Graded Representations of GL(n,q)

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$G_n = GL(n, q)$, K a sufficiently large field of characteristic 0.

Unipotent characters of G_n are constituents of $M = \mathrm{Ind}_B^{G_n}(1)$ (B a Borel) and are indexed by partitions of n.

$\mathrm{End}_{KG_n}(M)$ is isomorphic to H_n, Hecke algebra of type A

The Hecke algebra H_n of type A over a field F, $q \in F$, has generators $\{T_1, T_2, \ldots T_{n-1}\}$ and relations:

$T_i^2 = (q-1)T_i + q.1,$

$T_iT_j = T_jT_i, 1 \leqslant i < j-1 \leqslant n-2,$

$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, i = 1, 2, \ldots n-2.$

G a finite group

Brauer theory: $K, \mathcal{O}, k$, ℓ-modular system, defined as

ℓ	a prime integer
K	a sufficiently large field of characteristic 0
$\mathcal{O}$	a complete discrete valuation ring with quotient field K
k	residue field of $\mathcal{O}$, $\mathrm{char} k = \ell$

A representation of G over K is equivalent to a representation over $\mathcal{O}$, can be reduced mod ℓ to get a modular representation of G over k.

Can compare ordinary and ℓ-modular (Brauer) characters. The decomposition matrix D (over $\mathbf{Z}$) is the transition matrix between ordinary and Brauer characters. Entries of D are decomposition numbers.

The matrix D for G_n when ℓ does not divide q is our main object of study.

Describe the ℓ-modular decomposition matrix of G_n.

Work done on blocks and decomposition matrices of finite reductive groups: Dipper-James, Geck, Gruber, Hiss, Kessar, Malle .. by modular Harish-Chandra theory.

Three objects: $G_n, H_n, \mathcal{S}_n$

H_n and $\mathcal{S}_n$ (q-Schur algebra) defined over a field F, $q \in F$

$\lambda \vdash n$. Let $x_\lambda = \sum_{w \in W_\lambda} T_w \in H_n$, W_λ a parabolic

Define $\mathcal{S}_n = \mathrm{End}_{H_n}(\oplus_\lambda x_\lambda H_n)$, the q-Schur algebra

Theorem (Dipper-James): $\mathcal{S}_n \cong \mathrm{End}_{G_n}(\oplus_\lambda M^\lambda)$, $M^\lambda = \mathrm{Ind}_{P_\lambda}^{G_n}(1)$ (P_λ a parabolic)

Note: $H_n = \mathrm{End}_{G_n}(x_\lambda M)$ for $\lambda = 1^n$.

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Dipper-James theory (continued):

H_n has Specht modules, irreducible modules

$\mathcal{S}_n$ has Weyl modules, irreducible modules

G_n has Specht modules (for unipotent representations), irreducible modules

Characteristic of F is 0, or ℓ not dividing q,

$q \in F$ a primitive e-th root of unity,

$1 + q + q^2 \dots q^{e-1} = 0$.

Can talk of D for

- G_n, prime ℓ
- H_n and $\mathcal{S}_n$, q as above

Dipper-James theory (continued):

Characteristic of F is ℓ:

When q is a primitive e-th root of unity, the decomposition matrix of $\mathcal{S}_n$ is square, has entries the multiplicities of irreducibles in Weyl modules.

There is a square part of the ℓ-decomposition matrix of G_n, rows indexed by unipotent characters, columns by Brauer characters.

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Brundan-Kleshchev: Consider H_n over a field F.

H_n can be made into a $\mathbf{Z}$-graded F-algebra. Uses KLR-algebras, which are naturally graded. Thus can talk of Graded Representation Theory of (cyclotomic) Hecke algebras.

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Ariki (2009): F=field, $q \in F$, $\mathcal{S}_n$ and H_n as before. Using work of Brundan-Kleshchev and Hemmer-Nakano,

$\mathcal{S}_n$ is a **Z**-graded F-algebra.

Weyl modules $W(\lambda)$, irreducible modules $L(\mu)$ are graded.

Graded decomposition number $d_{\lambda\mu}(v) = \sum_{k \in Z} (W(\lambda) : L(\mu)[k]) v^k$, polynomial in v.

New modular representation theory connects decomposition numbers for symmetric groups, (cyclotomic) Hecke algebras, with Lie theory.

The quantized Kac-Moody algebra $\mathcal{U}_v(\widehat{sl_e})$ over $\mathbf{Q}(v)$ is generated by $e_i, f_i, k_i, k_i^{-1}, \ldots,$ $(0 \leqslant i \leqslant e-1)$ with some relations.

Consider the Fock space $\oplus_{n \geqslant 0} K_0(FH_n - mod)$, F a field of characteristic 0. Then $\mathcal{U}_v(\widehat{sl_e})$ acts on this space!

e_i, f_i are functors on the Fock space, called i-induction, i-restriction.

Work of Ariki (1996), Grojnowski, Vazirani, Lascoux-Leclerc-Thibon, Varagnolo-Vasserot, ...

Decomposition matrix D for H_n with q a e-th root of unity, appears as transition between two bases of the Fock space.

Ariki (2009):

$d_{\lambda\mu}(v) = e^{+}_{\lambda\mu}(v^{-1}),$

where $e^{+}_{\lambda\mu}$ are entries of the transition matrix.

Over a field of characteristic 0, $d_{\lambda\mu}(v)$ can be computed by the LLT algorithm when μ is e-restricted.

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Special cases:

Varagnolo-Vasserot: $v = 1$ gives decomposition numbers of $\mathcal{S}_n$

Brundan-Kleshchev: Graded decomposition numbers of H_n given by the formula, so μ is e-restricted $(\mathrm{char} F = 0)$

Consider G_n, F has characteristic ℓ.

There are functors, from bimodules defined above:

- $H_n - mod \to FG_n - mod$
- $H_n - mod \to \mathcal{S}_n - mod$
- $\mathcal{S}_n - mod \to fG_n - mod$

Extend these to the graded case.

Theorem. (i) FG_n is graded.

(ii) We can define graded decomposition numbers $d_{\lambda\mu}(v)$.

Comparison with decomposition numbers for $\mathcal{S}_n$ holds in graded case.

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Ingredients in the proof:

- (CPS) Morits equivalence from $FG_n/J - mod$ to $\mathcal{S}_n - mod$, J the ideal annihilating unipotent characters
- Functors of Dipper connecting FG_n/J-modules and $\mathcal{S}_n$-modules

Extend these to the graded case.

An example of a decomposition matrix D for $n = 4$, $e = 4$:

$$\begin{pmatrix} 4|| & 1 & 0 & 0 & 0 \\ 31|| & 1 & 1 & 0 & 0 \\ 211|| & 0 & 1 & 1 & 0 \\ 1111|| & 0 & 0 & 1 & 1 \end{pmatrix}$$

From such a matrix we can read:

- (Part of) transition between two bases of the Fock space as $\widehat{sl_e}$-module
- Decomposition numbers for H_n (also cyclotomic) over characteristic 0
- (conjecturally) part of ℓ-decomposition matrix of $GL(n, q)$

Summary

- Known: Graded Decomposition numbers for H_n (also cyclotomic) over characteristic 0
- Known: Decomposition numbers for $GL_n(q)$, ℓ large
- Not known: Decomposition numbers for S_n (symmetric group), $GL_n(q)$, all ℓ

An example of a v-decomposition matrix D for $n = 4$, $e = 4$:

$$\begin{pmatrix} 4|| & 1 & 0 & 0 & 0 \\ 31|| & v & 1 & 0 & 0 \\ 211|| & 0 & v & 1 & 0 \\ 1111|| & 0 & 0 & v & 1 \end{pmatrix}$$

An example of the inverse of a v-decomposition matrix D for $n = 6$,

$$e = 2: \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -v & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v^2 & -v & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -v & 0 & -v & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -v^3 & v^2 & -v & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ v^2 & -v & v^2 & -v & -v & 1 & 0 & 0 & 0 & 0 \\ v^2 & 0 & v^2 & -v & -v & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -v^3 & v^2 & v^2 & -v & -v & 1 & 0 & 0 \\ 0 & 0 & v^2 & -v & 0 & 0 & v^2 & -v & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -v^3 & v^2 & -v & 1 \end{pmatrix}$$

Here the rows are indexed as: $6, 51, 42, 41^2, 3^2, 31^3, 2^3, 2^21^2, 21^4, 1^6$

Source: GAP, MAPLE

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