# Wardrop equilibria in a continuous framework, theoretical and numerical aspects

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Transportation: what are we talking about? Depends on applications!! static vs dynamic, different scales.

Mathematical models: (at least) 3 distinct lines of research.

- hyperbolic systems of conservation laws (road traffic: LWR, Rascle...),
- congested networks,
- optimal transport (Monge-Kantorovich),

The two last ones will be of particular interest here.

# Monge-Kantorovich

Optimal transportation, $\mu_0$, $\mu_1$ prob. measures on $\mathbb{R}^d$, $c$ transp. cost. Transport $s : \mathbb{R}^d \to \mathbb{R}^d$ pushing $\mu_0$ forward to $\mu_1$. ($s\sharp\mu_0 = \mu_1$ with $s\sharp\mu_0(B) := \mu_0(s^{-1}(B))$.

Monge's problem:

$$\inf\{\int_{\mathbb{R}^d} c(x, s(x))d\mu_0(x) \ : \ s\sharp\mu_0 = \mu_1\},$$

relaxation : Monge-Kantorovich

$$\inf\{\int_{\mathbb{R}^d\times\mathbb{R}^d} c(x, y)d\gamma(x, y) \ : \ \pi_0\sharp\gamma = \mu_0, \ \pi_1\sharp\gamma = \mu_1\}$$

$\gamma$: transport plan.

In the classical Monge-Kantorovich problem, the transportation cost only depends on the amount of mass sent from sources to destinations and not on the paths followed by this mass. Thus, it does not allow for congestion effects. Congestion effect: the travelling cost of a trip depends on "how crowded" this trip is.

There exist well-known (finite-dimensional) congested networks models (Wardrop, Beckmann...).

Aim of this talk: optimal transportation model with congestion, generalization of the finite-dimensional case.

# The classical congested network model

$G = (N, A)$ finite oriented and connected graph, $P \subset N \times N$ (sources/dest.), $\gamma_{s,d} \geq 0$ mass to be sent from $s$ to $d$, $C_{s,d}$ (nonempty) set of simple paths connecting $s$ to $d$ ($(s, d) \in P$) and $C$ their union. Travelling time functions(congestion), for $a \in A$ $w \mapsto t_a(w)$ ($w \geq 0$ flow on arc $a$), $t_a$ nonnegative, nondecreasing.

Cost of a path $r \in C$ given the flows $(w_a)_{a \in A}$:

$$T_w(r) := \sum_{a \in r} t_a(w_a)$$

Unknown: arc flows $(w_a)_{a\in A}$ and mass travelling on each road $(h_r)_{r\in C}$, constraints:

$$\gamma_{s,d} = \sum_{r\in C_{s,d}} h_r,\ w_a = \sum_{r\ni a} h_r,\ w_a \geq 0, h_r \geq 0. \tag{1}$$

Pbm: what is a long-term steady state or equilibrium flow-configuration?

Wardrop: used paths have to be shortest paths, given the flow configuration (similar to Nash equilibrium).

Wardrop equilibrium (1952): $(w_a)_{a\in A}$, $(h_r)_{r\in C}$ satisfying (1) such that, $\forall (s,d)\in P$, $\forall r\in C_{s,d}$, if $h_r>0$, then:

$$T_w(r)=\min\{T_w(r'),\ r'\in C_{s,d}\}$$

Beckman, McGuire, Winsten (1956) noticed that $(w_a)_{a\in A}$, $(h_r)_{r\in C}$ is a Wardrop equilibrium iff it minimizes

$$C(w):=\sum_{a\in A}\int_0^{w_a} t_a$$

subject to (1).

## Outline

① Continuous congestion model

② Existence and characterization of minimizers

③ Back to equilibria

④ Other formulations

⑤ Numerical approximation

# Congestion modelling

Basic ideas: path-dependent model, traffic intensity (transport density), and metric depending on the traffic intensity.

$\Omega$ open bounded convex subset of $\mathbb{R}^2$, two probability measures, $\mu_0$ and $\mu_1$ in $\mathcal{M}^1_+(\overline{\Omega})$, of residents and services in the city $\overline{\Omega}$, transportation plans

$$\Pi(\mu_0, \mu_1) := \{\gamma \in \mathcal{M}^1_+(\overline{\Omega} \times \overline{\Omega}) : \pi_0 \sharp \gamma = \mu_0, \ \pi_1 \sharp \gamma = \mu_1\} \quad (2)$$

sets of paths

- $C := W^{1,\infty}([0,1],\overline{\Omega})$, viewed as a subset of $C^0([0,1],\mathbb{R}^2)$,
- $C^{x,y} := \{\sigma \in C : \ \sigma(0) = x, \ \sigma(1) = y\}$ ($x, \ y$ in $\overline{\Omega}$),
- $l(\sigma) := \int_0^1 |\dot{\sigma}(t)| \, dt$, the length of $\sigma \in C$,
- for $\sigma \in C$, $\widetilde{\sigma}$ denotes the arclength reparameterization of $\sigma$ belonging to $C$, hence $|\dot{\widetilde{\sigma}}(t)| = l(\sigma) = l(\widetilde{\sigma})$ for a.e. $t \in [0,1]$,
- $\widetilde{C} := \{\sigma \in C \ : \ |\dot{\sigma}| \text{ is constant}\} = \{\widetilde{\sigma}, \ \sigma \in C\}$,
- for $Q \in \mathcal{M}^1_+(C)$, $\widetilde{Q} \in \mathcal{M}^1_+(\widetilde{C})$ is the push forward of $Q$ through the map $\sigma \mapsto \widetilde{\sigma}$,

- for $\varphi \in C^0(\overline{\Omega}, \mathbb{R})$ and $\sigma \in C$,

$$L_\varphi(\sigma) := \int_0^1 \varphi(\sigma(t))|\dot{\sigma}(t)|dt = l(\sigma)\int_0^1 \varphi(\widetilde{\sigma}(t))dt,$$

- $e_0(\sigma) := \sigma(0)$, $e_1(\sigma) := \sigma(1)$, for all $\sigma \in C^0([0,1], \mathbb{R}^2)$.

$$\mathcal{Q}(\mu_0, \mu_1) = \{Q \in \mathcal{M}^1_+(C) \ : \ e_0\sharp Q = \mu_0, \ e_1\sharp Q = \mu_1\}.$$

**Définition 1** *A transportation strategy consists of a pair* $(\gamma, p)$ *with* $\gamma \in \Pi(\mu_0, \mu_1)$ *and* $p = (p^{x,y})_{(x,y)\in\overline{\Omega}\times\overline{\Omega}}$ *is a Borel family of probability measures on* $C$ *such that* $p^{x,y}(C^{x,y}) = 1$ *for* $\gamma$*-a.e.* $(x, y) \in \overline{\Omega} \times \overline{\Omega}$*.*

*Traffic intensity* $I_{\gamma,p} \in \mathcal{M}_+(\overline{\Omega})$: $\forall \varphi \in C^0(\overline{\Omega}, \mathbb{R})$

$$\int_{\overline{\Omega}} \varphi(x) dI_{\gamma,p}(x) := \int_{\overline{\Omega}\times\overline{\Omega}} \left( \int_{C^{x,y}} L_\varphi(\sigma) dp^{x,y}(\sigma) \right) d\gamma(x, y) \qquad (3)$$

Probability over paths $Q_{\gamma,p} \in \mathcal{M}^1_+(C)$ given by $Q_{\gamma,p} = p \otimes \gamma$

Setting $Q = Q_{\gamma,p} = p \otimes \gamma$, we simply have $I_{\gamma,p} = i_Q$ where

$$\int_{\overline{\Omega}} \varphi(x) di_Q(x) = \int_C L_\varphi(\sigma) dQ(\sigma), \quad \forall \varphi \in C^0(\overline{\Omega}, \mathbb{R}). \tag{4}$$

Note that

$$i_Q(\overline{\Omega}) = \int_C l(\sigma) dQ(\sigma)$$

Congestion: metric which depends on the traffic intensity $I \mapsto G_I$:

$$c_{\gamma,p}(x,y) := \int_{C^{x,y}} L_{G_{I_{\gamma,p}}}(\sigma) dp^{x,y}(\sigma)$$

($G_{I_{\gamma,p}}$ is a nonnegative function which depends-in a way specified later on-on the traffic intensity $I_{\gamma,p}$). Total transportation cost:

$$C_{\gamma,p} = \int_{\overline{\Omega}\times\overline{\Omega}} c_{\gamma,p}(x,y) d\gamma(x,y)$$

## Optimal transportation with congestion

Optimal transportation problem with traffic congestion:

$$\inf\left\{\int_{\overline{\Omega}\times\overline{\Omega}} c_{\gamma,p}(x,y)d\gamma(x,y)\ :\ (\gamma,p)\ \text{transp. strategy}\right\}. \tag{5}$$

Setting $Q = Q_{\gamma,p} = p\otimes\gamma$, this can (formally) be rewritten as:

$$\inf\left\{\int_{\overline{\Omega}} G_{i_Q}(x)di_Q(x)\ :\ Q\in\mathcal{Q}(\mu_0,\mu_1)\right\}. \tag{6}$$

where

$$\mathcal{Q}(\mu_0,\mu_1) = \{Q\in\mathcal{M}^1_+(C)\ :\ e_0\sharp Q = \mu_0,\ e_1\sharp Q = \mu_1\}.$$

In problem (6) allows for more general forms of congestion through $i \mapsto G_i$ (no need for $G_i$ to be l.s.c.). From now on, we assume that $G$ has the following local form:

$$G_i(x) = g\left(\frac{di}{d\mathcal{L}^2}(x)\right), \text{ for } i << \mathcal{L}^2, \tag{7}$$

with $g$ nondecreasing $\mathbb{R}_+ \to \mathbb{R}_+$ such that the function $H$ defined by $H(z) = zg(z)$ for all $z \in \mathbb{R}_+$ is convex and superlinear (i.e. $\lim_{z\to+\infty} g(z) = +\infty$).

The optimization problem reads as:

$$\inf_{Q \in \mathcal{Q}(\mu_0,\mu_1)} \mathcal{H}(i_Q) \text{ where } \mathcal{H}(i) = \begin{cases} \int_\Omega H(i(x))dx \text{ if } i << \mathcal{L}^2, \\ +\infty \text{ otherwise.} \end{cases} \tag{8}$$

In the sequel, we say that a transportation strategy $(\gamma, p)$ is optimal if $Q_{\gamma,p}$ solves (8).

# Existence of minimizers

Assumptions:

- $H$ is convex and nondecreasing on $\mathbb{R}_+$ with $H(0) = 0$,
- there exists $q > 1$, and positive constants $a$ and $b$ such that $az^q \leq H(z) \leq b(z^q + 1)$ for all $z \in \mathbb{R}_+$,
- $H$ is differentiable on $\mathbb{R}_+$, and there exists a positive constant $c$ such that $0 \leq H'(z) \leq c(z^{q-1} + 1)$, for all $z \in \mathbb{R}_+$,
- the following set

$$\mathcal{Q}^q(\mu_0, \mu_1) := \{Q \in \mathcal{Q}(\mu_0, \mu_1) \ : \ i_Q \in L^q\} \tag{9}$$

is nonempty.

The optimal transportation with congestion problem then reads as:

$$\inf_{Q \in \mathcal{Q}^q(\mu_0,\mu_1)} \int_\Omega H(i_Q(x))dx. \tag{10}$$

Not easy to check a priori that $\mathcal{Q}^q(\mu_0, \mu_1) \neq \emptyset$, but

- it holds whenever $\mu_0$ and $\mu_1$ are $L^q$ (De Pascale, Pratelli),
- also when $\overline{\Omega} = [0,1]^2$ and $\mu_0$ and $\mu_1$ are respectively the one-dimensional Hausdorff measures of the vertical sides of the square.

**Lemma 1** *Let* $(Q_n)_n \in \mathcal{M}^1_+(C^0([0,1],\mathbb{R}^2))^{\mathbb{N}}$ *such that* $Q_n(C) = 1$ *for all* $n$ *and there exists a constant* $M > 0$ *such that:*

$$\sup_n \int_C l(\sigma)\ dQ_n(\sigma) \leq M.$$

*Then the sequence* $(\widetilde{Q}_n)_n$ *is tight and admits a subsequence that converges weakly* $*$ *to a probability* $Q$ *such that* $Q(C) = 1$*.*

**Lemma 2** *Let* $(Q_n)_n$ *be a sequence in* $\mathcal{M}^1_+(C)$ *that converges weakly* $*$ *to some* $Q \in \mathcal{M}^1_+(C)$. *If there exists* $i \in \mathcal{M}_+(\overline{\Omega})$ *such that* $i_{Q_n}$ *converges weakly* $*$ *to* $i$ *in* $\mathcal{M}_+(\overline{\Omega})$ *then we have* $i_Q \leq i$.

Hence

**Theorem 1** *The minimization problem (10) admits a solution.*

## Characterization of minimizers

$\overline{Q} \in \mathcal{Q}^q(\mu_0, \mu_1)$ solves (10) if and only if

$$\int_\Omega \overline{\xi} i_{\overline{Q}} = \inf \left\{ \int_\Omega \overline{\xi} i_Q \ : \ Q \in \mathcal{Q}^q(\mu_0, \mu_1) \right\} \text{ with } \overline{\xi} := H'(i_{\overline{Q}}) \in L^{q^*}. \tag{11}$$

What does it mean on (the optimal transport strategies)? If $\overline{Q} = \overline{p} \otimes \overline{\gamma}$, what can be said on $(\overline{p}, \overline{\gamma})$? link with classical Monge-Kantorovich theory?

(Very) Formal manipulations,

$$\begin{aligned}
\int_\Omega \overline{\xi} i_{\overline{Q}} &= \int_C L_{\overline{\xi}}(\sigma) d\overline{Q}(\sigma) \\
&= \int_{\overline{\Omega}\times\overline{\Omega}} \left( \int_{C^{x,y}} L_{\overline{\xi}}(\sigma) d\overline{p}^{x,y}(\sigma) \right) d\overline{\gamma}(x,y) \\
&= \inf_{(\gamma,p)} \int_{\overline{\Omega}\times\overline{\Omega}} \left( \int_{C^{x,y}} L_{\overline{\xi}}(\sigma) dp^{x,y}(\sigma) \right) d\gamma(x,y) \\
&= \inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \left( \inf_{p\in\mathcal{M}^1_+(C^{x,y})} \int_{C^{x,y}} L_{\overline{\xi}}(\sigma) dp(\sigma) \right) d\gamma(x,y) \\
&= \inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \left( \inf_{\sigma\in C^{x,y}} L_{\overline{\xi}}(\sigma) \right) d\gamma(x,y)
\end{aligned}$$

define:

$$c_{\overline{\xi}}(x,y) := \inf_{\sigma\in C^{x,y}} L_{\overline{\xi}}(\sigma)$$

So that firstly:

$$\int_{\overline{\Omega}\times\overline{\Omega}} c_{\overline{\xi}}(x,y)d\overline{\gamma}(x,y) \leq \int_C L_{\overline{\xi}} d\overline{Q}$$
$$= \inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} c_{\overline{\xi}}(x,y)d\gamma(x,y)$$

so that $\overline{\gamma}$ solves the Monge-Kantorovich problem:

$$\inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} c_{\overline{\xi}}(x,y)d\gamma(x,y).$$

Secondly,

$$\int_C L_{\overline{\xi}}(\sigma)d\overline{Q}(\sigma) = \int_{\overline{\Omega}\times\overline{\Omega}} c_{\overline{\xi}}(x,y)d\overline{\gamma}(x,y)$$
$$= \int_C c_{\overline{\xi}}(\sigma(0),\sigma(1))d\overline{Q}(\sigma)$$

and since $L_{\overline{\xi}}(\sigma) \geq c_{\overline{\xi}}(\sigma(0),\sigma(1))$, we get

$$L_{\overline{\xi}}(\sigma) = c_{\overline{\xi}}(\sigma(0),\sigma(1)) \quad \text{for } \overline{Q}\text{-a.e. } \sigma.$$

or, in an equivalent way, for $\overline{\gamma}$-a.e. $(x,y)$ one has:

$$L_{\overline{\xi}}(\sigma) = c_{\overline{\xi}}(x,y) \quad \text{for } \overline{p}^{x,y}\text{-a.e. } \sigma.$$

Of course, all this is only formal since $\overline{\xi}$ is only $L^{q^*}$.

Aim: make the previous manipulations rigorous.

For a non-negative continuous $\xi \in C^0(\Omega)$ we define

$$c_\xi(x, y) = \inf\{L_\xi(\sigma) \, : \, \sigma \in C^{x,y}\}.$$

**Proposition 1** *Let us assume that* $q < 2$ *and define* $\alpha := 1 - 2/q^*$, *then there exists a non-negative constant* $C$ *such that for every* $\xi \in C^0(\overline{\Omega}, \mathbb{R}^+)$ *and every* $(x_1, y_1, x_2, y_2) \in \Omega^4$, *one has:*

$$|c_\xi(x_1, y_1) - c_\xi(x_2, y_2)| \leq C \|\xi\|_{L^{q^*}(\Omega)} \left(|x_1 - x_2|^\alpha + |y_1 - y_2|^\alpha\right). \tag{12}$$

*Consequently, if* $(\xi_n)_n \in C^0(\overline{\Omega}, \mathbb{R}^+)^{\mathbb{N}}$ *is bounded in* $L^{q^*}$, *then* $(c_{\xi_n})_n$ *admits a subsequence that converges in* $C^0(\overline{\Omega} \times \overline{\Omega}, \mathbb{R}_+)$.

From, now on, we assume $q < 2$, and for a non-negative $\xi \in L^{q^*}(\Omega)$ we define

$$\overline{c}_\xi(x,y) = \sup\left\{c(x,y)\ : c \in \mathcal{A}(\xi)\right\},$$

where

$$\mathcal{A}(\xi) = \left\{\lim_n c_{\xi_n} \text{ in } C^0\ :\ (\xi_n)_n \in C^0(\overline{\Omega}),\ \xi_n \geq 0,\ \xi_n \to \xi \text{ in } L^{q^*}\right\}.$$

As for $L_\xi$, we have:

**Lemma 3** *Let us assume that $q < 2$. Let $Q \in \mathcal{Q}^q(\mu_0, \mu_1)$, $\xi$ be a non-negative element of $L^{q^*}$, and $(\xi_n)_n$ be a sequence of non-negative continuous functions that converges to $\xi$ in $L^{q^*}$, then:*

*(i)* *$(L_{\xi_n})_n$ converges strongly in $L^1(C, Q)$ to some limit which is independent of the approximating sequence $(\xi_n)_n$ and which will again be denoted $L_\xi$.*

*(ii)* *The following equality holds:*

$$\int_\Omega \xi(x) i_Q(x)\, dx = \int_C L_\xi(\sigma)\, dQ(\sigma). \tag{13}$$

*(iii)* *The following inequality holds for $Q$-a.e. $\sigma \in C$:*

$$L_\xi(\sigma) \geq \overline{c}_\xi(\sigma(0), \sigma(1)). \tag{14}$$

Characterization of optimal transport strategies:

**Theorem 2** *Let us assume that* $q < 2$ *and that* $H$ *is strictly convex. A transportation strategy* $(\overline{\gamma}, \overline{p})$ *is optimal if and only if, setting* $\overline{Q} := Q_{\overline{\gamma},\overline{p}}$ *and* $\overline{\xi} := H'(i_{\overline{Q}})$*, one has:*

1. $\overline{\gamma}$ *solves the Monge-Kantorovich problem:*

$$\inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\xi}}(x,y)d\gamma(x,y), \tag{15}$$

2. *for* $\overline{Q}$*-a.e.* $\sigma \in C$*, one has:*

$$L_{\overline{\xi}}(\sigma) = \overline{c}_{\overline{\xi}}(\sigma(0), \sigma(1)). \tag{16}$$

# Wardrop-like equilibria

congestion function $g : \mathbb{R}_+ \to \mathbb{R}_+$ continuous increasing and satisfies $az^{q-1} \leq g \leq b(z^{q-1}+1)$ for all $z \in \mathbb{R}_+$ and some $q \in (1,2)$. Then for any transportation strategy $(\gamma, p)$ such that $I_{\gamma,p} \in L^q$, the transportation cost function resulting from the strategy $(\gamma, p)$ is $\overline{c}_\xi$ for $\xi := g \circ I_{\gamma,p} \in L^{q^*}$.

**Définition 2** *A transportation strategy $(\overline{\gamma}, \overline{p})$ is an equilibrium if $I_{\overline{\gamma},\overline{p}} \in L^q$ and, setting $\overline{\xi} := g \circ I_{\overline{\gamma},\overline{p}}$ one has*

1. $L_{\overline{\xi}}(\sigma) = \overline{c}_{\overline{\xi}}(\sigma(0), \sigma(1))$ *for $Q_{\overline{\gamma},\overline{p}}$-a.e. $\sigma \in C$,*
2. $\overline{\gamma}$ *solves the Monge-Kantorovich problem:*

$$\inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\overline{\xi}}(x,y)\,d\gamma(x,y).$$

**Theorem 3** *Under the assumptions of this paragraph, there exists an equilibrium. Moreover* $(\overline{\gamma}, \overline{p})$ *is an equilibrium if and only if* $\overline{Q} := Q_{\overline{\gamma},\overline{p}}$ *solves the minimization problem:*

$$\inf_{Q \in \mathcal{Q}^q(\mu_0,\mu_1)} \int_\Omega H_g(i_Q(x))dx \quad \text{with } H_g(z) := \int_0^z g(s)ds, \ \forall z \in \mathbb{R}_+.$$

More general than Wardrop equilibrium (where $\gamma$ is fixed). Easily adapts to the case of a fixed $\gamma$, only changes the constraint on $Q$:

$$\int_C f(\sigma(0), \sigma(1))dQ(\sigma) = \int_{\overline{\Omega}\times\overline{\Omega}} f(x,y)d\gamma(x,y).$$

# Other formulations

## Duality

$\overline{\xi} := H'(i_{\overline{Q}})$ solves

$$\sup_{\xi \in L^{q^*},\, \xi \geq g(0)} W(\xi) - \int_\Omega H^*(\xi(x))dx \tag{17}$$

where $H^*$ is the Legendre-Fenchel transform of $H$ and:

$$W(\xi) := \inf_{\gamma \in \Pi(\mu_0, \mu_1)} \int_{\overline{\Omega} \times \overline{\Omega}} \overline{c}_\xi(x,y)d\gamma(x,y).$$

## Vector-valued measure formulation

For $Q \in \mathcal{Q}^q(\mu_0, \mu_1)$, define the vector measure $\vec{i}_Q$ by,
$\forall X \in C^0(\overline{\Omega}, \mathbb{R}^2)$:

$$\int_\Omega X(x) d\vec{i}_Q(x) = \int_C \left( \int_0^1 X(\sigma(t)) \cdot \dot{\sigma}(t) dt \right) dQ(\sigma), \ .$$

It is easy to check:

$$\operatorname{div}(\vec{i}_Q) = \mu_0 - \mu_1, \text{ and } \|\vec{i}_Q\| \leq i_Q.$$

Suggests to consider

$$\inf\left\{\int_\Omega H(p)\ :\ \operatorname{div}(p)=\mu_0-\mu_1\right\} \tag{18}$$

(slight abuse of notations $H(p)=H(|p|)$), unique minimizer $\overline{p}=\nabla H^*(\nabla\overline{u})$ where $\overline{u}$ solves the (typically degenerate) elliptic equation

$$\operatorname{div}(\nabla H^*(\nabla\overline{u}))=\mu_0-\mu_1.$$

By monotonicity of $H$, one has inf(10)$\geq$ inf(18).

Heuristic construction of an optimizer $\overline{Q}$: consider (as in Evans and Gangbo) the ODE

$$\dot{X}(t,x) = \frac{\overline{p}(X(t,x))}{(1-t)\mu_0(X(t,x)) + t\mu_1(X(t,x))}, \; X(0,x) = x.$$

and define $\overline{Q}$

$$\int_C F(\sigma) d\overline{Q}(\sigma) = \int_{\overline{\Omega}} F(X(.,x)) d\mu_0(x)$$

by construction, $i_{\overline{Q}} = |\overline{p}|$ and then

$$\int_\Omega H(i_{\overline{Q}}(x))dx = \inf(18)$$

so that formally $\overline{Q}$ is optimal. By suitable regularization inf(10)= inf(18).

# Numerical approximation

Start with the dual formulation

$$\inf_{\xi \in L^{q^*},\, \xi \geq \xi_0 = g(.,0)} J(\xi) = \int_\Omega H^*(x, \xi(x))dx - W(\xi)$$

to compute the optimal metric $\xi = \partial_i H(x, i_Q(x))$. Case of a fixed (discrete) transport plan $\gamma = \sum \gamma_{\alpha\beta}\delta_{(S_\alpha, T_\beta)}$:

$$W(\xi) := \sum \gamma_{\alpha\beta} c_\xi(S_\alpha, T_\beta).$$

Where $c_\xi(S, .)$ is the *viscosity* solution (or largest $W^{1,q^*}$ a.e. subsolution) of the Eikonal equation

$$\|\nabla \mathcal{U}\| = \xi; \quad \mathcal{U}_\xi(S) = 0 \tag{19}$$

(recall $q^* > 2$).

Space discretization, mesh size $h$, consistent (Souganidis, Barles-Souganidis, Rouy-Tourin) discretization of the Eikonal equation:

$$\left(\frac{\max\{(\mathcal{U}_{i,j}-\mathcal{U}_{i-1,j}),(\mathcal{U}_{i,j}-\mathcal{U}_{i+1,j}),0\}}{h_x}\right)^2 + \left(\frac{\max\{(\mathcal{U}_{i,j}-\mathcal{U}_{i,j-1}),(\mathcal{U}_{i,j}-\mathcal{U}_{i,j+1}),0\}}{h_y}\right)^2 = (\xi_{i,j})^2.$$

can be solved efficiently by Sethian's Fast Marching Method.

Notation : $c^h_\xi(S,T)$, discrete functional

$$J^h(\xi) = h^2 \sum_{i,j} H^*(i,j;\xi_{i,j}) - \sum_{r,s} c^h_\xi(S_\alpha,T_\beta)\gamma_{\alpha,\beta},$$

Note that each $J^h$ is convex.

$\Gamma$-convergence:

**Theorem 4** *The sequence of functionals* $J^h$ $\Gamma-$*converges with respect to the weak* $L^{q^*}$ *convergence to the limit functional* $J$. *Moreover, as the sequence* $(J^h)_h$ *is equi-coercive and every functional,* $J$ *included, is strictly convex, (strong) convergence of the unique minimizers and of the values of the minima is guaranteed.*

Solving the discrete problem by a subgradient descent method, $J^h$ involves a differentiable part and a convex homogenous one. Problem : compute at each iteration a subgradient of the second part. Not straightforward but possible recursively by a method that uses the same recursivity as the FMM.

Several other applications of this strategy to compute by FMM a subgradient of distances with respect to metrics: inverse problems in tomography, optimal design of obstacles to prevent mass transfer (Buttazzo)...