Simple representations of simple Lie algebras remain Indecomposable restricted to some Abelian Subalgebras

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Motivations

This work was inspired by the papers of:

Douglas, Premat, de Guise and Repka.

In particular from the paper of Douglas and Premat:

A class of nonunitary, Finite Dimensional Representations of the Euclidian Lie Algebra $\mathfrak{e}(2)$ Communication in Algebra **35** (2007) 1433–1448

follows that there exist a 2–dimensional abelian Lie algebra such that how any $\mathfrak{sl}(3,\mathbb{C})$–module restricted to these algebra remains indecomposable.

This is true for any simple A_n–modules

P. Casati *Irreducible SL_{n+1}–Representations remain Indecomposable restricted to some Abelian Subalgebras* Journal of Lie Theory Volume **20** (2010) 393–407

Is this fact still true for the other simple Lie algebras?

The answer is yes!

For the simple Lie algebras C_2, G_2 this was proved by A. Premat in an unpublished paper.

Notations

Let $\mathfrak{g}$ be the simple Lie algebra of classical type of rank n. Let $\mathfrak{h}$ be its Cartan subalgebra, $\mathfrak{h}^*$ its complex dual, and let $\Delta = \Delta(\mathfrak{sl}(n, \mathbb{C}), \mathfrak{h}) \in \mathfrak{h}^*$ the corresponding set of roots. Then we may decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} |\ [H, X] = \alpha(H)X\ \forall H \in \mathfrak{h}\}$. Let $\Delta = \Delta^+ \cup -\Delta^+$ be further the decomposition of the root system in positive and negative roots and define

$$\begin{array}{ll} \mathfrak{n}^+ & = \sum_{\beta \in \Delta^+} \mathfrak{g}_\beta \\ \mathfrak{n}^+ & = \sum_{\beta \in -\Delta^+} \mathfrak{g}_\beta. \end{array}$$

We denote by $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ the corresponding set of simple roots and accordingly we fix a Chevalley basis of $\mathfrak{g}$: $X_\beta \in \mathfrak{g}_\beta$ and $Y_\beta \in \mathfrak{g}_{-\beta}$ for $\beta \in \Delta^+$, and $H_\alpha \in \mathfrak{h}$ for α simple.

The abelian Lie algebra $\mathfrak{a}_n$

Let $\Pi = \Pi_Y \cup \Pi_X$ the decomposition of the set of simple roots Π:

$$\begin{aligned} \Pi_Y &= \{\alpha_{2i+1}\}_{i=0,\ldots,\left[\frac{n}{2}\right]} \\ \Pi_X &= \{\alpha_{2i}\}_{i=1,\ldots,\left[\frac{n}{2}\right]} \end{aligned}$$

where $[x]$ denotes the integer part of x.

Then n–dimensional subalgebra $\mathfrak{a}_n$
spanned by $\{X_\alpha, Y_\beta\}_{\alpha\in\Pi_X,\beta\in\Pi_Y}$ is abelian.

Littelman Basis case A_n

For a dominat weight λ of $\mathfrak{g}$, let $V(\lambda)$ be the corresponding irreducible finite dimensional $\mathfrak{g}$–module of highest weight λ and $u_\lambda \in V(\lambda)$ be a heighest weight vector.

Denote by λ_i^j the weight of

$$Y^{(\mathbf{a}_\mathbf{n}^\mathbf{i})} u_\lambda = \left(Y_i^{(a_j^i)} \cdots Y_1^{(a_1^j)}\right)\left(\cdots\right)\left(Y_i^{(a_i^i)} Y_{i-1}^{(a_{i-1}^i)} \cdots Y_1^{(a_1^i)}\right)\left(\cdots\right)\left(Y_n^{(a_n^n)} \cdots Y_2^{(a_2^n)} Y_1^{(a_1^n)}\right) u_\lambda$$

$(Y_i = Y_{\alpha_i})$ and set

$\lambda_0^n := \lambda$, and $\lambda_0^{j-1} := \lambda_j^j$ for $i \leq j \leq n$. Then the elements of $V(\lambda)$

$$Y^{(\mathbf{a})} u_\lambda = Y_1^{(a_1^1)} \left(Y_2^{(a_2^2)} Y_1^{(a_1^2)}\right)\left(\cdots\right)\left(Y_i^{(a_i^i)} Y_{i-1}^{a_{(i-1)}^i)} \cdots Y_1^{(a_1^i)}\right)\left(\cdots\right)\left(Y_n^{(a_n^n)} \cdots Y_2^{a_2^n} Y_1^{a_1^n}\right) u_\lambda$$

with $\mathbf{a} \in \mathcal{S}$ such that

$$\begin{array}{llllll}
\lambda_0^n(H_1) \geq a_1^n & \lambda_1^n(H_2) \geq a_2^n & \ldots & \lambda_{i-1}^n(H_i) \geq a_i^n & \ldots & \lambda_{n-1}^n(H_n) \geq a_n^n \\
\ldots & \ldots & \ldots & \ldots & \ldots & \\
\lambda_0^j(H_1) \geq a_1^j & \ldots & \ldots & \lambda_{i-1}^j(H_j) \geq a_j^j & & \\
\ldots & \ldots & \ldots & & & \\
\lambda_0^2(H_1) \geq a_1^2 & \lambda_1^2(H_2) \geq a_2^2 & & & & \\
 & & & & & \\
\lambda_0^1(H_1) \geq a_1^1 & & & & &
\end{array}$$

form a basis $\mathfrak{L}_\lambda$ of $V(\lambda)$.

Littelman Type Basis for C_n (B_n)

For a dominat weight λ of $\mathfrak{g}$, let $V(\lambda)$ be the corresponding irreducible finite dimensional $\mathfrak{g}$–module of highest weight λ and $u_\lambda \in V(\lambda)$ be a heighest weight vector.

Denote by λ_i^j the weight of

$$Y^{(\mathbf{a}_\mathbf{n}^\mathbf{i})}u_\lambda = \left(Y_i^{(a_j^i)} \cdots Y_n^{(a_n^j)}\right)\left(\cdots\right)\left(Y_1^{(a_1^i)} Y_2^{(a_2^i)} \cdots Y_n^{(a_n^i)}\right)\left(\cdots\right)\left(Y_1^{(a_1^n)} \cdots Y_2^{(a_{n-1}^n)} Y_1^{(a_n^n)}\right) u_\lambda$$

($Y_i = Y_{\alpha_i}$) and set

$\lambda_0^n := \lambda$, and $\lambda_0^{j-1} := \lambda_j^j$ for $i \le j \le n$. Then the elements of $V(\lambda)$

$$Y^{(\mathbf{a})}u_\lambda = \left(Y_1^{(a_1^1)} Y_2^{(a_2^2)} \cdots Y_n^{(a_n^2)}\right)\left(\cdots\right)\left(Y_1^{(a_1^i)} Y_2^{a_{(2}^i)} \cdots Y_n^{(a_n^i)}\right)\left(\cdots\right)\left(Y_1^{(a_1^n)} \cdots Y_{n-1}^{a_{n-1}^n} Y_1^{a_n^n}\right) u_\lambda$$

with $\mathbf{a} \in \mathcal{S}$ such that

$$\begin{array}{lllll}
\lambda_0^n(H_n) \ge a_n^n & \cdots & \lambda_{i-1}^n(H_i) \ge a_i^n & \cdots & \lambda_0^{n-1}(H_n) \ge a_n^{n-1} \\
\lambda_1^{n-1}(H_1) \ge a_1^{n-1} & \cdots & \lambda_{i-1}^n(H_i) \ge a_i^n & \cdots & \lambda_0^{n-2}(H_2) \ge a_n^{n-1} \\
\cdots & \cdots & \cdots & \cdots & \\
\lambda_j^{n-j}(H_j) \ge a_j^{n-j} & \cdots & \lambda_{i-1}^j(H_1) \ge a_1^j & & \\
\cdots & \cdots & & & \\
\lambda_{n-1}^1(H_{n-1}) \ge a_{n-1}^1 & & & &
\end{array}$$

form a basis $\mathfrak{L}_\lambda$ of $V(\lambda)$.

Example C_2

Let $\lambda \in \mathfrak{h}^*$ such that $\lambda(\alpha_1) = \lambda_1$, $\lambda(\alpha_2) = \lambda_2$, then the basis for $V(\lambda)$ is:

$$Y_1^{a_1^1} Y_2^{a_2^1} Y_1^{a_1^2} Y_2^{a_2^2} u_\lambda$$

with

$$\begin{aligned} a_2^2 &\le \lambda_2 \\ a_1^2 &\le \lambda_1 + 2a_2^2 \\ a_2^1 &\le \lambda_2 - 2a_2^2 + a_1^2 \\ a_1^1 &\le \lambda_1 + 2a_2^2 - 2a_1^2 + 2a_2^1 \end{aligned}$$

and

$$0 \le 2a_2^2 \le a_1^2 \le 2a_2^1 \quad a_1^1 \ge 0$$

The Set of Generators

Let V be a $\mathfrak{a}_n$–modules, a subset of elements $\{v_1, \dots v_m\}$ in V is said to be a set of generators of V if $V = \mathcal{U}(\mathfrak{a}_n)\{v_1, \dots v_m\}$. The set is called a minimal set of generators if fewer than m vectors will not generate V. In the case of the $\mathfrak{a}_n$–modules $V(\lambda)$ a set of generators $\mathfrak{W}$ is a set of homogeneous generators if any element in $\mathfrak{W}$ is a $\mathfrak{g}$–weight vector.

There exist a (complicate) subset of $\mathfrak{L}_\lambda$ which is a set of minimal generators of the $\mathfrak{a}_n$–modules $V(\lambda)$.

Case A_n

Theorem Let $\mathcal{G}_\lambda$ be the subset of $\mathcal{L}_\lambda = \{\mathbf{a} \in \mathcal{S} |\ Y^{(\mathbf{a})} u_\lambda \in \mathfrak{L}_\lambda\}$ given by:

$$\mathcal{G}_\lambda = \left\{ \mathbf{g} \in \mathcal{L}_\lambda \;\middle|\; \begin{array}{l} a_{2j}^{2j} = \lambda_{2j-1}^{2j}(H_{2j}) \\ a_{2j}^{2j+1} \neq 0 \Rightarrow a_{2j-1}^{2j} \neq 0 \\ a_1^{2j+1} = 0 \\ \lambda_{2i-1}^{2j+1}(H_{2i}) \neq 0 \text{ and } \lambda_0^{2j+1}(H_1) = 0,\ \lambda_{2r-1}^{2j+1}(H_{2r}) = a_{2r}^{2j+1}, \\ \Rightarrow a_{2i-1}^{2j} = \lambda_{2i-2}^{2j}(H_{2i-1}) \end{array} \right\}$$

then the corresponding subset $\mathfrak{G}_\lambda = \{Y^{(\mathbf{a})} u_\lambda |\ \ \mathbf{a} \in \mathcal{G}_\lambda\}$ of the Littelmann basis $\mathfrak{L}_\lambda$ is a set of homogeneous $\mathfrak{a}_n$–generators of $V(\lambda)$.

The Properties of the set $\mathfrak{G}_\lambda$

Lemma No proper subset $\mathfrak{G}'$ of $\mathfrak{G}_\lambda$ ($\mathfrak{G}' \subsetneq \mathfrak{G}$) generates $\mathfrak{G}_\lambda$.

Theorem The set $\mathfrak{G}_\lambda$ is a minimal set of generators.

Proposition Any set $\mathfrak{W} = \{w_1, \dots w_k\}$ of homogeneous generators contains an element $w_{\overline{g}}$ such that:

$$w_{\overline{g}} = a_{\overline{g}}\overline{g} \qquad a_{\overline{g}} \neq 0 \in \mathbb{C}.$$

where $\overline{g}$ is the element of the set of generators $\mathfrak{G}_\lambda$ given by

$$\overline{g} = Y_{2[\frac{n}{2}]}^{\left(\lambda_{2[\frac{n}{2}]-1}^{2[\frac{n}{2}]}\left(H_{[\frac{n}{2}]}\right)\right)} \dots Y_{2j}^{(\lambda_{2j-1}^{2j}(H_{2j}))} \dots Y_2^{(\lambda_1^2(H_2))} u_\lambda.$$

Lemma Let $\mathfrak{W} = \{w_1, \dots w_k\}$ be a set (non necessarily minimal) of generators then there exists a $w_{\overline{k}}$ in $\mathfrak{W}$ such that

$$w_{\overline{k}} = \overline{a}\,\overline{g} + \sum_{g \in \mathfrak{G}_\lambda, g \neq \overline{g}} P_{lg}(X_{2j}, Y_{2j+1}) Y^{(\mathbf{a_g})} u_\lambda.$$

The Lie algebra $\mathfrak{s}_n$

Let $\mathfrak{s}_n = \mathfrak{h} \rtimes \mathfrak{a}_n$ be the subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ given by the semidirect product between the Cartan subalgebra $\mathfrak{h}$ and the subalgebra $\mathfrak{a}_n$. The $\mathfrak{sl}(n+1, \mathbb{C})$–module ($\mathfrak{a}_n$–module) $V(\lambda)$ is also a $\mathfrak{s}_n$–module, on which the subalgebra $\mathfrak{h}$ acts diagonally. Obviously any set of generators of the $\mathfrak{a}_n$–module $V(\lambda)$ is also a set of generators of the $\mathfrak{s}_n$–module $V(\lambda)$. Moreover for what said above any $\mathfrak{s}_n$–submodule of $V(\lambda)$ is a $\mathfrak{sl}(n+1, \mathbb{C})$–weight module, i.e., it can decomposed as a direct sum of $\mathfrak{sl}(n+1, \mathbb{C})$–weight spaces.

Proposition If the $\mathfrak{s}_n$–module $V(\lambda)$ decomposes in a direct sum of two subspaces: $V(\lambda) = U \oplus T$, then $\overline{g}$ belongs either to U or to T.

The Main Theorem

Theorem The $\mathfrak{a}_n$–module $V(\lambda)$ is indecomposable.

Hint of the Proof

Use the theory of the irreducible finite dimensional $\mathfrak{sl}(n, \mathbb{C})$–modules together with an induction argument over the lenght of the elements of the Littelman basis.