On the Chevalley Property of a Finite Dimensional Group Algebra and Its Drinfeld Double

Gerhard Hiss
Joint work with Hui-Xiang Chen

RWTH Aachen University

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Theorem (Chevalley, $\leq$ 1968)

If $\operatorname{char}(k) = 0$*, then kG has the Chevalley property.*

Theorem (Molnar, 1981)

Suppose that $\dim(H) < \infty$*. Then H has the Chevalley property if and only if* $J(H)$ *is a Hopf ideal.*

Theorem (Various authors)

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Thus: $V \otimes V$ is semisimple for each simple $V \in kG$-mod.

But: kG does not have the Chevalley property since G does not have a normal Sylow 3-subgroup.

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If $M \in D(kG)$-mod is labelled by (V, g_i), then

$$\dim(M) = |G \colon C_G(g_i)| \dim(V).$$

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(6) *$G = S \times K$ with S an abelian Sylow p-subgroup of G.*

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If $p \mid \dim(V)$, then $V \otimes V^*$ is not semisimple (Exercise).

Okuyama's argument

Proposition (Okuyama)

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There is $1 \neq x \in \langle t, t^g \rangle$ with $|x|$ odd and $t^{-1}xt = x^{-1}$, a contradiction.

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Lemma

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The above imply $J(D(kG)) \cong J(D(kS)) \otimes D(kK)$.

The latter is a Hopf ideal of $D(kG)$.

Thank you for your attention!