Modular representations of symmetric groups: An Overview

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The outline of my talk

- Ordinary Representations of S_n
- Combinatorics of tableaux
- Modular Representations of S_n
- Blocks
- Decomposition Numbers
- New Methods: Lie Theory

G is a finite group, K a field.

If K has characteristic 0, a homomorphism $\rho : G \to GL(n, K)$ is an ordinary representation of G. Equivalently, we have a KG-module of dimension n.

ρ is reducible if for a fixed T $T^{-1}\rho(g)T$ is of the form

$$\begin{pmatrix} \rho_1(g) & * \\ 0 & \rho_2(g) \end{pmatrix}$$ Or; The G-module has a proper submodule.

Otherwise, ρ is irreducible, G-module is simple.

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Important invariant of Representation Theory: Character Table, rows show characters, columns show classes

Important classical tool: Frobenius induction: Given a character of a subgroup H of G, get a character of G.

$$Ind_H^G : K_0(KH) \to K_0(KG)$$
$$(Ind_H^G(\psi))(g) = 1/|H| \sum_{x \in G} \psi(x^{-1}gx)$$

$K_0(KG)$ is the Grothendieck group, i.e. free abelian group with basis (isomorphism classes of) simple modules of G. If K has characteristic 0 can use characters for a basis.

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Richard Brauer developed the modular representation theory of finite groups, starting in the thirties.

G	a finite group
p	a prime integer
K	a sufficiently large field of characteristic 0
$\mathcal{O}$	a complete discrete valuation ring with quotient field K
k	residue field of $\mathcal{O}$, char k=p

A representation of G over K is equivalent to a representation over $\mathcal{O}$, and can then be reduced mod p to get a modular representation of G over k.

Brauer character of a modular representation: a complex-valued function on the p-regular elements (order prime to p) of G. Then we can compare ordinary and p-modular (Brauer) characters.

Number of irreducible Brauer characters = number of p-regular classes,

The decomposition matrix D is the transition matrix between ordinary and Brauer characters.

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Consider the algebras KG and kG. KG is semisimple, kG is not semisimple if p divides the order of G.

$$kG = B_1 \oplus B_2 \oplus \ldots \oplus B_n$$

where the B_i are "block algebras", indecomposable ideals of kG, reductions mod p of some ideals in $\mathcal{O}G$.

Each irreducible representation over K or k belongs to precisely one block.

At the character level:

χ_1, χ_2 are in the same block if there is a chain $\chi_1, \chi_{11}, \chi_{12} \ldots \chi_2$ such that any two consecutive characters have a Brauer character in common when reduced mod p.

Leads to:

- a partition of the ordinary characters, or KG-modules, into blocks
- a partition of the Brauer characters, or kG-modules, into blocks
- a partition of the decomposition matrix into blocks

Some main problems of modular representation theory:

- Describe the blocks as sets of characters, or as algebras
- Describe the irreducible modular representations, e.g. their degrees
- Find the decomposition matrix D, the transition matrix between ordinary and Brauer characters.
- Global to local: Describe information on the block B by "local information", i.e. from blocks of subgroups of the form $N_G(P)$, P a p-group

The symmetric group S_n is generated by the transpositions $\{(12),(23),\dots(n-1,n)\}$.

Partitions λ of n play a big role.

With each $\lambda \vdash n$, a Young subgroup H_λ, used to construct representations of S_n.

Example: $\lambda = \{5,4,2\}$, $H_\lambda = S_{\{1,2,3,4,5\}} \times S_{\{6,7,8,9\}} \times S_{\{10,11\}}$

Example: Character table of S_3.

order of element	1	2	3
classsize	1	3	2
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Example: Character table of S_4.

order of element	1	2	2	3	4
classsize	1	6	3	8	6
$\chi_{[4]}$	1	1	1	1	1
$\chi_{[31]}$	3	1	-1	0	-1
$\chi_{[2^2]}$	2	0	2	-1	0
$\chi_{[21^2]}$	3	-1	-1	0	-1
$\chi_{[1^4]}$	1	-1	1	1	-1

Character Table of S_5

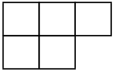
order of element	1	2	2	3	6	4	5
classsize	1	10	15	20	20	24	24
χ_5	1	1	1	1	1	1	1
χ_{1^5}	1	-1	1	1	-1	-1	1
χ_{41}	4	2	0	1	-1	0	-1
χ_{21^3}	4	-2	0	1	1	0	-1
χ_{32}	5	1	1	-1	1	-1	0
χ_{2^21}	5	-1	1	-1	-1	1	0
χ_{31^2}	6	0	-2	0	0	0	1

- Ordinary characters of S_n: parametrized by partitions of n
- Given partition λ , have $\chi_\lambda \in Irr(S_n)$
- λ has associated Young tableaux
- Hook length formula:

$$\chi_\lambda(1) = n!/\prod h_{ij}$$

Here h_{ij} is the hook length of node (i,j), i.e. from the node (i,j) go right and down.

Example:



Hook lengths are $\{4, 3, 1, 2, 1\}$, Dimension of $\chi_{3,2} = 120/4.3.2 = 5$

Dimension of $\chi_{3,2}$ = Number of Standard tableaux of shape $3, 2$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

Example of reduction mod 2: The irreducible representation of dimension 2 of S_3 becomes reducible mod 3, since $(123) \to \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$ where ω is a primitive cube root of unity, and $(12) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, becomes $(123) \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $(12) \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,mod 3.

Example: In S_3, $p = 2$, $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$

Module version: Specht modules indexed by partitions of n, defined over any field F. Denote by S_λ. S_λ has a unique irreducible quotient D_λ.

If F has characteristic 0 then $S_\lambda = D_\lambda$, these are all the simples up to isomorphism.

If F has characteristic p, take λ to be p-regular (each part appears less than p times). Then D_λ are all the simples up to isomorphism.

Decomposition matrix has entries $(S_\lambda : D_\mu)$.

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Example: S_5, $p = 2$: Brauer characters

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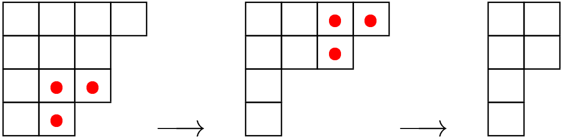
Decomposition matrix of S_5, $p = 2$:

$$
\begin{array}{c}
 \\ 5 \\ 41 \\ 32 \\ 31^2 \\ 2^2 1 \\ 21^3 \\ 1^5
\end{array}
\begin{array}{c}
\begin{array}{ccc} 5 & 41 & 32 \end{array} \\
\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
2 & 0 & 1 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array}\right)
\end{array}
$$

- Given λ , have $\chi_\lambda \in Irr(S_n)$
- p positive integer: Have p-hooks, p-core of λ.

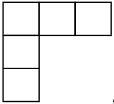
The p-core is obtained from the Young diagram of λ by removing as many p-hooks as you can.

Example: **Removing 3-hooks to get a 3-core**



Theorem (Brauer-Robinson) χ_λ, χ_μ are in the same p-block if and only if λ, μ have the same p-core.

Example: $S_5, p = 3$, blocks are $\{\chi_5, \chi_{2^21}, \chi_{21^3}\}$, $\{\chi_{41}, \chi_{32}, \chi_{1^5}\}$, $\{\chi_{31^2}\}$



, no 3-hooks, D with 3 block matrices along diagonal.

Summary: Blocks of S_n known.

Decomposition matrix not known in general, but shape is known.

Describe new methods, from Lie theory.

Recall: Irreducible KS_n-modules indexed by partitions of n

Partition of n is represented by a diagram with n nodes.

Induction (resp. restriction) from S_n to S_{n+1} (resp. S_n *to* S_{n-1}) is combinatorially represented by adding (resp. removing) a node from a partition.

residue r at the (i, j)-node of a diagram is defined as $r \equiv (j - i)(mod\ p)$.

Define r-induction (resp. r-restriction) from S_n to S_{n+1} (resp. $S_n\ to\ S_{n-1}$), for various r. Represent induction or restriction and cutting to a block.

GdeB.Robinson: Operators e_i and f_i, $0 \leqslant i \leqslant (p-1)$, which move only nodes with residue i. Then $\mathrm{Ind} = \sum_0^{l-1} f_i$ and $\mathrm{Res} = \sum_0^{l-1} e_i$.

Example of $0-\text{ind}$:

$$\begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 \\ \cline{1-3} 0 \\ \cline{1-1} \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|}\hline 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \cline{1-3} 0 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ \hline 0 \\ \cline{1-1} \end{array}$$

Example of $1-\text{ind}$:

$$\begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 \\ \cline{1-3} 0 \\ \cline{1-1} \end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 \\ \cline{1-3} 0 & 1 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 \\ \cline{1-3} 0 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array}$$

k a field of characteristic p.

$\mathcal{A} = \oplus_{n \geqslant 0} K_0(kS_n - \mathrm{mod})$

Then $\widehat{sl_p}$ acts on $\mathcal{A}$. Generators e_i, f_i of $U(\widehat{sl_p})$ act like i-induction and i-restriction defined above.

Generators h_i act by characters called *weights*.

The affine Weyl group acts on the set of weights.

Theorem (Lascoux, Leclerc and Thibon) The decomposition of $\mathcal{A}$ into p-blocks coincides with the decomposition into weight spaces for $\widehat{sl_p}$. Two blocks of symmetric groups have the same p-weight if and only if they are in the same orbit under the action of the affine Weyl group.

Recall: p-weight of a block of kS_n= number of hooks removed to get core

Here consider blocks as algebras over k.

If A is a k-algebra, $\mathcal{D}^b(A)$ is the bounded derived category of the category $A - mod$ of finitely generated A-modules. It is a triangulated category. Two algebras A and B are derived equivalent if $\mathcal{D}^b(A)$ and $\mathcal{D}^b(B)$ are equivalent as triangulated categories.

Theorem (Chuang-Rouquier) Two p-blocks of symmetric groups which have the same p-weight are derived equivalent.

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Dimension of $S_\lambda = f_\lambda$, number of standard Young tableaux.

Application to the Ballot Problem: (R.Stanley, Enumerative Combinatorics, Vol 2)

$\lambda_1, \lambda_2 \ldots$ are the parts of λ.

Ways in which n voters can vote sequentially for candidates $A_1, A_2, \ldots$ so that A_1 gets λ_1 votes, A_2 gets λ_2 votes,... and so that A_i never trails A_{i+1} if the votes are counted consecutively. How many such sequences are there?

Answer: the number f_λ of standard tableaux.

Example: 5 voters, 2 candidates: $\lambda = \{3, 2\}$

$\{11122, 11212, 11221, 12112, 12121\}$

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