Extensions of tensor categories

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Representation Theory of Groups, Lie Algebras, and Hopf Algebras

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Outline

Exact sequences of tensor categories
- Definitions
- Examples

Exact sequences and Hopf monads
- Classification of exact sequences
- Applications

Tensor categories

Let k be a field.

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- Hom spaces are finite dimensional.
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A tensor functor preserves duals. Moreover it is automatically faithful.

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- Suppose $\mathcal{C}, \mathcal{D}$ are fusion categories. F is normal iff $\forall$ simple object X, $\mathrm{Hom}(\Bbbk, F(X)) \neq 0 \Rightarrow X \in \mathfrak{Ker}_F$.

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$$\mathfrak{Ker}_F \to \mathcal{C} \xrightarrow{F} \mathcal{C}''.$$

- *Suppose $\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$ exact $\Rightarrow \exists$ a Hopf algebra H st:*

$$\mathcal{C}' \simeq \text{comod-}H.$$

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where $\mathcal{C}(G)$: finite dimensional G-graded vector spaces over k.

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A *G-equivariant morphism* $f : (X, u) \to (Y, v)$ is a morphism $f : X \to Y$ st: $fu_g = v_g f, \forall g \in G$.

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Equivariantization gives rise canonically to an exact sequence

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If so, then $\forall$ *simple object* $Y \in \mathcal{C}''$*,*

$$\operatorname{FPdim} Y = \frac{1}{\operatorname{FPdim} \mathcal{C}'} \sum_{X \in \operatorname{Irr}(\mathcal{C})} m_Y(F(X)) \operatorname{FPdim} X,$$

where $m_Y(F(X)) = \dim \operatorname{Hom}_{\mathcal{C}''}(Y, F(X))$*.*

Hopf monads and their modules

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$\mathcal{C}^T$: T-modules in $\mathcal{C}$. Objects: (X, r), $X \in \mathcal{C}$, $r : T(X) \to X$, st:

$$rT(r) = r\mu_X, \qquad r\eta_X = \mathrm{id}_X .$$

Consider the forgetful functor $\mathcal{U} : \mathcal{C}^T \to \mathcal{C}$.

A monad T is a *bimonad* iff $\mathcal{C}^T$ is a monoidal category st $\mathcal{U}$ is strict monoidal.

Equivalently, T is a *comonoidal endofunctor*:

$$T_2(X, Y) : T(X \otimes Y) \to T(X) \otimes T(Y), \quad T_0 : T(\mathbb{1}) \to \mathbb{1},$$

$\forall X, Y \in \mathcal{C}$, satisfying:

$$(T_2(X,Y)\otimes \mathrm{id}_{T(Z)})T_2(X\otimes Y,Z) = (\mathrm{id}_{T(X)}\otimes T_2(Y,Z))T_2(X,Y\otimes Z),$$

$$(\mathrm{id}_{T(X)}\otimes T_0)T_2(X,\mathbb{1}) = \mathrm{id}_{T(X)} = (T_0\otimes \mathrm{id}_{T(X)})T_2(\mathbb{1},X),$$

and st μ, η are monoidal transformations, that is:

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Suppose $\mathcal{C}$ is rigid.

A bimonad T on $\mathcal{C}$ is a *Hopf monad* if $\mathcal{C}^T$ is rigid.

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Therefore in this case F is monadic.

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- The induced Hopf algebra of T is k^{G}.

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In fact, conjugation under $\gamma \in G' = G/G''$ defines an action of G' on $\mathrm{rep}\, G''$ by tensor autoequivalences.

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Theorem

Suppose $\mathrm{char}\, k = 0$*. Let* $\mathcal{C}, \mathcal{D}$ *be fusion categories over* k *and let* $F : \mathcal{C} \to \mathcal{D}$ *be a dominant tensor functor. If* $\mathrm{FPind}(\mathcal{C} : \mathcal{D}) = 2$*, then* F *is an equivariantization.*

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Theorem
Suppose $\mathrm{char}\, k = 0$*. Let* $\mathcal{C}, \mathcal{D}$ *be fusion categories over* k *st* $\mathcal{C}$ *is weakly integral. Let* $F : \mathcal{C} \to \mathcal{D}$ *be a dominant tensor functor. If* $\mathrm{FPind}(\mathcal{C} : \mathcal{D}) = p$*, where* p *is the smallest prime divisor of* $\mathrm{FPdim}\,\mathcal{C}$*, then* F *is an equivariantization.*

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