \in G$.

Algorithm-GB. **(Buchberger-Bergman-Mora)**

Input: $G_0 = \{g_1, \ldots, g_s\}$

Output: $\mathcal{G} = \{g_1, \ldots, g_m, \ldots\}$, a Gröbner basis for $I = \langle G_0\rangle$

Initialization: $\mathcal{G} := G_0$, $S := \{o(g_i, g_j) \mid g_i \neq g_j \in G_0\}$

While $S \neq \emptyset$ **Do**

Choose any $o(g_i, g_j) \in S$,

$S := S - \{o(g_i, g_j)\}$

$N(o(g_i, g_j)) \mod \mathcal{G} = r$

If $r \neq 0$ **Then**

$\mathcal{G} := \mathcal{G} \cup \{r\}$

$S := S \cup \{o(g, r), o(r, g) \mid g \in \mathcal{G}\}$

Remark. This algorithm may not terminate, since the resulting Gröbner basis may not be finite.

Universal associative enveloping algebra

Definition. Let $\mathfrak{J}$ be an anti-Jordan triple system over a field F and let $\mathfrak{A}$ be an associative algebra. Let $i : \mathfrak{J} \longrightarrow \mathfrak{A}$ be an anti-Jordan homomorphism. The pair $(\mathfrak{A}, i)$ is called a *universal enveloping algebra* of $\mathfrak{J}$ if for any associative algebra A and every anti-Jordan homomorphism $f : \mathfrak{J} \longrightarrow A$ there exists a unique algebra homomorphism $\bar{f} : \mathfrak{A} \longrightarrow A$ such that the map $\bar{f} \circ i = f$.

$$
\begin{tikzcd}
\mathfrak{J} \arrow[r, "i"] \arrow[dr, "f"'] & \mathfrak{A} \arrow[d, dashed, "\bar{f}"] \\
 & A
\end{tikzcd}
$$

G. Benkart and T. Roby (1998) introduced the down-up algebras over complex numbers. We define it over an arbitrary field.

Definition. Let F be a field and $\alpha, \beta, \gamma \in F$ be fixed arbitrary parameters. Then the **down-up algebra** $A(\alpha, \beta, \gamma)$ is the unital associative F-algebra with generators a, b and relations,

$$b^2a = \alpha bab + \beta ab^2 + \gamma b, \quad ba^2 = \alpha aba + \beta a^2b + \gamma a.$$

Example. Consider the 2-dimensional anti-Jordan triple system $\mathfrak{J}$ with basis $\mathcal{B} = \{a = E_{12}, b = E_{21}\}$ of 2×2 matrix units and triple product $\langle a, b, c\rangle = abc - cba$. The multiplication table of $\mathfrak{J}$ is zero. The universal enveloping algebra $\mathfrak{A}(\mathfrak{J})$ has two generators a, b and relations $b^2a = ab^2$, $ba^2 = a^2b$. Thus, $\mathfrak{A}(\mathfrak{J}) \cong A(0, 1, 0)$.

Universal envelope of the anti-Jordan of $n \times n$ matrices

Let $\mathfrak{J}$ be the anti-Jordan triple system of all $n \times n$ $(n \geq 2)$ matrices over an algebraically closed field F of characteristic 0 with the triple product $\langle a, b, c\rangle = abc - cba$.

- We write $\delta_{i,j}$ for the Kronecker delta, and $\widehat{\delta}_{i,j} = 1 - \delta_{i,j}$.
- Let $\Omega = \{1, 2, \ldots, n\}$ be a finite index set.
- Let $B = \{E_{i,j}\}_{i,j\in\Omega}$ be an ordered basis of $\mathfrak{J}$, where $E_{i,j}$ is the matrix with a single 1 in the ith row and jth column, and zeros elsewhere.
- The structure constants for $\mathfrak{J}$ are

$$\langle E_{i,j}, E_{k,\ell}, E_{m,t}\rangle = \delta_{j,k}\delta_{\ell,m}E_{i,t} - \delta_{t,k}\delta_{\ell,i}E_{m,j} \quad (i, j, k, \ell, m, t \in \Omega).$$

- Consider the bijection $\phi \colon B \to X = \{e_{i,j}\}_{i,j\in\Omega}$ defined by $\phi(E_{i,j}) = e_{i,j}$. We extend ϕ to a linear map $\phi \colon \mathfrak{J} \to F\langle X\rangle$.
- We use the deglex order $<$ where $e_{i,j} < e_{k,\ell}$ if either $i < k$, or $i = k$ and $j < \ell$.

Let $G \subset F\langle X\rangle$ consist of these elements $(i, j, k, r, s, t \in \Omega)$:

$$\begin{aligned}
\mathcal{R}_1^{(i,j,k,t)} &= e_{i,j}e_{j,k}e_{k,t} - e_{k,t}e_{j,k}e_{i,j} - e_{i,t} \ (k < i),\\
\mathcal{R}_2^{(i,j,t)} &= e_{i,j}e_{j,i}e_{i,t} - e_{i,t}e_{j,i}e_{i,j} - e_{i,t} \ (t < j),\\
\mathcal{R}_3^{(i,j,k,t)} &= e_{i,j}e_{k,i}e_{t,k} - e_{t,k}e_{k,i}e_{i,j} + e_{t,j} \ (t < i),\\
\mathcal{R}_4^{(i,j,k)} &= e_{i,j}e_{k,i}e_{i,k} - e_{i,k}e_{k,i}e_{i,j} + e_{i,j} \ (k < j),\\
\mathcal{R}_5^{(i,j,k,t,r,s)} &= e_{i,j}e_{k,t}e_{r,s} - e_{r,s}e_{k,t}e_{i,j} \ \ (r < i,\, j \neq k \textbf{ or } t \neq r,\, s \neq k \textbf{ or } t \neq i)\,,\\
\mathcal{R}_6^{(i,j,k,t,s)} &= e_{i,j}e_{k,t}e_{i,s} - e_{i,s}e_{k,t}e_{i,j} \ \ (s < j,\, j \neq k \textbf{ or } t \neq i,\, s \neq k \textbf{ or } t \neq i)\,.
\end{aligned}$$

Let $I \subset F\langle X\rangle$ be the ideal generated by G. We write $\mathfrak{A} = F\langle X\rangle/I$ with surjection $\pi\colon F\langle X\rangle \to \mathfrak{A}$ sending f to $f + I$, and $i = \pi \circ \phi$ for the natural map $i\colon \mathfrak{J} \to \mathfrak{A}$.

Lemma. The unital associative algebra $\mathfrak{A}$ and the linear map i form the universal associative envelope of the anti-Jordan triple system $\mathfrak{J}$.

Our goal is to complete the set G to a Gröbner basis for the ideal I.

Lemma. The set of all normal forms modulo G of nontrivial compositions among elements of G includes the set G_1 which consists of these elements:

$$\begin{aligned}
\mathcal{G}_1^{(r,t,m)} &= e_{r,t}e_{t,m} - e_{r,1}e_{1,m} \quad (m \neq r,\, t \neq 1),\\
\mathcal{G}_2^{(i,t,\ell)} &= e_{i,t}e_{\ell,i} - e_{1,t}e_{\ell,1} \quad (t \neq \ell,\, i \neq 1),\\
\mathcal{G}_3^{(i,j,k,\ell)} &= e_{i,j}e_{k,\ell} \quad (i \neq \ell,\, j \neq k).
\end{aligned}$$

Lemma. The set of all normal forms modulo $G \cup G_1$ of nontrivial compositions among elements of $G \cup G_1$ includes the set G_2 which consists of these elements:

$$\mathcal{G}_4^{(r,i)} = e_{r,i}e_{i,r} - e_{r,1}e_{1,r} + e_{1,1}^2 - e_{1,i}e_{i,1} \quad (r, i \in \Omega \setminus \{1\}).$$

Lemma. The set of all normal forms modulo $G \cup G_1 \cup G_2$ of nontrivial compositions among elements of $G \cup G_1 \cup G_2$ includes the set G_3 which consists of these elements:

$$
\begin{aligned}
\mathcal{G}_5^{(r,i)} &= e_{r,1}e_{1,i}e_{i,1} - e_{1,1}^2 e_{r,1} - e_{r,1} \quad (r < i;\ i, r \in \Omega \setminus \{1\}),\\
\mathcal{G}_6^{(i,r)} &= e_{i,1}e_{1,i}e_{r,1} - e_{1,1}^2 e_{r,1} \quad (i < r;\ i, r \in \Omega \setminus \{1\}),\\
\mathcal{G}_7^{(t,\ell)} &= e_{1,t}e_{t,1}e_{1,\ell} - e_{1,1}^2 e_{1,\ell} \quad (t < \ell;\ t, \ell \in \Omega \setminus \{1\}),\\
\mathcal{G}_8^{(\ell,t)} &= e_{1,\ell}e_{t,1}e_{1,t} - e_{1,1}^2 e_{1,\ell} + e_{1,\ell} \quad (\ell < t; \ell, t \in \Omega \setminus \{1\}),\\
\mathcal{G}_9^{(r)} &= e_{r,1}e_{1,r}e_{r,1} - 2e_{1,1}^2 e_{r,1} - e_{r,1} \quad (r \in \Omega \setminus \{1\}),\\
\mathcal{G}_{10}^{(r)} &= e_{1,r}e_{r,1}e_{1,r} - 2e_{1,1}^2 e_{1,r} + e_{1,r} \quad (r \in \Omega \setminus \{1\}),\\
\mathcal{G}_{11}^{(r,i,\ell)} &= e_{r,1}e_{1,i}e_{\ell,1} \quad (r \neq i \neq \ell),\\
\mathcal{G}_{12}^{(\ell,i,r)} &= e_{1,\ell}e_{i,1}e_{1,r} \quad (r \neq i \neq \ell),\\
\mathcal{G}_{13}^{(i)} &= e_{1,1}e_{1,i}e_{i,1} - \tfrac{1}{2}e_{1,1}^3 - \tfrac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{14}^{(i)} &= e_{1,1}e_{i,1}e_{1,i} - \tfrac{1}{2}e_{1,1}^3 + \tfrac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}).
\end{aligned}
$$

Lemma. The set of all normal forms modulo $G \cup G_1 \cup G_2 \cup G_3$ of nontrivial compositions among elements of $G \cup G_1 \cup G_2 \cup G_3$ includes the set G_4 which consists of these elements:

$$\mathcal{G}_{17}^{(i)} = e_{1,1}^3 e_{1,i} - e_{1,1} e_{1,i}, \quad \mathcal{G}_{18}^{(i)} = e_{1,1}^3 e_{i,1} + e_{1,1} e_{i,1} \qquad (i \in \Omega \setminus \{1\}),$$
$$\mathcal{G}_{19} = e_{1,1}^5 - e_{1,1}.$$

Lemma. The self-reduced form $\mathfrak{G}$ of the set $G \cup \bigcup_{i=1}^{4} G_i$ consists of these elements:

$$\begin{aligned}
\mathcal{G}_0^{(i,j)} &= e_{i,1}e_{1,1}e_{1,j} - e_{1,j}e_{1,1}e_{i,1} - e_{i,j} \quad (i,j \in \Omega \setminus \{1\}),\\
\mathcal{G}_1^{(i,j,k)} &= e_{i,j}e_{j,k} - e_{i,1}e_{1,k} \quad (i,j,k \in \Omega;\, k \neq i, j \neq 1),\\
\mathcal{G}_2^{(i,j,k)} &= e_{i,j}e_{k,i} - e_{1,j}e_{k,1} \quad (i,j,k \in \Omega;\, j \neq k,\, i \neq 1),\\
\mathcal{G}_3^{(i,j,k,\ell)} &= e_{i,j}e_{k,\ell} \quad (i,j,k,\ell \in \Omega;\, i \neq \ell,\, j \neq k),\\
\mathcal{G}_4^{(i,j)} &= e_{i,j}e_{j,i} - e_{i,1}e_{1,i} - e_{1,j}e_{j,1} + e_{1,1}^2 \quad (i,j \in \Omega \setminus \{1\}),\\
\mathcal{G}_5^{(i,j)} &= e_{i,1}e_{1,j}e_{j,1} - e_{1,1}^2e_{i,1} - e_{i,1} \quad (i,j \in \Omega;\, i \neq 1, j \neq i),\\
\mathcal{G}_6^{(i,j)} &= e_{j,1}e_{1,j}e_{i,1} - e_{1,1}^2e_{i,1} \quad (i,j \in \Omega \setminus \{1\}; i \neq j),\\
\mathcal{G}_7^{(i,j)} &= e_{1,i}e_{i,1}e_{1,j} - e_{1,1}^2e_{1,j} \quad (i,j \in \Omega \setminus \{1\};\, i \neq j),\\
\mathcal{G}_8^{(i,j)} &= e_{1,i}e_{j,1}e_{1,j} - e_{1,1}^2e_{1,i} + e_{1,i} \quad (i,j \in \Omega;\, i \neq j, i \neq 1),
\end{aligned}$$

$$
\begin{aligned}
\mathcal{G}_9^{(i)} &= e_{i,1}e_{1,i}e_{i,1} - 2e_{1,1}^2e_{i,1} - e_{i,1} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{10}^{(j)} &= e_{1,j}e_{j,1}e_{1,j} - 2e_{1,1}^2e_{1,j} + e_{1,j} \quad (j \in \Omega \setminus \{1\}),\\
\mathcal{G}_{11}^{(i,j,k)} &= e_{i,1}e_{1,j}e_{k,1} \quad (k,i,j \in \Omega;\ k,i \neq j),\\
\mathcal{G}_{12}^{(i,j,k)} &= e_{1,i}e_{j,1}e_{1,k} \quad (k,i,j \in \Omega;\ i,k \neq j),\\
\mathcal{G}_{13}^{(i)} &= e_{1,1}e_{1,i}e_{i,1} - \tfrac{1}{2}e_{1,1}^3 - \tfrac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{14}^{(i)} &= e_{1,1}e_{i,1}e_{1,i} - \tfrac{1}{2}e_{1,1}^3 + \tfrac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{15}^{(i)} &= e_{1,i}e_{i,1}e_{1,1} - \tfrac{1}{2}e_{1,1}^3 - \tfrac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{16}^{(i)} &= e_{i,1}e_{1,i}e_{1,1} - \tfrac{1}{2}e_{1,1}^3 + \tfrac{1}{2}e_{1,1} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{17}^{(i)} &= e_{1,1}^3e_{1,i} - e_{1,1}e_{1,i} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{18}^{(i)} &= e_{1,1}^3e_{i,1} + e_{1,1}e_{i,1} \quad (i \in \Omega \setminus \{1\}),\\
\mathcal{G}_{19} &= e_{1,1}^5 - e_{1,1}.
\end{aligned}
$$

Gröbner basis and finite dimensionality

The following lemma plays a crucial role in proving that the set $\mathfrak{G}$ is a Gröbner basis for the ideal I.

Lemma. For the universal enveloping algebra $\mathfrak{A}$, either

$$(i)\ \dim \mathfrak{A} = \infty, \text{or} \qquad (ii)\ \dim \mathfrak{A} < \infty \text{ and } \dim \mathfrak{A} \geq 4n^2 + 1.$$

Theorem. With notation as above. If $\mathfrak{J}$ is the anti-Jordan triple system of all $n \times n$ matrices $(n \geq 2)$ then:

(i) The set $\mathfrak{G}$ is a Gröbner basis for the ideal I.

(ii) The universal enveloping algebra $\mathfrak{A}$ of $\mathfrak{J}$ is finite dimensional with basis $\mathfrak{B}$ consists of $4n^2 + 1$ monomials:

$$\mathfrak{B} = \left\{1,\ e_{i,j},\ e_{i,1}e_{1,j},\ e_{1,1}^2 e_{1,j},\ e_{1,1}^4 \mid i,j \in \Omega\right\}$$
$$\cup \{e_{1,i}e_{j,1} \mid i,j \in \Omega, (i,j) \neq (1,1)\} \cup \{e_{1,i}e_{1,1}e_{j,1} \mid i,j \in \Omega, j \neq 1\}.$$

Lemma. Define an anti-automorphism $\eta\colon F\langle X\rangle \to F\langle X\rangle$ of the free associative algebra generated by $X = \{e_{i,j}\}_{i,j\in\Omega}$ by $\eta(e_{i,j}) = e_{j,i}$. Then η induces an anti-automorphism of order 2 on $\mathfrak{A}$.

Proposition. Let $i, j, k, \ell \in \Omega$. Then in $\mathfrak{A}$, we have

$$\begin{aligned} e_{i,j}\cdot e_{k,\ell} = {} & \delta_{j,k}\big[\delta_{i,\ell}\big\{ \left(\delta_{i,1}+\widehat{\delta}_{i,1}\widehat{\delta}_{j,1}\right) e_{1,j}e_{j,1} + \left(\delta_{j,1}\widehat{\delta}_{i,1}+\widehat{\delta}_{i,1}\widehat{\delta}_{j,1}\right) e_{i,1}e_{1,i} \\ & - \widehat{\delta}_{i,1}\widehat{\delta}_{j,1}e_{1,1}^2\big\} + \widehat{\delta}_{i,\ell}e_{i,1}e_{1,\ell}\big] + \widehat{\delta}_{j,k}\delta_{i,\ell}e_{1,j}e_{k,1}, \end{aligned} \tag{1}$$

$$\begin{aligned} e_{i,j}\cdot e_{k,1}e_{1,\ell} = {} & \delta_{i,1}\Big[\left(\delta_{j,k}\delta_{\ell,1}\left(\delta_{\ell,j}+\tfrac{1}{2}\widehat{\delta}_{\ell,j}\right) + \tfrac{1}{2}\widehat{\delta}_{j,k}\delta_{k,\ell}\delta_{j,1}\right) e_{1,1}^3 \\ & + \widehat{\delta}_{j,k}\delta_{k,\ell}\widehat{\delta}_{j,1}e_{1,1}^2e_{1,j} + \delta_{j,k}\left(2\delta_{\ell,j}\widehat{\delta}_{\ell,1}+\widehat{\delta}_{\ell,j}\left(\delta_{j,1}+\widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\right)\right) e_{1,1}^2e_{1,\ell} \\ & + \tfrac{1}{2}\left(\delta_{\ell,1}\delta_{j,k}\widehat{\delta}_{\ell,j} - \widehat{\delta}_{j,k}\delta_{k,\ell}\delta_{j,1}\right) e_{1,1} - \left(\delta_{j,k}\delta_{\ell,j}\widehat{\delta}_{\ell,1}+\widehat{\delta}_{j,k}\delta_{k,\ell}\widehat{\delta}_{j,1}\right) e_{1,j}\Big] \\ & + \widehat{\delta}_{i,1}\delta_{j,k}(e_{1,\ell}e_{1,1}e_{i,1}+e_{i,\ell}), \end{aligned} \tag{2}$$

The structure constants of $\mathfrak{A}$

$$
\begin{aligned}
e_{\ell,1}e_{1,k}\cdot e_{j,i} = {} & \delta_{i,1}\Big[\left(\delta_{j,k}\delta_{\ell,1}\left(\delta_{\ell,j}+\tfrac{1}{2}\widehat{\delta}_{\ell,j}\right)+\tfrac{1}{2}\widehat{\delta}_{j,k}\delta_{k,\ell}\delta_{j,1}\right)e_{1,1}^{3}+\widehat{\delta}_{j,k}\delta_{k,\ell}\widehat{\delta}_{j,1}e_{1,1}^{2}e_{j,1}\\
& +\delta_{j,k}\left(2\delta_{\ell,j}\widehat{\delta}_{\ell,1}+\widehat{\delta}_{\ell,j}\left(\delta_{j,1}+\widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\right)\right)\left(e_{1,1}^{2}e_{\ell,1}+e_{\ell,1}\right)\\
& +\tfrac{1}{2}\left(\delta_{\ell,1}\delta_{j,k}\widehat{\delta}_{\ell,j}-\widehat{\delta}_{j,k}\delta_{k,\ell}\delta_{j,1}\right)e_{1,1}-\delta_{j,k}\delta_{\ell,j}\widehat{\delta}_{\ell,1}e_{j,1}\Big]\\
& +\widehat{\delta}_{i,1}\delta_{j,k}\left(\delta_{\ell,1}e_{1,1}^{2}e_{1,i}+\widehat{\delta}_{\ell,1}\left(e_{1,i}e_{1,1}e_{\ell,1}+e_{\ell,i}\right)\right), \qquad (3)\\
e_{i,j}\cdot e_{1,k}e_{\ell,1} = {} & \delta_{j,1}\Big[\tfrac{1}{2}\left(\delta_{i,k}\delta_{\ell,1}\widehat{\delta}_{i,1}+\widehat{\delta}_{i,k}\widehat{\delta}_{\ell,1}\delta_{k,\ell}\delta_{i,1}\right)e_{1,1}^{3}\\
& +\widehat{\delta}_{i,k}\delta_{k,\ell}\widehat{\delta}_{i,1}\widehat{\delta}_{\ell,1}\,e_{1,1}^{2}e_{i,1}+\delta_{i,k}\left(\delta_{i,1}+\widehat{\delta}_{i,1}\widehat{\delta}_{\ell,1}\left(2\delta_{i,\ell}+\widehat{\delta}_{i,\ell}\right)\right)e_{1,1}^{2}e_{\ell,1}\\
& +\tfrac{1}{2}\left(\widehat{\delta}_{i,k}\widehat{\delta}_{\ell,1}\delta_{k,\ell}\delta_{i,1}-\delta_{i,k}\delta_{\ell,1}\widehat{\delta}_{i,1}\right)e_{1,1}+\widehat{\delta}_{i,1}\widehat{\delta}_{\ell,1}\left(\delta_{i,k}\delta_{i,\ell}+\widehat{\delta}_{i,k}\delta_{k,\ell}\right)e_{i,1}\Big]\\
& +\widehat{\delta}_{j,1}\delta_{i,k}\left(\delta_{\ell,1}\left(e_{1,1}^{2}e_{1,j}-e_{1,j}\right)+\widehat{\delta}_{\ell,1}e_{1,j}e_{1,1}e_{\ell,1}\right); \quad (k,\ell)\neq(1,1), \qquad (4)
\end{aligned}
$$

The structure constants of $\mathfrak{A}$

$$
\begin{aligned}
e_{1,\ell}e_{k,1}\cdot e_{j,i} = \delta_{j,1}\Big[&\tfrac{1}{2}\left(\delta_{i,k}\delta_{\ell,1}\widehat{\delta}_{i,1}+\widehat{\delta}_{i,k}\widehat{\delta}_{\ell,1}\delta_{k,\ell}\delta_{i,1}\right)e_{1,1}^{3}+\widehat{\delta}_{i,k}\delta_{k,\ell}\widehat{\delta}_{i,1}\widehat{\delta}_{\ell,1}\,e_{1,1}^{2}e_{1,i}\\
&+\delta_{i,k}\left(\delta_{i,1}+\widehat{\delta}_{i,1}\widehat{\delta}_{\ell,1}\left(2\delta_{i,\ell}+\widehat{\delta}_{i,\ell}\right)\right)\left(e_{1,1}^{2}e_{1,\ell}-e_{1,\ell}\right)\\
&+\tfrac{1}{2}\left(\widehat{\delta}_{i,k}\widehat{\delta}_{\ell,1}\delta_{k,\ell}\delta_{i,1}-\delta_{i,k}\delta_{\ell,1}\widehat{\delta}_{i,1}\right)e_{1,1}+\widehat{\delta}_{i,1}\widehat{\delta}_{\ell,1}\delta_{i,k}\delta_{i,\ell}e_{1,i}\Big]\\
&+\widehat{\delta}_{j,1}\delta_{i,k}\left(\delta_{\ell,1}e_{1,1}^{2}e_{j,1}+\widehat{\delta}_{\ell,1}e_{1,\ell}e_{1,1}e_{1,j}\right);\quad (k,\ell)\neq(1,1), \qquad (5)
\end{aligned}
$$

$$
e_{i,j}\cdot e_{1,k}e_{1,1}e_{\ell,1} = \delta_{i,k}\left[-e_{1,j}e_{\ell,1}+\widehat{\delta}_{j,1}\delta_{j,\ell}\tfrac{1}{2}\left(e_{1,1}^{4}+e_{1,1}^{2}\right)\right];\quad \ell\neq 1, \qquad (6)
$$

$$
\begin{aligned}
e_{i,j}\cdot e_{1,1}^{2}e_{1,k} = \delta_{j,1}\Big[&\delta_{i,1}\left(\delta_{k,1}e_{1,1}^{4}+\widehat{\delta}_{k,1}e_{1,1}e_{1,k}\right)\\
&+\widehat{\delta}_{i,1}\left(\delta_{i,k}\tfrac{1}{2}(e_{1,1}^{4}-e_{1,1}^{2})+e_{i,1}e_{1,k}\right)\Big]-\widehat{\delta}_{j,1}\delta_{i,1}\delta_{k,1}e_{1,j}e_{1,1}, \qquad (7)
\end{aligned}
$$

$$
e_{i,j}\cdot e_{1,1}^{4} = \delta_{j,1}\left[\delta_{i,1}e_{1,1}+\widehat{\delta}_{i,1}\left(e_{1,1}^{2}e_{i,1}+e_{i,1}\right)\right]-\widehat{\delta}_{j,1}\delta_{i,1}\left(e_{1,1}^{2}e_{1,j}-e_{1,j}\right), \qquad (8)
$$

The structure constants of $\mathfrak{A}$

$$
\begin{aligned}
e_{i,1}e_{1,j}\cdot e_{k,1}e_{1,\ell} = {} & \delta_{j,k}\delta_{\ell,1}\left(\delta_{\ell,j}\delta_{i,1}e_{1,1}^{4}+\tfrac{1}{2}\widehat{\delta}_{\ell,j}\delta_{i,1}\left(e_{1,1}^{4}+e_{1,1}^{2}\right)\right)\\
& + \tfrac{1}{2}\Big[\delta_{j,k}\widehat{\delta}_{i,1}\delta_{i,\ell}\left(2\delta_{\ell,j}\widehat{\delta}_{\ell,1}+\widehat{\delta}_{\ell,j}\left(\delta_{j,1}+\widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\right)\right)\\
& \qquad + \widehat{\delta}_{j,k}\delta_{k,\ell}\left(\widehat{\delta}_{j,1}\widehat{\delta}_{i,1}\delta_{i,j}+\delta_{j,1}\delta_{i,1}\right)\Big]\left(e_{1,1}^{4}-e_{1,1}^{2}\right)\\
& + \delta_{j,k}\left(\delta_{\ell,1}\widehat{\delta}_{i,1}+\delta_{\ell,j}\widehat{\delta}_{\ell,1}+\widehat{\delta}_{\ell,j}\left(\delta_{j,1}+\widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\right)\right)e_{i,1}e_{1,\ell}, \qquad (9)\\
e_{i,1}e_{1,j}\cdot e_{1,k}e_{\ell,1} = {} & \delta_{j,1}\widehat{\delta}_{k,1}\delta_{k,\ell}\tfrac{1}{2}\left(\delta_{i,1}\left(e_{1,1}^{4}+e_{1,1}^{2}\right)+2\widehat{\delta}_{i,1}e_{i,1}e_{1,1}\right)\\
& - \delta_{k,1}\left(\delta_{j,1}\delta_{i,1}+\widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\delta_{i,j}\right)e_{1,1}e_{\ell,1}; \quad (k,\ell)\neq(1,1), \qquad (10)\\
e_{i,1}e_{1,j}\cdot e_{1,k}e_{1,1}e_{\ell,1} = {} & -\delta_{k,1}\left(\delta_{i,1}\delta_{j,1}+\widehat{\delta}_{j,1}\delta_{i,j}\right)e_{1,1}^{2}e_{\ell,1}; \quad \ell\neq 1, \qquad (11)\\
e_{i,1}e_{1,j}\cdot e_{1,1}^{2}e_{1,k} = {} & \delta_{j,1}\Big[\delta_{k,1}\left(\delta_{i,1}e_{1,1}+\widehat{\delta}_{i,1}\left(e_{1,1}^{2}e_{i,1}+e_{i,1}\right)\right)\\
& \qquad + \widehat{\delta}_{k,1}\left(\delta_{i,1}e_{1,1}^{2}e_{1,k}+\widehat{\delta}_{i,1}\left(e_{1,k}e_{1,1}e_{i,1}+e_{i,k}\right)\right)\Big]\\
& - \widehat{\delta}_{j,1}\delta_{k,1}\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^{3}-e_{1,1}\right). \qquad (12)
\end{aligned}
$$

Proposition. Let $i, j, k, \ell \in \Omega$ and $(i, j) \neq (1, 1)$. Then in $\mathfrak{A}$, we have

$$e_{i,1}e_{1,j} \cdot e_{1,1}^4 = \delta_{j,1}e_{i,1}e_{1,1} + \widehat{\delta}_{j,1}\delta_{j,i}\tfrac{1}{2}\left(e_{1,1}^2 - e_{1,1}^4\right), \tag{13}$$

$$\begin{aligned} e_{1,i}e_{j,1} \cdot e_{1,k}e_{\ell,1} = \tfrac{1}{2}\Big[&\delta_{i,1}\delta_{j,k}\delta_{\ell,1}\widehat{\delta}_{j,1}\left(e_{1,1}^4 - e_{1,1}^2\right) \\ &+ \Big\{\delta_{j,k}\widehat{\delta}_{i,1}\delta_{i,\ell}\left(\delta_{j,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{\ell,1}\left(2\delta_{j,\ell} + \widehat{\delta}_{j,\ell}\right)\right) \\ &+ \widehat{\delta}_{j,k}\widehat{\delta}_{\ell,1}\delta_{k,\ell}\left(\widehat{\delta}_{j,1}\widehat{\delta}_{j,\ell}\delta_{i,j} + \delta_{i,1}\delta_{j,1}\right)\Big\}\left(e_{1,1}^4 + e_{1,1}^2\right)\Big] \\ &- \delta_{j,k}\left(\delta_{j,1} + \widehat{\delta}_{j,1}\left(\delta_{\ell,1}\widehat{\delta}_{i,1} + \widehat{\delta}_{\ell,1}\right)\right)e_{1,i}e_{\ell,1}; \quad (k,\ell) \neq (1,1), \end{aligned} \tag{14}$$

$$\begin{aligned} e_{1,i}e_{j,1} \cdot e_{1,1}^2e_{1,k} = &\left(-\delta_{j,1}\widehat{\delta}_{i,1}\delta_{k,1} + \widehat{\delta}_{j,1}\delta_{i,j}\delta_{k,i}\right)e_{1,1}^2e_{1,i} + \delta_{j,1}\delta_{k,1}\widehat{\delta}_{i,1}e_{1,i} \\ &+ \delta_{i,j}\widehat{\delta}_{j,1}\left(\tfrac{1}{2}\delta_{k,1}\left(e_{1,1}^3 + e_{1,1}\right) + \widehat{\delta}_{k,i}\widehat{\delta}_{k,1}e_{1,1}^2e_{1,k}\right), \end{aligned} \tag{15}$$

$$e_{1,i}e_{j,1} \cdot e_{1,1}^4 = \delta_{j,1}e_{1,i}e_{1,1} + \widehat{\delta}_{j,1}\widehat{\delta}_{i,1}\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right), \tag{16}$$

$$e_{1,i}e_{j,1} \cdot e_{1,k}e_{1,1}e_{\ell,1} = -\delta_{j,k}e_{1,i}e_{1,1}e_{\ell,1}; \quad \ell \neq 1. \tag{17}$$

Proposition. Let $i, j, k \in \Omega$. Then in $\mathfrak{A}$, we have

$$\begin{aligned} e_{1,1}^2 e_{1,k} \cdot e_{i,j} &= \delta_{k,i}\left[\delta_{j,1}\left(\left(\delta_{k,1}+\tfrac{1}{2}\widehat{\delta}_{k,1}\right)e_{1,1}^4+\tfrac{1}{2}\widehat{\delta}_{k,1}e_{1,1}^2\right)+\widehat{\delta}_{j,1}e_{1,1}e_{1,j}\right] \\ &\quad - \widehat{\delta}_{k,i}\delta_{j,1}\delta_{k,1}e_{1,1}e_{i,1}, \qquad (18) \\ e_{1,1}^2 e_{1,k} \cdot e_{i,1}e_{1,j} &= \left[\delta_{k,i}\delta_{j,1}\left(\delta_{k,1}+\tfrac{1}{2}\widehat{\delta}_{k,1}\right)+\tfrac{1}{2}\widehat{\delta}_{k,i}\delta_{k,1}\widehat{\delta}_{i,1}\delta_{i,j}\right]e_{1,1} \\ &\quad + \tfrac{1}{2}\left(\delta_{k,i}\widehat{\delta}_{k,1}\delta_{j,1}-\widehat{\delta}_{k,i}\delta_{k,1}\widehat{\delta}_{i,1}\delta_{i,j}\right)e_{1,1}^3 \\ &\quad + \left(\delta_{k,1}\left(\delta_{k,i}\widehat{\delta}_{j,1}-\widehat{\delta}_{k,i}\delta_{i,1}\right)+\widehat{\delta}_{k,1}\widehat{\delta}_{j,1}\right)e_{1,1}^2 e_{1,j}, \qquad (19) \\ e_{1,1}^2 e_{1,k} \cdot e_{1,i}e_{j,1} &= \delta_{k,1}\left(-\delta_{i,1}e_{1,1}^2 e_{j,1}+\widehat{\delta}_{i,1}\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^3+e_{1,1}\right)\right);\ (i,j)\neq(1,1), \qquad (20) \\ e_{1,1}^2 e_{1,k} \cdot e_{1,j}e_{1,1}e_{\ell,1} &= \delta_{k,1}\delta_{j,1}e_{1,1}e_{\ell,1}; \quad \ell \neq 1, \qquad (21) \\ e_{1,1}^2 e_{1,k} \cdot e_{1,1}^2 e_{1,j} &= \delta_{k,1}e_{1,1}e_{1,j}, \qquad (22) \\ e_{1,1}^2 e_{1,j} \cdot e_{1,1}^4 &= \delta_{j,1}e_{1,1}^3. \qquad (23) \end{aligned}$$

The structure constants of $\mathfrak{A}$

Proposition. Let $i, j, k, \ell \in \Omega$ and $\ell \neq 1$. Then in $\mathfrak{A}$, we have

$$\begin{aligned} e_{1,k}e_{1,1}e_{\ell,1} \cdot e_{i,j} = \delta_{j,\ell}\Big[\tfrac{1}{2}\Big\{ &\left(\delta_{k,1}\widehat{\delta}_{j,1}\delta_{i,1} + \widehat{\delta}_{i,1}\delta_{i,k}\right) e_{1,1}^4 \\ &+ \left(\widehat{\delta}_{i,1}\delta_{i,k} - \delta_{k,1}\widehat{\delta}_{j,1}\delta_{i,1}\right) e_{1,1}^2 \Big\} - \widehat{\delta}_{i,1}\delta_{k,1}e_{1,1}e_{i,1} \\ &- \widehat{\delta}_{k,1}e_{1,k}e_{i,1}\Big], \end{aligned} \tag{24}$$

$$\begin{aligned} e_{1,k}e_{1,1}e_{\ell,1} \cdot e_{1,i}e_{j,1} = \delta_{i,\ell}\Big[&\delta_{k,1}\left(\delta_{j,1}\tfrac{1}{2}\left(e_{1,1} - e_{1,1}^3\right) - \widehat{\delta}_{j,1}e_{1,1}^2e_{j,1}\right) \\ &- \widehat{\delta}_{k,1}\left(\widehat{\delta}_{j,1}e_{1,k}e_{1,1}e_{j,1} + \delta_{j,1}\left(e_{1,1}^2e_{1,k} - e_{1,k}\right)\right)\Big]; \\ &(i,j) \neq (1,1), \end{aligned} \tag{25}$$

$$e_{1,i}e_{1,1}e_{j,1} \cdot e_{1,k}e_{1,1}e_{\ell,1} = \delta_{j,k}\left(e_{1,i}e_{\ell,1} - \widehat{\delta}_{i,1}\delta_{i,\ell}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right)\right); \quad j \neq 1, \tag{26}$$

$$e_{1,k}e_{1,1}e_{\ell,1} \cdot e_{i,1}e_{1,j} = 0, \quad e_{1,k}e_{1,1}e_{\ell,1} \cdot e_{1,1}^2e_{1,j} = 0, \quad e_{1,k}e_{1,1}e_{\ell,1} \cdot e_{1,1}^4 = 0. \tag{27}$$

Proposition. Let $i, j \in \Omega$. Then in $\mathfrak{A}$, we have

$$e_{1,1}^4 \cdot e_{i,j} = \delta_{i,1}\left(\delta_{j,1}e_{1,1} + \widehat{\delta}_{j,1}e_{1,1}^2 e_{1,j}\right) - \widehat{\delta}_{i,1}\delta_{j,1}e_{1,1}^2 e_{j,1}, \tag{28}$$

$$e_{1,1}^4 \cdot e_{1,i}e_{j,1} = \delta_{i,1}e_{1,1}e_{j,1} + \widehat{\delta}_{i,1}\widehat{\delta}_{j,1}\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^4 + e_{1,1}^2\right); \quad (i,j) \neq (1,1), \tag{29}$$

$$e_{1,1}^4 \cdot e_{i,1}e_{1,j} = \delta_{i,1}e_{1,1}e_{1,j} + \widehat{\delta}_{i,1}\delta_{i,j}\tfrac{1}{2}\left(e_{1,1}^2 - e_{1,1}^4\right), \tag{30}$$

$$e_{1,1}^4 \cdot e_{1,1}^2 e_{1,j} = e_{1,1}^2 e_{1,j}, \tag{31}$$

$$e_{1,1}^4 \cdot e_{1,i}e_{1,1}e_{j,1} = \delta_{i,1}e_{1,1}^2 e_{j,1}; \quad j \neq 1, \tag{32}$$

$$e_{1,1}^4 \cdot e_{1,1}^4 = e_{1,1}^4. \tag{33}$$

The center of $\mathfrak{A}$

Our next aim is to determine the center of $\mathfrak{A}$:
$Z(\mathfrak{A}) = \{z \in \mathfrak{A} \mid zu = uz, \text{ for all } u \in \mathfrak{A}\}$.

Theorem. The center $Z(\mathfrak{A})$ of the (unital) universal enveloping algebra $\mathfrak{A}$ has dimension 5 with basis:

$$z_1 = \tfrac{(n-2)}{n} e_{1,1}^2 - \tfrac{2}{n} \sum_{i=2}^{n} e_{1,i} e_{i,1} + e_{1,1}^4,$$

$$z_2 = (2-n) e_{1,1}^2 + \sum_{i=2}^{n} e_{1,i} e_{i,1} + \sum_{i=2}^{n} e_{i,1} e_{1,i},$$

$$z_3 = -\tfrac{1}{2} e_{1,1} + \tfrac{1}{2} e_{1,1}^3 + \sum_{i=2}^{n} e_{1,i} e_{1,1} e_{i,1},$$

$$z_4 = \sum_{i=1}^{n} e_{i,i}, \quad z_5 = 1.$$

Explicit decomposition of $\mathfrak{A}$

Theorem. The universal enveloping algebra $\mathfrak{A}$ of the anti-Jordan triple system $\mathfrak{J}$ can be decomposed as follows:

$$\mathfrak{A} = F \oplus M_{n,n}(F) \oplus M_{n,n}(F) \oplus M_{n,n}(F) \oplus M_{n,n}(F),$$

where $M_{n,n}$ is the ordinary associative algebra of all $n \times n$ matrices.

Corollary. The universal enveloping algebra of the simple anti-Jordan triple system of all $n \times n$ matrices over an algebraically closed field of characteristic 0 is semisimple.

Irreducible representations of the anti-Jordan triple system of $n \times n$ matrices

- There are 5 irreducible representations of the simple anti-Jordan triple system of $n \times n$ matrices:

$$1 = A_{1,1} + \sum_{i=1}^{n} B_{i,i}^{(0)} + \sum_{i=1}^{n} B_{i,i}^{(1)} + \sum_{i=1}^{n} D_{i,i}^{(0)} + \sum_{i=1}^{n} D_{i,i}^{(1)},$$

$$e_{1,1} = B_{1,1}^{(0)} - B_{1,1}^{(1)} - \mathrm{I}\, D_{1,1}^{(0)} + \mathrm{I}\, D_{1,1}^{(1)},$$

$$e_{i,j} = B_{i,j}^{(0)} - B_{i,j}^{(1)} - \mathrm{I}\, D_{j,i}^{(0)} + \mathrm{I}\, D_{j,i}^{(1)}; \quad i,j \neq 1, i \neq j,$$

$$e_{i,i} = B_{i,i}^{(0)} - B_{i,i}^{(1)} - \mathrm{I}\, D_{i,i}^{(0)} + \mathrm{I}\, D_{i,i}^{(1)}; \quad i \neq 1,$$

$$e_{1,i} = \tfrac{1}{2} B_{1,i}^{(0)} - \tfrac{1}{2} B_{1,i}^{(1)} - \mathrm{I}\, D_{i,1}^{(0)} + \mathrm{I}\, D_{i,1}^{(1)}; \quad i \neq 1,$$

$$e_{i,1} = 2B_{i,1}^{(0)} - 2B_{i,1}^{(1)} - \mathrm{I}\, D_{1,i}^{(0)} + \mathrm{I}\, D_{1,i}^{(1)}; \quad i \neq 1,$$

where $I = \sqrt{-1}$.

Conjecture

Conjecture. If the universal enveloping algebra of a simple finite dimensional anti-Jordan triple system is finite dimensional, then it is semisimple.

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Thank You