Global crystal bases and q-Schur algebras

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The quantum enveloping algebra $U_q(\mathfrak{gl}_n)$

- $U_q(\mathfrak{gl}_n)$ is the associative algebra over $\mathbb{C}(q)$ with generators

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- $U_q(\mathfrak{sl}_n)$ is the subalgebra generated by $e_i, f_i, K_i K_{i+1}^{-1}$ and $K_i^{-1} K_{i+1}$.

Comultiplication on $U_q(\mathfrak{gl}_\mathfrak{n})$

- The comultiplication $\Delta : U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_n)$ gives an action on tensor products of $U_q(\mathfrak{gl}_n)$-modules:

$$\Delta(e_i) = e_i \otimes 1 + K_i^{-1} K_{i+1} \otimes e_i, \quad \Delta(f_i) = f_i \otimes K_i K_{i+1}^{-1} + 1 \otimes f_i,$$

$$\Delta(K_i) = K_i \otimes K_i$$

Young tableaux

- A sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a **partition** of r if $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ and $\lambda_1 + \cdots + \lambda_n = r$.

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- A λ-**tableau** is obtained by filling the boxes of the Young diagram of shape λ with numbers from the set $\{1, 2, \ldots, n\}$.

Semistandard Young tableaux

$$T = \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 6 & 6 \\ \cline{1-2}\end{array}, \quad S = \begin{array}{|c|c|c|}\hline 2 & 1 & 5 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$$

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- T is *semistandard* since the entries in its rows are weakly increasing and entries in its columns are strictly increasing. S is column increasing but not semistandard.

Highest weight modules

- A $U_q(\mathfrak{gl}_n)$-module V is a *highest weight module* if it contains a highest weight vector v, where $e_i v = 0$ for $1 \leq i < n$, such that $U_q(\mathfrak{gl}_n)v = V$.

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- For $\chi = (\chi_1, \ldots, \chi_n)$, an n-tuple of nonnegative integers, the subspace $V(\lambda)^\chi = \{v \in V(\lambda) \mid K_i v = q^{\chi_i} v,\ i = 1, \ldots, n\}$ is a weight space and
$$V(\lambda) = \oplus_\chi V^\chi.$$

Fundamental modules

- Let $\Lambda_k = (1, \dots, 1, 0, \dots, 0) \dashv k \leq n$; can write $\lambda = \sum_{i=1}^{n} a_i \Lambda_i$.

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- The $U_q(\mathfrak{gl}_n)$-modules $V(\Lambda_k)$ are called *fundamental modules*.
- $V(\Lambda_k)$ is an $\binom{n}{k}$-dimensional vector space with basis $\{[T] \mid T \text{ column increasing}\}$ labeled by one-column Young tableau of shape $\Lambda_k = (1)^k$ with entries from $\{1, 2, \ldots, n\}$.

Fundamental modules cont'd

$$K_i[T] = \begin{cases} [T] & \text{if } i \notin T \\ q[T] & \text{otherwise} \end{cases}$$

$$f_i[T] = \begin{cases} 0 & \text{if } i+1 \in T \text{ or } i \notin T \\ [T'] & \text{otherwise, where } i \text{ is replaced with } i+1 \end{cases}$$

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Example. $n = 3$, $V(\Lambda_2)$ has basis

$$\left\{ \left[\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \right], \left[\begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} \right], \left[\begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} \right] \right\}; \quad f_2 \left[\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \right] = \left[\begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} \right], f_2 \left[\begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline \end{array} \right] = 0.$$

$\left[\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \right]$ is a highest weight vector.

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- $W(\lambda) = V(\Lambda_n)^{\otimes a_n} \otimes V(\Lambda_{n-1})^{\otimes a_{n-1}} \otimes \cdots \otimes V(\Lambda_1)^{\otimes a_1}$ is a $U_q(\mathfrak{gl}_n)$-module.

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- $V(\lambda) = U_q(\mathfrak{gl}_n) v_\lambda$

Example

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$$v_\lambda = \left[\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\right] \otimes \left[\,\boxed{1}\,\right], \quad V(\lambda) = U_q(\mathfrak{gl}_3)v_\lambda.$$

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- Basis for $W(\lambda)$ is indexed by column-increasing λ-tableaux; $B = \{\omega(T) \mid T \text{ column-increasing }\}$

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- Carter-Lusztig basis in terms of q-Schur algebra (G. Cliff, AS, 2010).

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- Define an involution $- : U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{gl}_n)$ by

$$\overline{e_i} = e_i,\ \overline{f_i} = f_i,\ \overline{K_j} = K_j^{-1},\ \overline{q} = q^{-1},\ 1 \leq i < n,\ 1 \leq j \leq n.$$

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- For $w = uv_\lambda \in V(\lambda)$, define $\overline{w} = \overline{u}v_\lambda$, $u \in U_q(\mathfrak{gl}_n)$.

Global crystal basis for $V(\lambda)$

Theorem. (Kashiwara) There exists a unique $\mathbb{Q}[q, q^{-1}]$-basis $\{G(T) \mid T \text{ semistandard}\}$ of $V(\lambda)$ with the properties that

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- In general, difficult to find.
- Leclerc, Toffin provided an intermediate basis for $V(\lambda)$, related to global crystal basis by an upper triangular matrix. Yields an algorithm for producing the global crystal basis.

(Quantum) Carter-Lusztig basis

- Let $f_{i,i+1} = f_i$ and for $j > i+1$, define

$$f_{ij} = f_{i+1,j} f_i - q^{-1} f_i f_{i+1,j}.$$

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- For T a semistandard λ-tableau, define
$$F_T = \prod_{1 \le i < k, i < j \le n} f_{ij}^{(\gamma_{ij})} = f_{12}^{(\gamma_{12})} F_{13}^{(\gamma_{13})} \dots f_{1k}^{(\gamma_{1k})} f_{23}^{(\gamma_{23})} \dots f_{2k}^{(\gamma_{2k})} \dots f_{k-1,k}^{(\gamma_{k-1,k})},$$
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- **Example.** Let $T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ and $T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$. Then $F_{T_1} = f_{12} f_{23}$ and $F_{T_2} = f_{13} = f_{23} f_{12} - q^{-1} f_{12} f_{23}$.

CL-basis

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  $V(\lambda)$ has one 2-dimensional weight space; Any basis for the weight space is indexed by tableaux $T_1 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$ and $T_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$.

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  CL-basis is $\{F_{T_1} v_\lambda, F_{T_2} v_\lambda\} = \{f_{13} v_\lambda = (f_2 f_1 - q^{-1} f_1 f_2) v_\lambda, f_1 f_2 v_\lambda\}$ which is not the same as the global basis.

Example cont'd

Replace $F_{T_1} v_\lambda$ with $F_{T_1} v_\lambda + q^{-1} F_{T_2} v_\lambda = f_2 f_1 v_\lambda$. Then $\overline{f_2 f_1} v_\lambda = f_2 f_1 v_\lambda$.

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$$f_2 f_1 v_\lambda = \left[\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\right] \otimes \left[\,\boxed{3}\,\right] + q \left[\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}\right] \otimes \left[\,\boxed{2}\,\right],$$

$$f_1 f_2 v_\lambda = \left[\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array}\right] \otimes \left[\,\boxed{2}\,\right] + q \left[\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}\right] \otimes \left[\,\boxed{1}\,\right].$$

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$\{f_2 f_1 v_\lambda, f_1 f_2 v_\lambda\}$ is the global crystal basis.

Definition of $\mathbb{C}_q[x_{ij} \mid 1 \leq i, j \leq n]$

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- Define $\mathbb{C}_q[x_{ij} \mid 1 \le i, j \le n]$ to be the associative $\mathbb{C}$-algebra generated by x_{ij}, $1 \le i, j \le n$ subject to the relations:

$$
\begin{array}{ll}
x_{il}x_{ik} = qx_{ik}x_{il} & 1 \le k < l \le n \\
x_{jk}x_{ik} = qx_{ik}x_{jk} & 1 \le i < j \le n \\
x_{il}x_{jk} = x_{jk}x_{il} & 1 \le i < j \le n, \\
 & 1 \le k < l \le n \\
x_{ik}x_{jl} - x_{jl}x_{ik} = (q^{-1} - q)x_{il}x_{jk} & 1 \le i < j \le n, \\
 & 1 \le k < l \le n.
\end{array}
$$

q-Schur algebras

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  Example. $x_{11}^2 + x_{22}x_{13} \in A_q(3, 2)$

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- Also...algorithm for producing global basis elements in terms of the q-Schur algebra.

The end

Thank you!