Coideal subalgebras of quantized enveloping algebras

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June 2012

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A: $\mathbb{C}$-vector space, A associative algebra if
$\exists\, m : A \otimes_{\mathbb{C}} A \longrightarrow A$ such that

$$\begin{tikzcd}
A \otimes_{\mathbb{C}} A \otimes_{\mathbb{C}} A \arrow[r, "m\otimes 1"] \arrow[d, "1\otimes m"'] & A \otimes_{\mathbb{C}} A \arrow[d, "m"] \\
A \otimes_{\mathbb{C}} A \arrow[r, "m"'] & A
\end{tikzcd}$$

$+$ unit $\mu : \mathbb{C} \longrightarrow A$.

A coassociative coalgebra if $\exists\, \Delta : A \longrightarrow A \otimes_{\mathbb{C}} A$ such

$$\begin{tikzcd}
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For $\mathfrak{gl}_n$, $\exists$ other definition.

Set

$$R = \sum_{i,j=1}^{n} q^{\delta_{ij}} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{\substack{i,j=1 \\ i>j}}^{n} E_{ij} \otimes E_{ji} \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)^{\otimes 2}$$

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Quantum Yang-Baxter equation:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

$$\mathfrak{U}_q\mathfrak{gl}_n = \langle t_{ij}, \overline{t}_{ij} | 1 \leqslant i,j \leqslant n \rangle$$

$$T = \sum_{i,j=1}^{n} t_{ij} \otimes E_{ij}, \ \overline{T} = \sum_{i,j=1}^{n} \overline{t}_{ij} \otimes E_{ij}$$

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Relations for $\mathfrak{U}_q\mathfrak{gl}_n$:
T: upper triangular, $\overline{T}$: lower triangular

$$t_{ii}\overline{t}_{ii} = \overline{t}_{ii}t_{ii} = 1, \ \ 1 \leqslant i \leqslant n$$

$$RT_2T_1 = T_1T_2R \text{ in } \mathfrak{U}_q\mathfrak{gl}_n \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)^{\otimes 2}$$

$+$ two more sets of relations.

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Explicitly:

$$q^{\delta_{ij}} t_{ia} t_{jb} - q^{\delta_{ab}} t_{jb} t_{ia} = (q - q^{-1})(\delta_{b<a} - \delta_{i<j}) t_{ja} t_{ib}$$

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Need

$$P = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji} : \overset{\curvearrowleftright}{\mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^n}$$

$$R(u,v) = u\widetilde{R} - vR \text{ where } \widetilde{R} = PR^{-1}P = R - (q - q^{-1})P$$

$\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n) = \langle t_{ij}^{(r)}, \overline{t}_{ij}^{(r)} | 1 \leqslant i,j \leqslant n, r \in \mathbb{Z}_{\geqslant 0} \rangle$ with relations:

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$$t_{ij}^{(0)} = \overline{t}_{ji}^{(0)} = 0, \text{ if } 1 \leqslant j < i \leqslant n, \quad t_{ii}^{(0)}\overline{t}_{ii}^{(0)} = \overline{t}_{ii}^{(0)} t_{ii}^{(0)} = 1$$

Set $T(u) = \sum_{i,j=1}^{n} t_{ij}(u) \otimes E_{ij}$ where $t_{ij}(u) = \sum_{r \geqslant 0} t_{ij}^{(0)} u^{-r}$.

$$R(u,v)\, T_2(v)\, T_1(u) = T_1(u)\, T_2(v)\, R(u,v)$$

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Coproduct and antipode:

$$\Delta(t_{ij}(u)) = \sum_{k=1}^{n} t_{ik}(u) \otimes t_{kj}(u), \; S(T(u)) = T(u)^{-1}$$

$Y(\mathfrak{gl}_n) = \langle T_{ij}^{(r)} | 1 \leqslant i, j \leqslant n, r \in \mathbb{Z}_{\geqslant 0} \rangle$ with relations

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v)$$

where

$$T_1(u) = \sum_{i,j=1}^{n} T_{ij}(u) \otimes E_{ij} \otimes 1, \;\; T_2(u) = \sum_{i,j=1}^{n} T_{ij}(u) \otimes 1 \otimes E_{ij}$$

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$Y(\mathfrak{gl}_n)$ is a deformation of $\mathfrak{U}(\mathfrak{gl}_n \otimes_{\mathbb{C}} \mathbb{C}[t])$.

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$\mathfrak{gl}_n(\mathbb{C}[t, t^{-1}])^\theta = \text{invariants} =$

$$\{X \otimes p(t) \in \mathfrak{gl}_n(\mathbb{C}[t, t^{-1}]) | \theta(X) \otimes p(t^{-1}) = X \otimes p(t)\}.$$

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$\mathfrak{gl}_n(\mathbb{C}[t])^\theta$ = invariants =

$$\{X \otimes p(t) \in \mathfrak{gl}_n(\mathbb{C}[t]) | \theta(X) \otimes p(-t) = X \otimes p(t)\}.$$

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Ex.: $1 \leqslant p \leqslant n$, $\Theta = \mathrm{diag}(\overbrace{1,\ldots,1}^{p}, \overbrace{-1,\ldots,-1}^{n-p})$.

$\theta_3(X) = \Theta X \Theta^{-1}$,

$$\mathfrak{gl}_n^\theta \cong \mathfrak{gl}_p \oplus \mathfrak{gl}_{n-p}$$

$$\theta_i \longleftrightarrow \mathfrak{gl}_n[t^{\pm 1}]^{\theta_i} \longleftrightarrow \mathfrak{U}_q^{\theta_i}\mathfrak{gl}_n[s^{\pm 1}]$$

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Definition

$\mathfrak{U}_q^{\theta_3}(\mathcal{L}\mathfrak{gl}_n) = \langle s_{ij}^{(r)} | 1 \leqslant i,j \leqslant n, r \in \mathbb{Z}_{\geqslant 0} \rangle \subset \mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n)$,
$S(u) = (s_{ij}(u))_{i,j=1}^n$

$$S(u) = T(u)G(u)\overline{T}(u^{-1})^{-1}$$

where

$$G(u) = \frac{uJ - u^{-1}J^{-1}}{u - u^{-1}}, \ J: \ \text{matrix}$$

Theorem (G-Ma)

$$\lim_{q \mapsto 1} \mathfrak{U}_q^{\theta_3}(\mathcal{L}\mathfrak{gl}_n) \cong \mathfrak{U}(\mathfrak{gl}_n(\mathbb{C}[t, t^{-1}])^{\theta_3})$$

$$\theta_i \longleftrightarrow \mathfrak{gl}_n[t]^{\theta_i} \longleftrightarrow Y^{\theta_i}(\mathfrak{gl}_n) : \text{ twisted Yangian.}$$

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Definition

$Y^{\theta_3}(\mathfrak{gl}_n) = \langle b_{ij}^{(r)} : 1 \leqslant i,j \leqslant n,\ r \in \mathbb{Z}_{\geqslant 0} \rangle \subset Y(\mathfrak{gl}_n)$,
$B(u) = (b_{ij}(u))_{i,j=1}^n$

$$B(u) = T(u)(\Theta + \tau u^{-1}) T(-u)^{-1}$$

$Y^{\theta_i}(\mathfrak{gl}_n)$ is a deformation of $\mathfrak{U}\mathfrak{gl}_n[t]^{\theta_i}$.

H: Hopf algebra, $A \subset H$ a subalgebra.

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Theorem (G-Ma, Molev-Ragoucy)

$\mathfrak{U}_q^{\theta_i}(\mathcal{L}\mathfrak{gl}_n)$ is a coideal subalgebra of $\mathfrak{U}_q(\mathcal{L}\mathfrak{gl}_n)$.

$Y^{\theta_i}(\mathfrak{gl}_n)$ is a coideal subalgebra of $Y(\mathfrak{gl}_n)$.

Twisted Yangians and twisted quantum loop algebras can be defined via the reflection equation.

Theorem (Molev-Ragoucy, G-Ma)

Definition relations for $Y^{\theta_3}(\mathfrak{gl}_n)$*:*

$$R(u-v)B_1(u)R(u+v)B_2(v) = B_2(v)R(u+v)B_1(u)R(u-v)$$

$$B(u)B(-u) = 1 - \tau^2 u^{-2}$$

Definition relations for $\mathfrak{U}_q^{\theta_3}(\mathcal{L}\mathfrak{gl}_n)$*:*

$$R_{21}(v,u)S_1(u)R(u^{-1},v)S_2(v) = S_2(v)R_{21}(u^{-1},v)S_1(u)R(v,u)$$

$+$ *vanishing condition for* $s_{ij}^{(0)}$.

$W_l = S_l \wr \mathbb{Z}_2$.

$\mathbf{H}_q(W_l)$: affine Hecke algebra $\supset \langle \sigma_1, \ldots, \sigma_{l-1}, \sigma_l \rangle$

$\mathtt{H}_\tau(W_l)$: degenerate affine Hecke algebra

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$\mathbf{H}_q(W_l)$: affine Hecke algebra $\supset \langle \sigma_1, \ldots, \sigma_{l-1}, \sigma_l \rangle$
$\mathtt{H}_\tau(W_l)$: degenerate affine Hecke algebra

Theorem (G-Ma)

$\exists$ *functors* $\mathrm{mod} - \mathbf{H}_q(W_l) \longrightarrow \mathrm{mod} - \mathfrak{U}_q^{\theta_3}(\mathcal{L}\mathfrak{gl}_n)$

$$M \mapsto M \otimes_{\mathbb{C}(q)} (\mathbb{C}^n)^{\otimes l} / \left(\sum_{i=1}^{l-1} \mathrm{Im}(\sigma_i + q^{-2}) + \mathrm{Im}(\sigma_l + q^{4l}) \right)$$

$$\mathrm{mod} - \mathtt{H}_\tau(W_l) \longrightarrow \mathrm{mod} - Y^{\theta_3}\mathfrak{gl}_n$$

Theorem (G-Ma)

The fermionic Fock space Λ^∞ (infinite wedge product with a stability condition) is a $Y^{\theta_3}\mathfrak{gl}_n$-module.

Theorem (Molev-Ragoucy, Chen-G)

The center of $Y^{\theta_3}\mathfrak{gl}_n$ is a polynomial algebra in infinitely many variables. It is generated by the coefficients of the Sklyanin determinant:

$$\mathrm{sdet} B(u) = \theta(u) \mathrm{qdet}\, T(u) \big(\mathrm{qdet}\, T(-u+n-1)\big)^{-1}$$

where $\theta(u)$: power series.

Similar result for $\mathfrak{U}_q^{\theta_3}(\mathcal{L}\mathfrak{gl}_n)$.