KPP Pulsating Traveling Fronts within Large Drift

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Recall: History and Definition of PTF

• Traveling fronts in the homogenous case:
The equation is

$$u_t(t,x) = \Delta u + f(u) \quad t \in \mathbb{R},\ x \in \mathbb{R}^N. \tag{1}$$

Diffusion= Id Matrix and Reaction is $f = f(u)$ and no advection term $(q \cdot \nabla u)$.

• In 1937 Kolmogorov, Petrovsky and Piskunov defined a Traveling front propagating in the direction of $-e$ (prefixed unitary direction in $\mathbb{R}^N$) with a speed c as:
a solution of (1) in the form $u(t,x) = \phi(x \cdot e + ct) = \phi(s)$ satisfying the limiting conditions $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$.

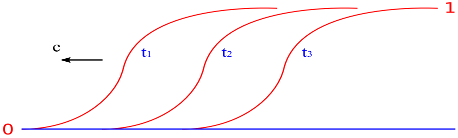


Figure: Traveling front, One dimensional case

TFs in the case of KPP nonlinearity

We say $f = f(u)$ is a homogenous KPP nonlinearity when

$$f(0) = f(1) = 0, f \equiv 0 \text{ in } \mathbb{R} \setminus [0,1], \ \forall s \in]0,1[, \ 0 < f(s) \leq f^{'}(0)s.$$

For example $f(u) = u(1-u)$ on $[0,1]$.

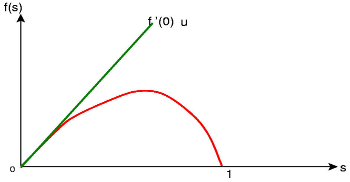


Figure: "KPP" homogenous nonlinearity

• Notice that (1) can be rewritten in this case as

$$\phi'' - c\phi' + f(\phi) = 0 \text{ for all } s \in \mathbb{R}! \tag{2}$$

Theorem

Having a KPP nonlinearity, a TF exists with a speed c iff $c \geq 2\sqrt{f'(0)}$=KPP speed. Moreover, this TF $u(t,x)$ is increasing in t.

- Main ideas of the proof:

1. Solving (1) is equivalent to solve (2).
2. Over any interval $(-a,a) \subset \mathbb{R}$, the nondecreasing function

$$\overline{\phi}^{a,r}(s) := \min(e^{\lambda(s+r)}, 1)$$

is a super-solution over $(-a,a)$ of (2) whenever $c \geq 2\sqrt{f'(0)}$ and $\lambda_1(c) \leq \lambda \leq \lambda_2(c)$.

3. The function $\underline{\phi}^{a,r} := \overline{\phi}^{a,r}(-a)$ is a subsolution.
4. Hence we get a solution $\phi^{a,r}$. For each a we take an r_a that ensures the non triviality of the limit function $\phi = \lim_{a\to+\infty} \phi^{a,r}$. For example we take $\phi^{a,r_a} = 1/2$.

• This idea can be adapted to prove existence in the more complicated heterogenous settings. Together with the KPP condition (sub-linearity) this will produce the Berestycki, Hamel and Nadirashvili variational formula for the minimal speed c^*.

In a More complicated setting, Pulsating traveling fronts...

Now suppose that our equation is no more homogenous:

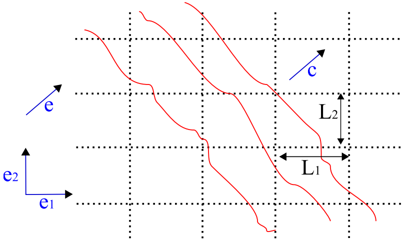
$$\begin{cases} u_t = \Delta u \; + q(z) \cdot \nabla u + f(z,u), \; t \in \mathbb{R}, \; z \in \Omega, \\ \nu \cdot \nabla u = 0 \;\text{ on } \mathbb{R} \times \partial\Omega, \end{cases} \tag{3}$$

where ν stands for the unit outward normal on $\partial\Omega$ whenever it is nonempty.

• Berestycki, Hamel (CPAM 2000) and J. Xin 2003 introduced after Shegisada, Teramoto and Kenzaki 1986 a rigorous mathematical definition of Pulsation traveling fronts in a Periodic framework:

• For instance, suppose that $\Omega = \mathbb{R}^N$. Let $\tilde{e} \in \mathbb{R}^N$ be a unitary direction. Suppose that q and f are $(L_1, \cdots, L_N)-$periodic with respect to z.

• A pulsating traveling front in the direction of $-\widetilde{e}$ with a speed c is a solution $u(t,z) = \phi(s,z) = \phi(z \cdot e + ct, z)$ of (3) with the limiting conditions $\phi(-\infty, z) = 0$ and $\phi(+\infty, z) = 1$ uniformly in z and ϕ is $L-$periodic in z.



- $u_t = \text{div}(A(x)\nabla u) + f(x,u)$
- $\Omega = \mathbb{R}^2$, $d = N = 2$, $e \parallel (1,1)$.

A precise description of the general periodic setting

The previous definition can be extended to unbounded domains $\Omega \subset \mathbb{R}^d \times \mathbb{R}^{N-d}$ where $1 \leq d \leq N$ such that:

• Each $z \in \Omega$ can be written as $z = (x, y) \in \mathbb{R}^d \times \mathbb{R}^{N-d}$.

• There exist $L_1, \cdots, L_d > 0$ such that $\boxed{\Omega = \Omega + k}$ whenever $k = (k_1, \cdots, k_d, 0, \cdots, 0) \in \prod_{i=1}^{d} L_i\mathbb{Z} \times \{0\}^{N-d}$.

• Ω is bounded in the y direction. That is, $\exists R > 0$ s.t $|y| \leq R$ for all $(x, y) \in \Omega$.

• In this case, we assume that $q = q(x, y)$ and $f = f(x, y, u)$ are $L-$periodic in x!

$$q(x + L, y) = q(x, y), \quad f(x + L, y, u) = f(x, y, u)$$

s.t $L = (L_1, \cdots, L_d)$.

The domain Ω may be for example:
1- The whole space $\mathbb{R}^N$ ($d = N$).

2- $\mathbb{R}^N$ except a periodic array of holes

3- For $d = 1$, Ω can be an infinite cylinder with a uniform boundary or with an oscillating boundary etc...

Definition of PTFs, Existence, Minimal speed...

Equation

$$\begin{cases} u_t = \Delta u \; + \; q(x,y)\cdot\nabla u + f(x,y,u), \; t\in\mathbb{R}, \; (x,y)\in\Omega, \\ \nu\cdot\nabla u = 0 \;\; \text{on } \mathbb{R}\times\partial\Omega, \end{cases} \tag{4}$$

- Let $e=(e^1,\cdots,e^d)\in\mathbb{R}^d$ be a unitary direction and denote by $\tilde{e}=(e,0,\cdots,0)\in\mathbb{R}^N$.

Definition

A PTF propagating in the direction of $-e$ with a speed c is a solution

$$u(t,x,y)=\phi(s,x,y)=\phi(x\cdot e+ct,x,y)$$

of (4) which is $L-$periodic in x and satisfies:

$$\phi(-\infty,\cdot,\cdot)=0, \;\; \phi(+\infty,\cdot,\cdot)=1 \text{ uniformly in } (x,y)\in\Omega.$$

Theorem of Berestycki & Hamel 2002

- The advection $q(x,y) = (q_1(x,y), \cdots, q_N(x,y))$ is a $C^{1,\delta}(\overline{\Omega})$ (with $\delta > 0$) vector field satisfying

$$q \quad \text{is } L\text{- periodic with respect to } \; x, \quad \nabla \cdot q = 0 \quad \text{in } \overline{\Omega},$$

$$q \cdot \nu = 0 \quad \text{on } \partial\Omega \; (\text{when } \partial\Omega \neq \emptyset), \quad \text{and } \int_C q \, dx \, dy = 0.$$

- Generalized KPP nonlinearity $f = f(x,y,u)$

$$f \geq 0, f \; \text{ is } L\text{-periodic with respect to } \; x, \text{ and of class } C^{1,\delta}(\overline{\Omega} \times [0,1]),$$

$$\forall (x,y) \in \overline{\Omega}, \quad f(x,y,0) = f(x,y,1) = 0,$$

$$f \text{ is decreasing in } u \text{ on } \Omega \times [1-\rho, 1] \text{ for some } \rho > 0$$

- With the additional "KPP" assumption

$$\forall (x,y,s) \in \overline{\Omega} \times (0,1), \; 0 < f(x,y,s) \leq f'_u(x,y,0) \times \; s.$$

- Simple example: $(x,y,u) \mapsto u(1-u)h(x,y)$ defined on $\overline{\Omega} \times [0,1]$ where h is a positive $C^{1,\delta}(\,\overline{\Omega}\,)$ L-periodic function.

Theorem

• *For any prefixed $e \in \mathbb{R}^d$, there exists a minimal speed $c^* := c^*_{\Omega,q,f}(e) > 0$ such that a PTF with a speed c exists* **if and only if** $c \geq c^*$.
• *Any PTF is increasing in time.*
• *Moreover, for any $c \geq c^*$, Hamel and Roques proved that the fronts $u(t, x, y)$ with a speed c are unique up to a translation in t.*

• A variational formula of this minimal speed was given in 2005.
• This formula shows that this minimal speed depends strongly on the coefficients of the equation (Reaction, diffusion and advection) and on the geometry of the domain.
• Many asymptotic behaviors of c^* and many homogenization results have been studied by El Smaily, J. Xin, Shigesada, Naderashvili, Hamel, Berestycki, S. Hienze, etc.
• In this talk, I will show many results about the asymptotic behavior of the minimal speed within large drift "M"q ($M \to +\infty$) and I will give some details about the limit in the case N=2.

Variational formula of the minimal speed involving Eigenvalue problems

$$c^*(M,e) = \min_{\lambda>0} \frac{k(\lambda,M)}{\lambda};$$

• $k(\lambda,M)$ is the principal eigenvalue of the elliptic operator L_λ defined by

$$\boxed{L_\lambda \psi := \Delta\psi + 2\lambda\tilde{e}\cdot\nabla\psi + M\,q\cdot\nabla\psi + [\lambda^2 + \lambda M\,q\cdot\tilde{e} + \zeta]\psi \text{ in } \Omega,}$$

$$E_\lambda = \left\{\psi(x,y)\in C^2(\overline{\Omega}), \psi \text{ is } L\text{-periodic in } x,\ \nu\cdot\nabla\psi = -\lambda(\nu\cdot\tilde{e})\psi \text{ on } \partial\Omega\right\}.$$

• Roughly $u(t,x,y) \sim e^{\lambda(x\cdot e+ct)+r}\psi(x,y) = e^{\lambda s+r}\psi(x,y)$.

• The principal eigenfunction $\psi^{\lambda,M}$ is positive in $\overline{\Omega}$. It is unique up to multiplication by a nonzero real number.

• $k(\lambda,M) > 0$ for all $(\lambda,M)\in(0,+\infty)\times(0,+\infty)$.

Main Results about the Minimal Speed within Large Advection

$$\begin{cases} u_t = \Delta u \ + Mq(x,y) \cdot \nabla u + f(x,y,u), \ t \in \mathbb{R}, \ (x,y) \in \Omega, \\ \nu \cdot \nabla u = 0 \ \text{ on } \mathbb{R} \times \partial\Omega. \end{cases}$$

$$c^*(M,e) \text{ as } M \to +\infty.$$

$C :=$ periodicity cell of $\Omega = \{(x,y) \in \Omega;\ x_1 \in (0,L_1), \cdots, x_d \in (0,L_d)\}$

First integrals

Definition (First integrals)

The family of first integrals of q is defined by

$$\mathcal{I} := \left\{ w \in H^1_{loc}(\Omega),\ w \neq 0,\ w \text{ is } L-\text{periodic in } x, \text{ and } q \cdot \nabla w = 0 \text{ almost everywhere in } \Omega \right\}.$$

We also define the two subsets $\mathcal{I}_1$ and $\mathcal{I}_2$:

$$\mathcal{I}_1 := \left\{ w \in \mathcal{I}, \text{ such that } \int_C \zeta w^2 \geq \int_C |\nabla w|^2 \right\}, \tag{5}$$

$$\mathcal{I}_2 := \left\{ w \in \mathcal{I}, \text{ such that } \int_C \zeta w^2 \leq \int_C |\nabla w|^2 \right\}.$$

$\zeta(x,y) := f'_u(x,y,0)$. $\zeta = f'(0)$ when $f = f(u)$.

1. About "first integrals" of a vector field q

- The set $\mathcal{I}$ is a closed subspace of $H^1_{loc}(\Omega)$.

Notice

One can see that if $w \in \mathcal{I}$ is a first integral of q and $\eta : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, then $\eta \circ w \in \mathcal{I}$.

2. Asymptotics within large advection in any space dimension

Theorem

We fix a unit direction $e \in \mathbb{R}^d$. Let q be an advection field which satisfies the previous assumptions. Then,

$$\lim_{M \to +\infty} \frac{c^*(Mq, e)}{M} = \max_{w \in \mathcal{I}_1} \frac{\displaystyle\int_C (q \cdot \tilde{e})\, w^2}{\displaystyle\int_C w^2}. \tag{6}$$

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2. A. Zlatos considered similar problems. He did not give a detailed NSC for which the limit is null.
3. We did this study in the case $N = 2$. The NSC that we gave is easy to verify!

Large advection mixed with small reaction or large diffusion

Theorem

- *If we have Mq as advection and ϵf as reaction then*

$$\lim_{\epsilon\to 0^+}\lim_{M\to+\infty}\frac{c^*_{\Omega,M\,q,\epsilon f}(e)}{M\sqrt{\epsilon}} = \left(2\,\frac{\sqrt{\int_C \zeta}}{|C|}\right)\max_{w\,\in\,\mathcal{I}}\frac{\displaystyle\int_C (q\cdot\tilde{e})\,w}{\sqrt{\displaystyle\int_C |\nabla w|^2}}, \qquad (7)$$

and

$$\lim_{B\to+\infty}\lim_{M\to+\infty}\frac{c^*_{\Omega,B,M\,q,f}(e)}{M\sqrt{B}} = \left(2\,\frac{\sqrt{\int_C \zeta}}{|C|}\right)\max_{w\,\in\,\mathcal{I}}\frac{\displaystyle\int_C (q\cdot\tilde{e})\,w}{\sqrt{\displaystyle\int_C |\nabla w|^2}}. \qquad (8)$$

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$$\lambda' = \lambda \times M, \text{ and } \mu(\lambda', M) = k(\lambda, M) \text{ and } \psi^{\lambda', M} = \psi^{\lambda, M}.$$

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$$\forall\, M > 0, \quad \frac{c^*(M)}{M} = \min_{\lambda'>0} \frac{\mu(\lambda', M)}{\lambda'}.$$

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$$(E)\begin{cases} \mu(\lambda', M)\psi^{\lambda', M} = & \Delta\psi^{\lambda', M} + 2\dfrac{\lambda'}{M}\tilde{e}\cdot\nabla\psi + M\,q\cdot\nabla\psi^{\lambda, M} \\ & + \left[\left(\dfrac{\lambda'}{M}\right)^2 + \lambda'\,q\cdot\tilde{e} + \zeta\right]\psi^{\lambda', M} \text{ in } \Omega, \\ \nu\cdot\nabla\psi^{\lambda', M} = & -\dfrac{\lambda'}{M}(\nu\cdot\tilde{e})\psi^{\lambda', M} \text{ on } \partial\Omega \text{ (whenever } \partial\Omega \neq \emptyset). \end{cases}$$

- We multiply (E) by $\dfrac{w^2}{\psi^{\lambda',M}}$ and integrate by parts over the periodicity cell C. (Any $w \in \mathcal{I}$)

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$$\frac{\mu(\lambda',M)}{\lambda'}\int_C w^2 = \underbrace{\frac{1}{\lambda'}\int_C \left|\frac{\nabla\psi^{\lambda',M}}{\psi^{\lambda',M}}\,w - \nabla w + \frac{\lambda'}{M}\tilde{e}\,w\right|^2}_{\geq 0} + \int_C (q\cdot\tilde{e})\,w^2 + \frac{1}{\lambda'}\int_C \left[\zeta w^2 - |\nabla w|^2\right], \tag{9}$$

for all $\lambda' > 0$ and $M > 0$, and for any $w \in \mathcal{I}$.

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for all $\lambda' > 0$ and $M > 0$, and for any $w \in \mathcal{I}$.

- $\forall\,\lambda', M > 0$, $\frac{\mu(\lambda',M)}{\lambda'} \geq h(\lambda') \geq \inf_{\lambda'>0} h(\lambda')$, where

$$h(\lambda) := \frac{g(\lambda)}{\lambda} = \sup_{w\in\mathcal{I}} \frac{\int_C \left(\zeta w^2-|\nabla w|^2\right)}{\lambda\int_C w^2}+\int_C (q\cdot\tilde{e})w^2. \tag{10}$$

for any $w \in \mathcal{I}$ with $||w||_{L^2(C)} = 1$,

$$\inf_{\lambda'>0} h(\lambda') \leq \frac{c^*(M)}{M} \leq \underbrace{\frac{1}{\lambda'} \int_C \left| \frac{\nabla \psi^{\lambda',M}}{\psi^{\lambda',M}} w - \nabla w + \frac{\lambda'}{M} \tilde{e}\, w \right|^2}_{D(\lambda',w,M)} + h(\lambda'). \quad (11)$$

Remark: Eigenfunctions converge to first integrals

For a fixed λ', we take a sequence $\left\{\psi^{\lambda',M_n}\right\}_{n\in\mathbb{N}}$ such that $\int_C \left(\psi^{\lambda',M_n}\right)^2 = 1$.

We get $\left\{\psi^{\lambda',M_n}\right\}_{n\in\mathbb{N}}$ is bounded in $H^1(C)$.

Hence $\psi^{\lambda',+\infty} \in H^1_{loc}(\Omega)$ $\psi^{\lambda',M_n} \to \psi^{\lambda',+\infty}$ in $H^1_{loc}(\Omega)$ weak, in $L^2_{loc}(\Omega)$ strong, and almost everywhere in Ω as $n \to +\infty$.

Elliptic eigenvalue problem implies that $\psi^{\lambda',+\infty}$ is a first integral.

Proposition: More about the function h

The functions g and h satisfy the following properties:
(i) The function g is convex on $[0,+\infty)$, and moreover, g and h are continuous on their domains and take values in $(0,+\infty)$.
(ii) $h(\lambda) \xrightarrow[\lambda\to+\infty]{} \sup_{w\in\mathcal{I}} \dfrac{\int_C (q\cdot\tilde{e})w^2}{\int_C w^2}$.
(iii) Either h is convex and decreasing on $(0,+\infty)$ or h attains a global minimum at some point $\lambda_0>0$.
(iv) If h is convex decreasing on $(0,+\infty)$, then we have

$$h(\lambda) \xrightarrow[\lambda\to+\infty]{} \max_{w\in\mathcal{I}_1} \frac{\int_C (q\cdot\tilde{e})w^2}{\int_C w^2} = \sup_{w\in\mathcal{I}} \frac{\int_C (q\cdot\tilde{e})w^2}{\int_C w^2}.$$

(v) If h attains its minimum at $\lambda_0>0$, then we have

$$h(\lambda_0) = \max_{w\in\mathcal{I}_1} \frac{\int_C (q\cdot\tilde{e})w^2}{\int_C w^2}. \tag{12}$$

End of the proof

Lemma

For a fixed $\lambda' > 0$,

$$D(\lambda', \psi^{\lambda', +\infty}, M_n) \to 0$$

as $M_n \to +\infty$.

(11) + Proposition+ Remark + Lemma finish the proof of the Theorem.

3. More details about the limit when $N = 2$

Proposition

Let $d = 1$ or 2 where d is defined before. Let $q \in C^{1,\delta}(\overline{\Omega})$, L-periodic with respect to x and verifying the conditions

$$\begin{cases} \int_C q &= 0, \\ \nabla \cdot q &= 0 \text{ in } \Omega, \\ q \cdot \nu &= 0 \text{ on } \partial\Omega. \end{cases} \tag{13}$$

Then, there exists $\phi \in C^{2,\delta}(\overline{\Omega})$, L-periodic with respect to x, such that

$$q = \nabla^{\perp}\phi \ \text{ in } \Omega. \tag{14}$$

Moreover, ϕ is constant on every connected component of $\partial\Omega$.

Remark

• *The representation* $q = \nabla^{\perp}\phi$ *is well-known in the case where the domain* Ω *is bounded and* **simply connected** *or equal to whole space* $\mathbb{R}^2$.
• *However, thanks to the condition* $q \cdot \nu = 0$ *on* $\partial\Omega$*, the above proposition applies in more cases.*

Examples

It applies when:
• Ω is the whole space $\mathbb{R}^2$ with a periodic array of holes
•• Ω is an infinite cylinder which may have an oscillating boundary and/or a periodic array of holes.

• $\nabla^{\perp}\phi \cdot \nu = q \cdot \nu = 0$ on $\partial\Omega \Rightarrow \phi$ is constant on every connected component of $\partial\Omega$.

• In the proof of existence of ϕ, ($d = 2$ let's say)

$$\hat{\Omega} := \Omega/(L_1\mathbb{Z} \times L_2\mathbb{Z}) \quad \text{and} \quad T := \mathbb{R}^2/(L_1\mathbb{Z} \times L_2\mathbb{Z}).$$

If $x \in \mathbb{R}^2$, we denote by $\hat{x}$ its class of equivalence in T, and if $\phi : \mathbb{R}^2 \to \mathbb{R}$ is L- periodic, we denote $\hat{\phi}$ the function $T \to \mathbb{R}^2$ verifying $\phi(x) = \hat{\phi}(\hat{x})$. we first get $\tilde{\phi}$ solution of

$$\Delta\tilde{\phi} = \nabla \cdot R\tilde{q} \text{ in } T$$

in the weak sense, where

$$\begin{array}{rcl} \tilde{q} : T & \longrightarrow & \mathbb{R}^2, \\ \hat{x} \in \overline{\hat{\Omega}} & \longmapsto & q(x), \\ \hat{x} \notin \overline{\hat{\Omega}} & \longmapsto & 0. \end{array}$$

• $\tilde{q}$ is a divergence free vector field on T in the sense of distributions:

$$\forall\, \psi \in C^\infty(T),$$

$$\begin{aligned} < div(\tilde{q}), \psi > \;&:=\; - < \tilde{q}, \nabla\psi > = -\int_T \tilde{q}\cdot\nabla\psi \\ &=\; -\int_{\hat{\Omega}} q\cdot\nabla\psi = -\int_{\partial\hat{\Omega}} \psi\, q\cdot\nu + \int_{\hat{\Omega}} \psi\nabla\cdot q \\ &=\; 0+0=0, \end{aligned}$$

• Then we define $\hat{\phi} = \tilde{\phi}|_{\hat{\Omega}}$ and we take ϕ the corresponding $L-$ periodic function on Ω.

• Also we get $\nabla^\perp\tilde{\phi} = \tilde{q} = 0$ on $T \setminus \hat{\Omega}$.

• Hence $\tilde{\phi}$ =Constant on $T \setminus \hat{\Omega}$.

Corollary (Now we know more about first integrals...)

Let

$$\mathcal{J} := \{\eta \circ \phi, \text{ such that } \eta : \mathbb{R} \to \mathbb{R} \text{ is Lipschitz}\}, \tag{15}$$

where ϕ*, such that* $q = \nabla^{\perp}\phi$*, is given by Proposition 8. Then,*

$$\mathcal{J} \subset \mathcal{I}.$$

The first integrals of the form $w = \eta \circ \phi$, $\mathcal{J}$

$$\forall\, w \in \mathcal{J},\ \text{ we have } \int_C (q \cdot \tilde{e}) w^2 = 0.$$

- Indeed, $w = \eta \circ \phi$ and $q = \nabla^\perp \phi$. This gives

$$\begin{aligned} \int_C (q \cdot \tilde{e}) w^2 &= \tilde{e} \cdot \int_C \left(\nabla^\perp \phi\right)\, \eta^2(\phi) \\ &= \tilde{e} \cdot R \int_C \nabla\left(F \circ \phi\right) = \tilde{e} \cdot R \int_{\hat{\Omega}} \nabla(F \circ \tilde{\phi}), \end{aligned}$$

where R the matrix of a direct rotation of angle $\pi/2$, $F' = \eta^2$,

and where

$$\begin{array}{lllllll} T & := & \mathbb{R}^2/(L_1\mathbb{Z} \times L_2\mathbb{Z}) & \text{and} & \hat{\Omega} & := & \Omega/(L_1\mathbb{Z} \times L_2\mathbb{Z}) \quad \text{if} \quad d = 2, \\ T & := & \mathbb{R}^2/\left(L_1\mathbb{Z} \times \{0\}\right) & \text{and} & \hat{\Omega} & := & \Omega/\left(L_1\mathbb{Z} \times \{0\}\right) \quad \text{if} \quad d = 1. \end{array}$$

• $\tilde{\phi}$ is constant on every connected component of $T\backslash\hat{\Omega}$, and so is $F\circ\tilde{\phi}$. We then have

$$\int_{T\backslash\hat{\Omega}} \nabla\left(F\circ\tilde{\phi}\right) = 0.$$

• Hence, $\int_C (q\cdot\tilde{e})w^2 = \tilde{e}\cdot R\int_T \nabla\left(F\circ\tilde{\phi}\right) = 0$, because T has no boundary. $\square$

We need the following preliminary lemma in order to give details about the limit when $N = 2$:

Lemma

Let $\hat{\Omega}$ be the set defined before, $\hat{U}$ be an open subset of $\hat{\Omega}$, and $\hat{\phi}$ given by (14). Suppose that:

(i) $\hat{q}(\hat{x}) \neq 0$ *for all* $\hat{x} \in \hat{U}$,

(ii) the level sets of $\hat{\phi}$ in $\hat{U}$ are all connected.

Then, for every $w \in \mathcal{I}$, there exists a continuous function $\eta : \hat{\phi}(\hat{U}) \to \mathbb{R}$ such that

$$\hat{w} = \eta \circ \hat{\phi} \text{ on } \hat{U}. \tag{16}$$

Trajectories of an $L-$periodic vector field, Periodicity of trajectories?

Definition (Trajectory of a vector field)

Assume that $N = 2$. Let $x \in \Omega$ such that $q(x) \neq 0$. The trajectory of q at x is the largest (in the sense of inclusion) connected differentiable curve $T(x)$ in Ω verifying:

(i) $x \in T(x)$,

(ii) $\forall y \in T(x),\ q(y) \neq 0$,

(iii) $\forall y \in T(x),\ q(y)$ is tangent to $T(x)$ at the point y.

The decision about the limit (null or positive) will depend on the existence of periodic unbounded trajs. for q!

Lemma (unbounded periodic trajectories)

Let $T(x)$ be an unbounded periodic trajectory of q in Ω, that is:
- *there exists $\mathbf{a} \in L_1\mathbb{Z} \times L_2\mathbb{Z} \setminus \{0\}$ (resp. $L_1\mathbb{Z} \times \{0\} \setminus \{0\}$) when $d = 2$ (resp. $d = 1$) such that $T(x) = T(x) + \mathbf{a}$.*
- *In this case, we say that $T(x)$ is $\mathbf{a}-$periodic.*

Then,

if $T(y)$ is another unbounded periodic trajectory of q, $T(y)$ is also $\mathbf{a}-$periodic.

Moreover,

in the case $d = 1$, $\mathbf{a} = L_1 e_1$. That is, all the unbounded periodic trajectories of q in Ω are $L_1 e_1-$periodic.

• There may exist **unbounded trajectories which are not periodic**, even though the vector field q **is periodic**.

A periodic vector field whose unbounded trajectories are not periodic!

Let

$$\phi(x,y) := \begin{cases} e^{-\dfrac{1}{\sin^2(\pi y)}} \sin(2\pi(x + \ln(y - [y]))) & \text{if } y \notin \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

• ϕ is C^∞ on $\mathbb{R}^2$, and 1-periodic in x and y.

• Hence the vector field $q = \nabla^\perp \phi$ is also C^∞, 1-periodic in x and y, and $\int_{[0,1]\times[0,1]} q = 0$ with $\nabla \cdot q \equiv 0$.

• The part of the graph of $x \mapsto e^{-x}$ lying between $y = 0$ and $y = 1$ is a trajectory of q, and is obviously unbounded and not periodic.

• There exist no periodic unbounded trajectory for this vector field, so the theorem asserts that for all $w \in \mathcal{I}$ we have

$$\int_C qw^2 = 0.$$

Theorem

Assume that $N = 2$ *and that* Ω *and* q *satisfy the assumptions. The two following statements are equivalent:*
(i) There exists $w \in \mathcal{I}$, *such that* $\int_C qw^2 \neq 0$.
(ii) There exists a periodic unbounded trajectory $T(x)$ *of* q *in* Ω.
Moreover, if (ii) is verified and $T(x)$ *is* $\mathbf{a}-$*periodic, then for any* $w \in \mathcal{I}$ *we have* $\int_C q\, w^2 \in \mathbb{R}\mathbf{a}$.

Definition

We define here the set of "regular trajectories" in $\hat{\Omega}$. Let
$\hat{U} := \left\{ \hat{x} \in \hat{\Omega} \text{ such that } T(\hat{x}) \text{ is well defined and closed in } \overline{\hat{\Omega}} \right\}$.
- We denote by $\hat{U}_i$ the connected components of $\hat{U}$.

- **Proposition**: The set $\hat{U}$ is exactly the union of the trajectories which are simple closed curves in $\hat{\Omega}$.

Proof of the Theorem

- $$\int_C q w^2 = R \int_C (\nabla\phi) w^2 = R \int_{\hat{\Omega}} (\nabla\hat{\phi}) \hat{w}^2.$$

- Let $W := \{\hat{x} \in \hat{\Omega} \text{ such that } \hat{\phi}(\hat{x}) \text{ is a critical value of } \hat{\phi}\}$.

- $$\boxed{\text{Co-area} \Rightarrow \left|\int_W \hat{w}^2 \nabla\hat{\phi}\right| \leq \int_W \hat{w}^2 |\nabla\hat{\phi}| = \int_{\hat{\phi}(W)} \left(\int_{\hat{\phi}^{-1}(t)} \hat{w}^2(x)\right) dt.}$$

- From Sard's theorem, since $\hat{\phi}$ is C^2, $\mathcal{L}^1(\hat{\phi}(W)) = 0$, where $\mathcal{L}^1$ denotes the Lebesgue measure on $\mathbb{R}$, one gets

$$\int_W \hat{w}^2 \nabla\hat{\phi} = 0.$$

• $\hat{\Omega}\backslash W \subset \hat{U} \subset \hat{\Omega}$, we get

$$\int_C qw^2 = R\int_{\hat{U}} (\nabla\hat{\phi})\hat{w}^2 = R\sum_i \int_{\hat{U}_i} (\nabla\hat{\phi})\hat{w}^2. \tag{17}$$

We now use Lemma 12 to get η_i continuous such that

$$\int_{\hat{U}_i} (\nabla\hat{\phi})\hat{w}^2 = \int_{\hat{U}_i} (\nabla\hat{\phi})\eta_i^2(\hat{\phi}).$$

We define the function F_i by $F_i' = \eta_i^2$ and $F_i(0) = 0$, and we obtain

$$\int_{\hat{U}_i} (\nabla\hat{\phi})\hat{w}^2 = \int_{\hat{U}_i} \nabla F_i(\hat{\phi}).$$

Lemma

Let $\hat{U}_i$ as in the previous definition. Then,
(i) all the level sets of $\hat{\phi}$ in $\hat{U}_i$ are connected,
(ii) all the level sets of $\hat{\phi}$ in $\hat{U}_i$ are homeomorphic,
(iii) $\partial\hat{U}_i$ has exactly two connected components $\hat{\gamma}_1$ and $\hat{\gamma}_2$ such that

$$\hat{\phi}(\hat{\gamma}_1) = \sup_{\hat{x}\in\hat{U}_i} \hat{\phi}(\hat{x}) \text{ and } \hat{\phi}(\hat{\gamma}_2) = \inf_{\hat{x}\in\hat{U}_i} \hat{\phi}(\hat{x}).$$

Due to the condition $q \cdot \nu = 0$ on $\partial\Omega$, we have

Trajs of q follow the boundary, and this led us to: γ_1 (resp. γ_2) is either a connected component of $\partial\hat{\Omega}$ or contains a critical point of $\hat{\phi}$.

If we define

$$\hat{U}_i^\varepsilon := \{\hat{x} \in \hat{U}_i \text{ such that } \inf_{\hat{U}_i} \hat{\phi} + \varepsilon < \hat{\phi}(x) < \sup_{\hat{U}_i} \hat{\phi} - \varepsilon\},$$

then it follows from dominated convergence theorem that

$$\int_{\hat{U}_i^\varepsilon} (\nabla\hat{\phi})\hat{w}^2 \xrightarrow[\varepsilon\to 0]{} \int_{\hat{U}_i} (\nabla\hat{\phi})\hat{w}^2. \tag{18}$$

• $\neg ii) \Longrightarrow \neg i)$ We suppose that there exist no periodic unbounded trajectories of q. In $\hat{U}_i$, the trajectories of q are exactly the level sets of $\hat{\phi}$. We consider the following set

$$U_i^\varepsilon := \Pi^{-1}(\hat{U}_i^\varepsilon).$$

Let $x_0 \in U_i^\varepsilon$ and let $U_{i,0}^\varepsilon$ be the connected component of U_i^ε containing x_0.

• We proved that Π is a measure preserving bijection from $U_{i,0}^\varepsilon$ to $\hat{U}_i^\varepsilon$.

• Thus $\int_{\hat{U}_i^\varepsilon}(\nabla\hat{\phi})\hat{w}^2 = \int_{U_{i,0}^\varepsilon}(\nabla\phi)w^2 = \int_{U_{i,0}^\varepsilon}\nabla F_i(\phi) = \int_{\partial U_{i,0}^\varepsilon} F_i(\phi)\mathbf{n}$,
• $\partial U_{i,0}^\varepsilon$ is the union of two level sets C_1 and C_2 of ϕ in Ω, which are both simple closed curves!
• So we can write

$$\int_{U_{i,0}^\varepsilon}(\nabla\phi)w^2 = F(\phi(C_1))\int_{C_1}\mathbf{n} + F(\phi(C_2))\int_{C_2}\mathbf{n},$$

with

$$\int_{C_1}\mathbf{n} = \int_{C_2}\mathbf{n} = 0,$$

because the integral of the unit normal on a C^1 closed curve in $\mathbb{R}^2$ is zero.
$\square$
ii) $\Longrightarrow$ *i*) was proved using the same technics.

Conclusions

As a direct consequence of the previous Theorems, we get the following about the asymptotic behavior of the minimal speed within large drift:

Assume that $N = 2$. Then,

(i) If there exists no periodic unbounded trajectory of q in Ω, then

$$\lim_{M \to +\infty} \frac{c^*_{\Omega, M\,q, f}(e)}{M} = 0,$$

for any unit direction e.

(ii) If there exists a periodic unbounded trajectory $T(x)$ of q in Ω (which will be $\mathbf{a}-$periodic for some vector $\mathbf{a} \in \mathbb{R}^2$) then

$$\lim_{M \to +\infty} \frac{c^*_{\Omega, M\,q, f}(e)}{M} > 0 \iff \tilde{e} \cdot \mathbf{a} \neq 0. \qquad (19)$$

Notice that in the case where $d = 1$, we have $\tilde{e} = \pm e_1$. Lemma 14 yields that $\tilde{e} \cdot \mathbf{a} = \pm L_1 \neq 0$.

Thus, for $d = 1$,

$$\lim_{M \to +\infty} \frac{c^*_{M\,q}(e)}{M} > 0 \Longleftrightarrow \exists \text{ a periodic unbounded traj. } T(x) \text{ of } q \text{ in } \Omega.$$

Thank you