The classification of uniserial $\mathfrak{sl}(2) \ltimes V(m)$-modules and a new interpretation of the Racah-Wigner $6j$-symbol

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$\mathfrak{sl}(2)$: Lie algebra over $\mathbb{C}$ with basis E, H, F, such that:

$$[H,E]=2E, \quad [H,F]=-2F, \quad [E,f]=H.$$

$V(m)$: irred $\mathfrak{sl}(2)$-module highest weight m, with basis $\{A_0,\dots,A_m\}$, considered as an abelian Lie algebra of dimension $m+1$.

$\mathfrak{g}_m = \mathfrak{sl}(2) \ltimes V(m)$ is a perfect Lie algebra ($m>0$),

$[H,A_i]=(m-2i)A_i, \quad [E,A_i]=(m-i+1)A_{i-1}, \quad [F,A_i]=(i+1)A_{i+1}.$

$\mathfrak{s}=\mathfrak{sl}(2)$, $\mathfrak{r}=V(m)$, and a faithful representation of $\mathfrak{g}_m$ (for $m=4$) is

$$X \mapsto \left(\begin{array}{ccccc|c} 4h & 4e & 0 & 0 & 0 & a_0 \\ f & 2h & 3e & 0 & 0 & a_1 \\ 0 & 2f & 0 & 2e & 0 & a_2 \\ 0 & 0 & 3f & -2h & e & a_3 \\ 0 & 0 & 0 & 4f & -4h & a_4 \\ \hline & & & & & 0 \end{array}\right),$$

for $X = e\,E + h\,H + f\,F + \sum_{i=0}^{m} a_i A_i \in \mathfrak{g}_m$.

Other representation of $\mathfrak{g}_m = \mathfrak{sl}(2) \ltimes V(2)$, $(m = 2)$:

$$
\begin{array}{cc|cccc|cccccc}
h & e & a_2 & -2a_1 & a_0 & 0 & & & & & & \\
f & -h & 0 & a_2 & -2a_1 & a_0 & & & & & & \\
\hline
 & & 3h & 3e & 0 & 0 & a_2 & -2a_1 & a_0 & 0 & 0 & 0 \\
 & & f & h & 2e & 0 & 0 & a_2 & -2a_1 & a_0 & 0 & 0 \\
 & & 0 & 2f & -h & e & 0 & 0 & a_2 & -2a_1 & a_0 & 0 \\
 & & 0 & 0 & 3f & -3h & 0 & 0 & 0 & a_2 & -2a_1 & a_0 \\
\hline
 & & & & & & 5h & 5e & 0 & 0 & 0 & 0 \\
 & & & & & & f & 3h & 4e & 0 & 0 & 0 \\
 & & & & & & 0 & 2f & h & 3e & 0 & 0 \\
 & & & & & & 0 & 0 & 3f & -h & 2e & 0 \\
 & & & & & & 0 & 0 & 0 & 4f & -3h & e \\
 & & & & & & 0 & 0 & 0 & 0 & 5f & -5h
\end{array}
$$

Here $V \simeq V(1) \oplus V(3) \oplus V(5)$ as $\mathfrak{sl}(2)$-modules.

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Key facts:

(1) $\mathfrak{r} = V(2) \hookrightarrow V(2)_1 \subset \mathrm{Hom}(V(3), V(1)) \simeq V(1) \otimes V(3)$,

(2) $\mathfrak{r} = V(2) \hookrightarrow V(2)_2 \subset \mathrm{Hom}(V(5), V(3)) \simeq V(3) \otimes V(5)$

(3) $[V(2)_1, V(2)_2] = 0$.

In this case, (3) follows from the fact:

$\Lambda^2\mathfrak{r}$ has no common factors with $\mathrm{Hom}(V(5), V(1)) \simeq V(1) \otimes V(5)$

Again $\mathfrak{g}_m = \mathfrak{sl}(2) \ltimes V(2)$, $(m = 2)$, now acting on V^*:

$V^* \simeq V(5) \oplus V(3) \oplus V(1)$ as $\mathfrak{sl}(2)$-modules.

$$
\left.\begin{array}{cccccc|cccc|cc}
5h & 5e & 0 & 0 & 0 & 0 & 10v_0 & 0 & 0 & 0 & & \\
f & 3h & 4e & 0 & 0 & 0 & 4v_1 & 6v_0 & 0 & 0 & & \\
0 & 2f & h & 3e & 0 & 0 & v_2 & 6v_1 & 3v_0 & 0 & & \\
0 & 0 & 3f & -h & 2e & 0 & 0 & 3v_2 & 6v_1 & v_0 & & \\
0 & 0 & 0 & 4f & -3h & e & 0 & 0 & 6v_2 & 4v_1 & & \\
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Recall that $V(k) \simeq V(k)^*$ as $\mathfrak{sl}(2)$-modules for all k.

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Uniserial module: a $\mathfrak{g}$-module V is uniserial if

- it has a unique composition series (i.e. submodules for a chain), or
- $\mathrm{soc}^i(V)/\mathrm{soc}^{i-1}(V)$ is irreducible for all i.

Our results (Lie algebras and reps are finite-dim over $\mathbb{C}$).

Theorem

Assume $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ *perfect,* $\mathfrak{r}$ *abelian,* $V_1, \ldots, V_n$ *irreducible* $\mathfrak{s}$*-modules.*

Then $V = V_1 \oplus \cdots \oplus V_n$ *admits a structure of uniserial* $\mathfrak{g}$*-module iff*

(1) *there exist non-zero* $\mathfrak{s}$*-homomorphisms* $f_{i+1,i} : \mathfrak{r} \to V_{i+1}^{*} \otimes V_i$,

(2) $\langle \{ f_{2,1}(X) + \cdots + f_{n,n-1}(X) : X \in \mathfrak{r} \} \rangle_{Lie} \subset \mathfrak{gl}(V)$ *is abelian.*

Item (2) is granted if $\Lambda^2 \mathfrak{r}$ *is* $\mathfrak{s}$*-disjoint with* $V_{i+2}^{*} \otimes V_i$ *for all* i*.*

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$V(\nu)$: is the irred $\mathfrak{s}$-module of highest weight ν.

Theorem

Let $\mathfrak{g} = \mathfrak{s} \ltimes V(\mu)$, μ a dominant weight of $\mathfrak{s}$.
Let λ be a dominant weight of $\mathfrak{s}$ and $b \in \mathbb{N}_0$.
Then $V = V(\lambda) \oplus V(\lambda + \mu^*) \oplus \cdots \oplus V(\lambda + b\mu^*)$ admits a unique structure of uniserial $\mathfrak{g}$-module, say $Z(\lambda, b)$. The dual $Z(\lambda, b)^*$ has socle factors $V(\lambda^* + b\mu) \oplus \cdots \oplus V(\lambda^* + \mu) \oplus V(\lambda^*)$.

Classification of the uniserial reps of $\mathfrak{g}_m = \mathfrak{sl}(2) \ltimes V(m)$

Clebsch-Gordan for $\mathfrak{sl}(2)$ leads to the classif. uniserial rep of length 2:

$$V(m) \hookrightarrow \mathrm{Hom}(V(b), V(a)) \simeq V(a) \otimes V(b)$$

if and only if $a + b \equiv m \mod 2$ and $0 \leq |a - b| \leq m \leq a + b$.

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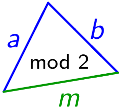
Thus $V(a) \oplus V(b)$ admits a structure of uniserial $\mathfrak{g}_m$-module if and only if $\{a, b, m\}$ is a *'triangular'* set:



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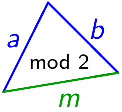

Key step: To determine the uniserial reps of length 3.

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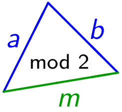

Key step: To determine the uniserial reps of length 3.

Lemma. An $\mathfrak{sl}(2)$-module $V(a) \oplus V(b) \oplus V(c)$ admits a structure of uniserial $\mathfrak{g}_m$-module if and only if

- $\{a, b, m\}$ and $\{b, c, m\}$ are triangular and
- $\{a, c, k\}$ is not triangular for all k such that $V(k) \hookrightarrow \Lambda^2 V(m)$ (this is $k = 2m - 2, 2m - 6, \ldots$).

In this case, the structure is unique.

Theorem

Let $\mathfrak{g}_m = \mathfrak{sl}(2) \ltimes V(m)$, where $m \geq 1$. Then the only $\mathfrak{sl}(2)$-modules admitting a structure of uniserial $\mathfrak{g}_m$-module are (apart from duals)

$$Z(\ell, b) = V(\ell) \oplus V(\ell + m) \oplus \cdots \oplus V(\ell + bm)$$

with the following exceptions:

Length 2. $V(\ell_1) \oplus V(\ell_2)$, *with* $\{\ell_1, \ell_2, m\}$ *triangular.*

Length 3. $V(0) \oplus V(m) \oplus V(\ell)$, *where* $\ell \equiv 2m \mod 4$ *and* $\ell \leq 2m$.

Length 4. $V(0) \oplus V(m) \oplus V(m) \oplus V(0)$, *in case that* $m \equiv 0 \mod 4$.

In all cases the structure of uniserial $\mathfrak{g}_m$-module is unique, except for $V(0) \oplus V(m) \oplus V(m) \oplus V(0)$, $m \equiv 0 \mod 4$; in this case the isomorphism classes are parametrized by the complex numbers.

Example. The one parameter family, parametrized by $z \in \mathbb{C}$, of non-isomorphic uniserial $\mathfrak{sl}(2) \ltimes V(4)$-modules with socle factors $V(0)$, $V(4)$, $V(4)$, $V(0)$ is given by:

$$
\left(\begin{array}{c|ccccc|ccccc|c}
0 & v_4 & -4v_3 & 6v_2 & -4v_1 & v_0 & & & & & & \\
\hline
 & 4h & 4e & 0 & 0 & 0 & 6v_2 & -12v_1 & 6v_0 & 0 & 0 & z\,v_0 \\
 & f & 2h & 3e & 0 & 0 & 3v_3 & -3v_2 & -3v_1 & 3v_0 & 0 & z\,v_1 \\
 & 0 & 2f & 0 & 2e & 0 & v_4 & 2v_3 & -6v_2 & 2v_1 & v_0 & z\,v_2 \\
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\hline
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 & & & & & & 0 & 0 & 3f & -2h & e & v_3 \\
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\hline
 & & & & & & & & & & & 0
\end{array}\right).
$$

Ingredients of the proof.

Assume that $\{a, b, p\}$ and $\{b, c, q\}$ are triangular. Let

$$f_1 : V(p) \hookrightarrow V(a) \otimes V(b)$$

$$f_2 : V(q) \hookrightarrow V(b) \otimes V(c)$$

$$A = \left\{ \begin{pmatrix} 0 & f_1(r_1) & 0 \\ 0 & 0 & f_2(r_2) \\ 0 & 0 & 0 \end{pmatrix} \middle| \begin{matrix} r_1 \in V(p) \\ r_2 \in V(q) \end{matrix} \right\} \subset End\big(V(a) \oplus V(b) \oplus V(c)\big)$$

What's the $\mathfrak{sl}(2)$-module structure of A (or, of the block $(1, 3)$ of A)?

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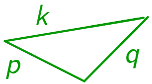

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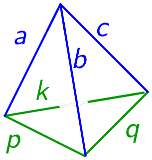

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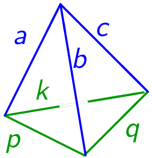

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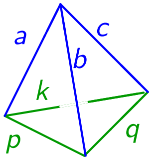

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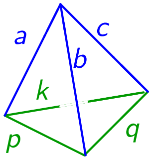

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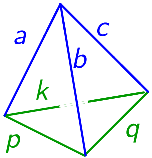

Answer: $h = \begin{Bmatrix} \frac{q}{2} & \frac{k}{2} & \frac{p}{2} \\ \frac{a}{2} & \frac{b}{2} & \frac{c}{2} \end{Bmatrix} \times$ canonical inclusion. (This is the $6j$-symbol)

If $\{x, y, z\}$ is triangular, let $\iota_z^{x,y} : V(z) \hookrightarrow V(x) \otimes V(y)$

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Implicit definition of the Racah-Wigner 6j-symbol:

– a real number $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$ associated to six half-integers $j_i \in \frac{1}{2}\mathbb{Z}_{\geq 0}$

– defined by the *transition matrix* between the following two basis of

$$\mathrm{Hom}_{\mathfrak{sl}(2)}\Big(V(2j_5)\ ,\ V(2j_1) \otimes V(2j_2) \otimes V(2j_4) \Big)$$

$$B_1 = \Big\{ V(2j_5) \hookrightarrow V(2j_3) \otimes V(2j_4) \hookrightarrow \Big(V(2j_1) \otimes V(2j_2) \Big) \otimes V(2j_4) \Big\}_{j_3 \geq 0}$$

$$B_2 = \Big\{ V(2j_5) \hookrightarrow V(2j_1) \otimes V(2j_6) \hookrightarrow V(2j_1) \otimes \Big(V(2j_2) \otimes V(2j_4) \Big) \Big\}_{j_6 \geq 0}$$

$$\left(\iota_{2j_3}^{2j_1, 2j_2} \otimes 1\right) \circ \iota_{2j_5}^{2j_3, 2j_4} = \sum_{j_6} \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \left(1 \otimes \iota_{2j_6}^{2j_2, 2j_4}\right) \circ \iota_{2j_5}^{2j_1, 2j_6}.$$