# *Invariant theory: differential vs. integral methods; the concept of observability*

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## *Abstract*

The purpose of these talks, is twofold:

1. To present a historical perspective of Invariant Theory comparing differential and integral methods. We start with the formulation of the *First and Second fundamental problem*, its inicial solution by Hilbert using differential methods for the case of binary forms. This was followed by the introduction of integral methods by Hurwitz and Weyl that solved completely the general problems in characteristic zero for the case of semisimple groups. Concerning positive characteristic we will talk about the results –along the 1960/70's– by Mumford, Nagata, Haboush and Popov (and many others) that gave a complete solution to the problems. At the end we will revisit the differential methods.
2. In the second part, time allowing, we will talk about the important concept of observable subgroup introduced in the 1960's with the purpose of studying extensions of representations from subgroups to groups. It can be considered as an intermediate step in the concept of reductivity and it has been recently generalized to the concept of observable action.

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It is very easy to see that in the case of finite groups, if $K \subset G$ is a group and a subgroup and $V$ is a representation of $K$, then $\mathbb{K}G \otimes_{\mathbb{K}H} V$ with its natural $G$–structure, is a $G$–module that contains $V$ as an $H$–submodule.

This extension problem, does not have a positive answer for other categories of groups –more about this in a minute–, and leads to the following definition for the category of algebraic groups.

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Let $G$ be an affine algebraic group and $K$ a closed subgroup. We say that $K$ is observable in $G$ if every finite dimensional rational $H$–module $V$ can be embedded as an $H$–submodule in a rational $G$–module.

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# Motivations

We write down the following definition:

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Assume that $K \subset G$ is an inclusion in the category of Lie groups –or finite, analytic, affine algebraic, etc. A representation $V$ of $K$ extends to a representation $W$ of $G$, if $V \subset W$ as $K$ modules.

A basic question many authors were trying to solve in the late 1950s is: Is it true that for an arbitrary pair of a group and a subgroup, every finite dimensional representation of $K$ admits a finite dimensional extension? If not, what restrictions one has to ask in order to guarantee that fact?

As I mentioned before in the case of finite groups, the problem has an obvious positive answer.

In the case of Lie groups the situation is much more akward, considering that the natural construction given by induction even if it were available, is not expected to produce a finite dimensional result.

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# The finite dimensionality for algebraic groups

We did not mention in the definition of observable subgroup, that the extension $W$ had to be finite dimensional, because it is unnecessary.

If we have an inclusion $V \subset W$ with $V$ a finite dimensional rational $K$–module and $W$ an arbitrary rational $G$–module, the $G$–submodule of $W$ generated by $V$ has to be also finite dimensional.

This is a consecuence of the very definition of rational representation.

If $G$ is an affine algebraic group and $W$ an arbitrary linear space, a rational linear action of $G$ on $W$ is a linear action satisfying the following additional properties: 1) If $w \in W$ then the orbit $G \cdot w$ generates a finite dimensional vector space. 2) Moreover, if we call $\{w_1, \cdots, w_n\}$ a basis of this space and write $x \cdot w = \sum_i f_i(x) w_i$ the functions $f_i : G \to \mathbb{K}$, are polynomials in $G$.

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# The case of Lie groups

Hochschild and Mostow in a couple of papers in 1957/58 only obtained positive answers for this situation under the strong hypothesis that $K$ is normal in $G$ and with additional restrictions. For example, one of their main result reads as follows. We write it down in detail to show the technicalities that appear in this situation.

Let $K \subset G$ be as above, with $K$ normal. Assume that there is an analytic subgroup $H$ of $G$ such that: $G = HK$, $H \cap K$ is compact and there is a finite dimensional representation of $H$ that is faithful on $H \cap K$.

Let $\rho$ be a representation of $K$, then $\rho$ can be extended to a representation of $G$ if and only if $\rho'([\mathrm{rad}(K), G]) = 1$, where the bracket represents the commutator subgroup, $\mathrm{rad}(K)$ is the radical of $K$ and $\rho'$ is the semisimple representation associated to $\rho$.

Even though the formulation of the theorem is rather restrictive, the authors show that it is very useful to simplify the proofs of some theorems concerning the existence of faithful representations due to E. Cartan, Gotô, Malcev, etc.

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# *From Lie groups to algebraic groups*

In the above result and in others, specially for the case of analytic groups, ideas and specific theorems from the theory of linear algebraic groups started to be systematically used.

In the case of an analytic group $G$ and a closed, normal subgroup $K$, theorems as the above started to be obtained. The main difference with the case of Lie groups was methodological as D. Mostow for example, started to use in a systematic way some of his own results in the theory of affine linear groups concerning linearly reductive subgroups and what is now called Mostow decomposition.

This decomposition theorem guarantees that a connected affine algebraic group in characteristic zero, is the semidirect product of its unipotent radical and a linearly reductive group.

Once the methods of the theory of affine algebraic groups were used to deal with the situation of analytic groups, it was clear that there were good perspectives that a reasonable theory of extensions could be developed in the algebraic environment.

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# *Observability and invariants*

The concept of observable subgroup has been extensively studied in connection with some important aspects of invariant theory.

There are many characterizations of observability, for example $K \subset G$ is observable, if and only if for an arbitrary rational character $\gamma$ of $K$, there is a $K$ semi–invariant polynomial $f$ on $G$ with respect to the character $\gamma$–i.e. for all $x \in K$, $x \cdot f = \gamma(x) f$–. Another characterization of observability has an important geometric content: $K \subset G$ is observable, if and only if the homogeneous space $G/K$ is a quasi–affine variety.

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# *Strong observability*

Let $G$ be an affine algebraic group and $K \subset G$ a closed subgroup. We say that $K$ is strongly observable (Cline, Parshall and Scott c. 1975) in $G$, if every finite dimensional rational $K$–module $V$, it can be emebedded in a rational $G$–module $W$ with the property that $V^K = W^G$. In other words when extending the representation the $K$–invariants become $G$–invariants.

The following theorem illustrates the relationship between strong observability and integrals –and consequently with a fundamental invariant theoretical tool–.

*Theorem*

*The subgroup $K$ is strongly observable in $G$ if and only if $K$ admits a total integral with values in $\mathbb{K}[G]$.*

Recall that an integral for an affine group $H$ is a linear map $\sigma : \mathbb{K}[H] \to \mathbb{K}$ that is invariant by the action of $H$ –i.e. $\sigma(x \cdot f) = \sigma(f)$ for $x \in H$ and $f \in \mathbb{K}[H]$–.

In the case that the $\sigma$ takes values in a rational $H$–module algebra $R$ and it is $H$–equivariant we say that it is an integral with values in $R$ and we say it is total if $\sigma(1) = 0$.

# *Strong observability*

Let $G$ be an affine algebraic group and $K \subset G$ a closed subgroup. We say that $K$ is strongly observable (Cline, Parshall and Scott c. 1975) in $G$, if every finite dimensional rational $K$–module $V$, it can be emebedded in a rational $G$–module $W$ with the property that $V^K = W^G$. In other words when extending the representation the $K$–invariants become $G$–invariants.

The following theorem illustrates the relationship between strong observability and integrals –and consequently with a fundamental invariant theoretical tool–.

*Theorem*

*The subgroup $K$ is strongly observable in $G$ if and only if $K$ admits a total integral with values in $\mathbb{K}[G]$.*

Recall that an integral for an affine group $H$ is a linear map $\sigma : \mathbb{K}[H] \to \mathbb{K}$ that is invariant by the action of $H$ –i.e. $\sigma(x \cdot f) = \sigma(f)$ for $x \in H$ and $f \in \mathbb{K}[H]$–.

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Let us say a few words about the proof of the above theorem.

Proof: Assume that $K$ is strongly observable and consider the $K$–module $\mathbb{K}[K]$. By the hypothesis of strong observability, one can find an inclusion $\mathbb{K}[K] \subset M$ where $M$ is a rational $G$–module and $\mathbb{K}[K]^K = \mathbb{K} = M^G$.

Take a linear functional $\alpha$ on $M$ such that $\alpha(1) = 1$ and define $f \mapsto \sigma(f) : \mathbb{K}[K] \to \mathbb{K}[G]$ as: $\sigma(f)(x) = \alpha(x \cdot f)$ for $x \in G$.

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## Observable subgroups

# From translations to general actions

# Motivation

Once the concept of observable subgroup was established, concept that deals with the case of an action of the subgroup $K$ acting on $G$ by *translations* and inducing a morphism $K \times G \to G$; it seemed natural to generalize it for the general situation where the affine algebraic group $K$ acts on a *general affine variety* $X$ by a rational map $K \times X \to X$. In other words, one wants to step up from the concept of observable subgrup to the concept of observable action of a group.

Even though at the end the generalization was not that hard, it took almost 50 years (from 1963 until 2009) to figure out the correct definition.

The first step was given when an equivalent definition of observability in the case of group subgroup –that made sense in a more general context– appeared in the mentioned book by A. Rittatore and WFS.

*Theorem (A.Rittatore, WFS; (2005))*

*Let $G$ be an affine algebraic group and $K$ a closed subgroup, then $K$ is observable in $G$ if and only if for every ideal $0 \neq I \subset \mathbb{K}[G]$ that is $K$–stable, then $I \cap K[G]^K \neq 0$.*

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*Definition (A. Rittatore, L. Renner; (2009))*

Let $K$ be an affine algebraic group and $X$ an affine variety endowed with a rational action $K \times X \to X$. We say that this action is observable if for an arbitrary, proper, closed and $K$–stable subvariety $Y \subset X$, there is an invariant function $0 \neq f \in \mathbb{K}[X]^K$ that is zero on $Y$. Equivalently, the action is observable if and only if for every ideal $0 \neq I \subset \mathbb{K}[X]$ that is $K$–stable, there is an invariant function $0 \neq f \in I \cap \mathbb{K}[X]^K$.

Unless explicitly mentioned, all the main results considered bellow are due to the authors mentioned above.

For simplicity, we assume that all the varieties are irreducible and all the affine algebraic groups are connected. Once this situation is understood, it is a technicality to go to the general situation.

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# First results on observable actions

The following result is well known for observable subgroups.

*Theorem (1963)*

*The subgroup $K$ of $G$ is observable if and only if every $K$–invariant rational function on $G$ is the quotient of two $K$–invariant polynomials. In symbolic terms that means that $[\mathbb{K}[G]]^K = [\mathbb{K}[G]^K]$.*

Sketch of proof: The inclusion $[\mathbb{K}[G]^K] \subset [\mathbb{K}[G]]^K$ is always true. Assume that $K$ is observable in $G$ and take an element $R \in [\mathbb{K}[G]]^K$. Consider the ideal of the numerators of $R$, or more explicitly $I_R = \mathbb{K}[G]R \cap \mathbb{K}[G]$. This is a non zero $K$–stable ideal that has a non zero fixed element $f_1$. We write then $f_1 = Rf_2$ and it is clear that we have in that manner obtained a representation of $R$ as a quotient of two $K$–invariant polynomials.

The proof of the converse, i.e. that $[\mathbb{K}[G]]^K = [\mathbb{K}[G]^K]$ implies that $K$ is observable in $G$ is more elaborate and can be found for example in the original 1963 paper.

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It is clear that the above proof can be generalized to observable actions. Hence we can guarantee that: If the action of $K$ on the affine variety $X$ is observable, then every $K$–invariant rational function on $X$ is the quotient of two $K$–invariant polynomials –i.e. $[\mathbb{K}[X]]^K = [\mathbb{K}[X]^K]$–.

The converse is not true in the case of arbitrary actions. We need one additional hypothesis that is invisible in the case of group subgroup because it is automatically verified. In that sense the following theorem was proved by Rittatore and Renner.

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*The action of $K$ on $X$ is observable if and only if every $K$–invariant rational function on $X$ is the quotient of two $K$–invariant polynomials and $X$ has an open set of closed orbits.*

In the case that the subgroup $K$ acts on $G$ by translations, all the orbits are closed as they are the right cosets of $K$ on $G$. So that the open set of closed orbits is the entire group $G$.

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It is clear that the above proof can be generalized to observable actions. Hence we can guarantee that: If the action of $K$ on the affine variety $X$ is observable, then every $K$–invariant rational function on $X$ is the quotient of two $K$–invariant polynomials –i.e. $[\mathbb{K}[X]]^K = [\mathbb{K}[X]^K]$–.

The converse is not true in the case of arbitrary actions. We need one additional hypothesis that is invisible in the case of group subgroup because it is automatically verified. In that sense the following theorem was proved by Rittatore and Renner.

*Theorem (2009)*

*The action of $K$ on $X$ is observable if and only if every $K$–invariant rational function on $X$ is the quotient of two $K$–invariant polynomials and $X$ has an open set of closed orbits.*

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# *Transitivity results*

There is a whole subtheme on invariant theory that has to do with transitivity results. One example is the following –goes under the name of Matsushima's criterion–.

Assume that $K \subset G$ is a closed subgroup of an affine algebraic group. If $G/K$ is affine and $G$ is reductive, then $K$ is reductive.

The above can be seen as a transitivity result if we read it as:

If $G$ is reductive and $K$ is reductive in $G$, then $K$ is reductive.

Rittatore and Renner in their 2009 paper, proved an interesting transitivity result. In order to explain it we need to describe an elementary geometric construction that generalizes the construction of the homogeneous space $G/K$.

Assume $K \subset G$ is a closed subgroup of the affine group $G$ and that $X$ is a rational affine $K$–variety. The induced variety $G \star_K X$ has also a natural $G$–action, and if $G/K$ is affine so is $G \star_K X$.

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# A transitivity result for observable actions

The authors formulated the following result on observability using geometric terminology.

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*Suppose that $K$ is a closed subgroup of $G$ with $G/K$ affine, and let $X$ be an affine variety equipped with an action of $K$. The induced $G$–action on the –affine– variety $G \star_K X$ is observable if and only if the original $K$ action on $X$ is observable.*

I will change it into an algebraic result. Assume that $G$ and $K$ are as above and that $R$ is a rational $K$–module algebra. Then the following two assertions are equivalent: (A) For every non trivial $K$–stable ideal $I \subset R$ then $I^K \neq 0$; (B) For every non trivial $G$–stable ideal $J \subset (\mathbb{K}[G] \otimes R)^K$ then $J^G \neq 0$.

First we notice that the fact that (B) implies (A) is valid assuming only that $K$ is observable in $G$. This condition is weaker than the one assumed by the authors as we have seen before that if $G/K$ is affine, then it is also quasi affine and hence $K$ is observable in $G$.

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