Optimal location of Dirichlet regions for elliptic PDEs

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"Transport, optimization, equilibrium in economics"
PIMS, July 14–20, 2008

We want to study shape optimization problems of the form

$$\min\Big\{F(\Sigma,u_\Sigma)\ :\ \Sigma\in\mathcal{A}\Big\}$$

where F is a suitable shape functional and $\mathcal{A}$ is a class of admissible choices. The function u_Σ is the solution of an elliptic problem

$$Lu=f \text{ in } \Omega \qquad u=0 \text{ on } \Sigma$$

or more generally of a variational problem

$$\min\Big\{G(u)\ :\ u=0 \text{ on } \Sigma\Big\}.$$

The cases we consider are when

$$G(u) = \int_\Omega \left(\frac{|Du|^p}{p} - f(x)u \right) dx$$

corresponding to the p-Laplace equation

$$\begin{cases} -\operatorname{div}\left(|Du|^{p-2} Du \right) = f & \text{in } \Omega \\ u = 0 \quad \text{on } \Sigma \end{cases}$$

and the similar problem for $p = +\infty$ with

$$G(u) = \int_\Omega \left(\chi_{\{|Du| \le 1\}} - f(x)u \right) dx$$

which corresponds to the Monge-Kantorovich equation

$$\begin{cases} -\operatorname{div}(\mu Du) = f & \text{in } \Omega \setminus \Sigma \\ u = 0 & \text{on } \Sigma \\ u \in \mathrm{Lip}_1 \\ |Du| = 1 & \text{on } \operatorname{spt} \mu \\ \mu(\Sigma) = 0. \end{cases}$$

We limit the presentation to the cases

$$p = +\infty \qquad \text{and} \qquad p = 2$$

occurring in mass transportation theory and in the equilibrium of elastic structures.

The case of mass transportation problems

We consider a given compact set $\Omega \subset \mathbf{R}^d$ (urban region) and a probability measure f on Ω (population distribution). We want to find Σ in an admissible class and to transport f on Σ in an optimal way.

It is known that the problem is governed by the Monge-Kantorovich functional

$$G(u) = \int_\Omega \Big(\chi_{\{|Du|\le 1\}} - f(x)u\Big)\,dx$$

which provides the shape cost

$$F(\Sigma) = \int_\Omega \mathrm{dist}(x, \Sigma)\, df(x).$$

Note that in this case the shape cost does not depend on the state variable u_Σ.

Concerning the class of admissible controls we consider the following cases:

• $\mathcal{A} = \Big\{\Sigma \ : \ \#\Sigma \leq n\Big\}$ called location problem;

• $\mathcal{A} = \Big\{\Sigma \ : \ \Sigma$ connected, $\mathcal{H}^1(\Sigma) \leq L\Big\}$ called irrigation problem.

Asymptotic analysis of sequences F_n
the Γ-convergence protocol

1. order of vanishing ω_n of $\min F_n$;

2. rescaling: $G_n = \omega_n^{-1} F_n$;

3. identification of $G = \Gamma$-limit of G_n;

4. computation of the minimizers of G.

The location problem

We call optimal location problem the minimization problem

$$L_n = \min \Big\{ F(\Sigma) \ : \ \Sigma \subset \Omega,\ \#\Sigma \le n \Big\}.$$

It has been extensively studied, see for instance

Suzuki, Asami, Okabe: Math. Program. 1991
Suzuki, Drezner: Location Science 1996
Buttazzo, Oudet, Stepanov: Birkhäuser 2002
Bouchitté, Jimenez, Rajesh: CRAS 2002
Morgan, Bolton: Amer. Math. Monthly 2002
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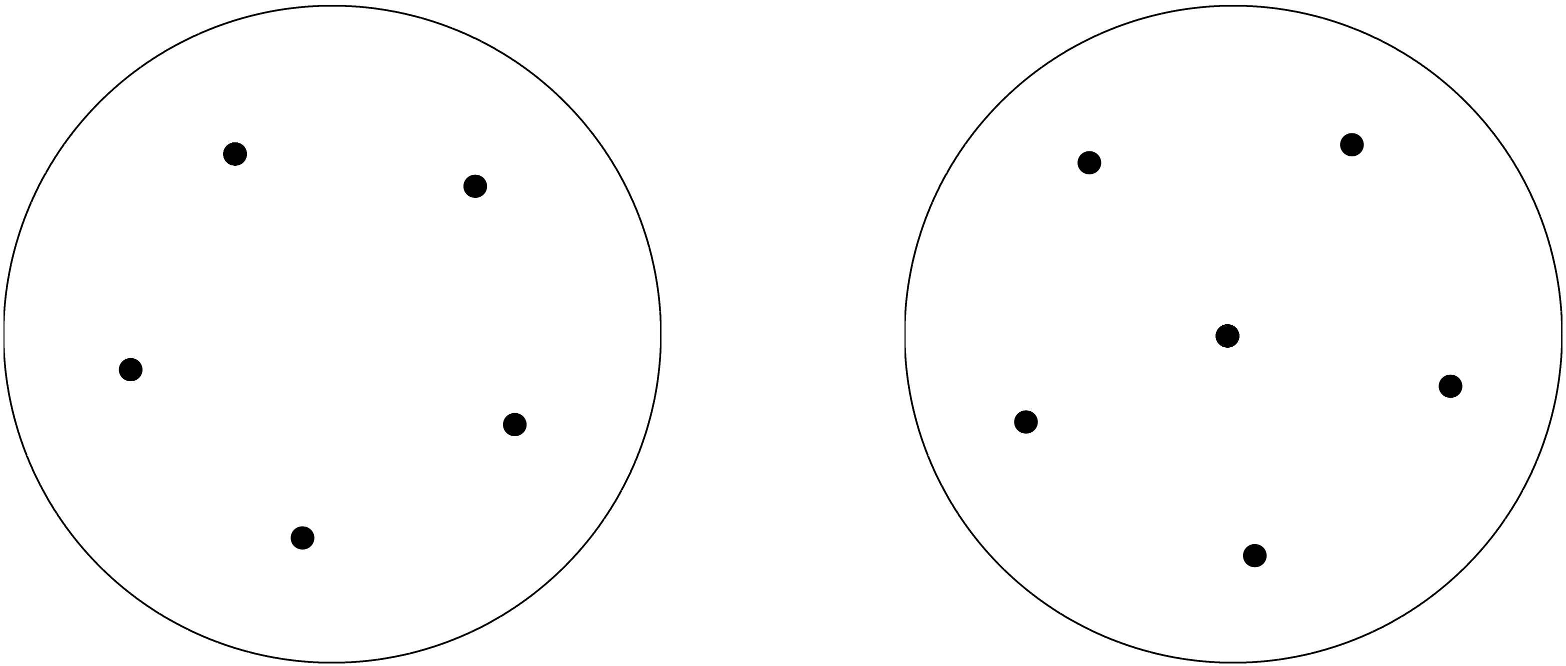


Optimal locations of 5 and 6 points in a disk for $f = 1$

We recall here the main known facts.

- $L_n \approx n^{-1/d}$ as $n \to +\infty$;
- $n^{1/d} F_n \to C_d \int_\Omega \mu^{-1/d} f(x)\, dx$ as $n \to +\infty$, in the sense of Γ-convergence, where the limit functional is defined on probability measures;

- $\mu_{opt} = K_d f^{d/(1+d)}$ hence the optimal configurations Σ_n are asymptotically distributed in Ω as $f^{d/(1+d)}$ and not as f (for instance as $f^{2/3}$ in dimension two).

- in dimension two the optimal configuration approaches the one given by the centers of regular exagons.

- In dimension one we have $C_1 = 1/4$.
- In dimension two we have

$$C_2 = \int_E |x|\,dx = \frac{3\log 3 + 4}{6\sqrt{2}\,3^{3/4}} \approx 0.377$$

where E is the regular hexagon of unit area centered at the origin.

- If $d \geq 3$ the value of C_d is not known.
- If $d \geq 3$ the optimal asymptotical configuration of the points is not known.
- The numerical computation of optimal configurations is very heavy.

• If the choice of location points is made randomly, surprisingly the loss in average with respect to the optimum is not big and a similar estimate holds, i.e. there exists a constant R_d such that

$$E\Big(F(\Sigma_N)\Big) \approx R_d N^{-1/d} \omega_d^{-1/d} \Big(\int_\Omega f^{d/(1+d)}\Big)^{(1+d)/d}$$

while

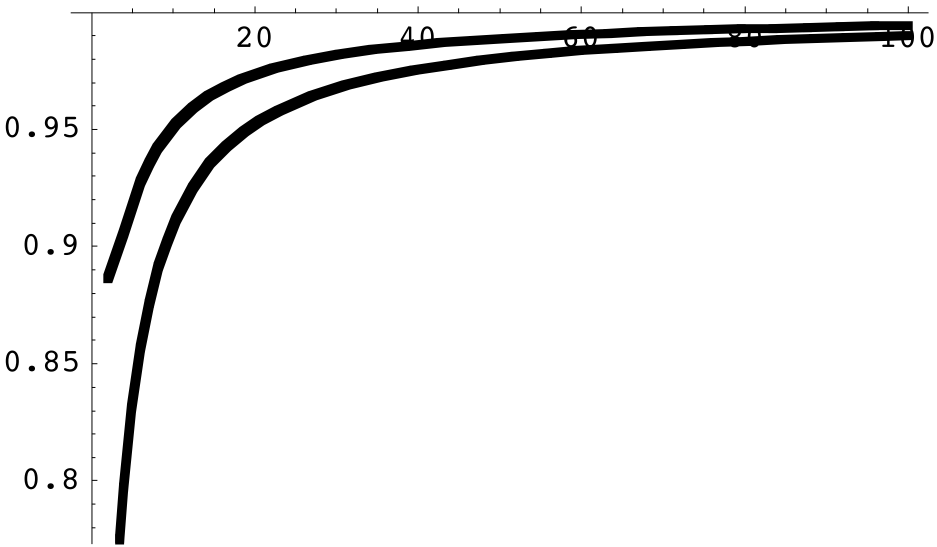
$$F(\Sigma_N^{opt}) \approx C_d N^{-1/d} \omega_d^{-1/d} \Big(\int_\Omega f^{d/(1+d)}\Big)^{(1+d)/d}$$

We have $R_d = \Gamma(1 + 1/d)$ so that

$C_1 = 0.5$ while $R_1 = 1$

$C_2 \simeq 0.669$ while $R_2 \simeq 0.886$

$\frac{d}{1+d} \leq C_d \leq \Gamma(1 + 1/d) = R_d$ for $d \geq 3$



Plot of $\frac{d}{1+d}$ and of $\Gamma(1 + 1/d)$ in terms of d

The irrigation problem

Taking again the cost functional

$$F(\Sigma) := \int_\Omega \mathrm{dist}(x, \Sigma)\, f(x)\, dx.$$

we consider the minimization problem

$$\min \Big\{ F(\Sigma) \ :\ \Sigma \text{ connected}, \mathcal{H}^1(\Sigma) \le \ell \Big\}$$

Connected onedimensional subsets Σ of Ω are called networks.

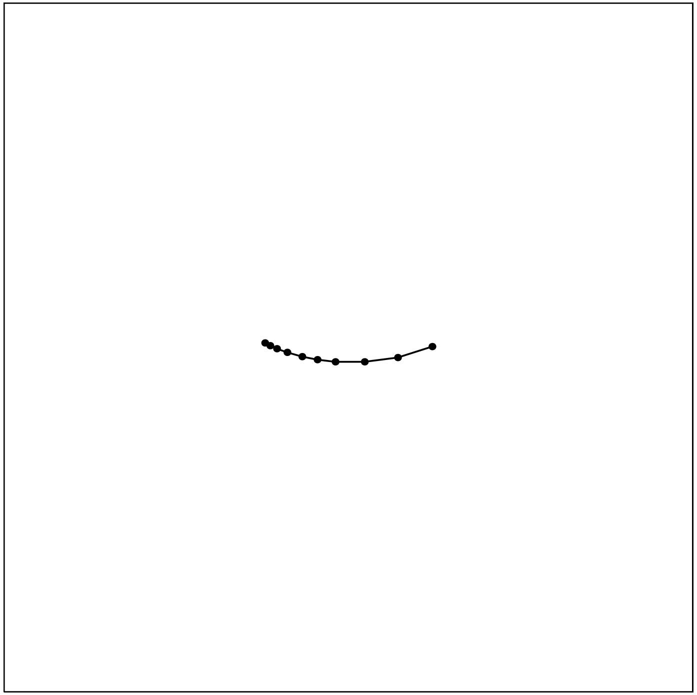
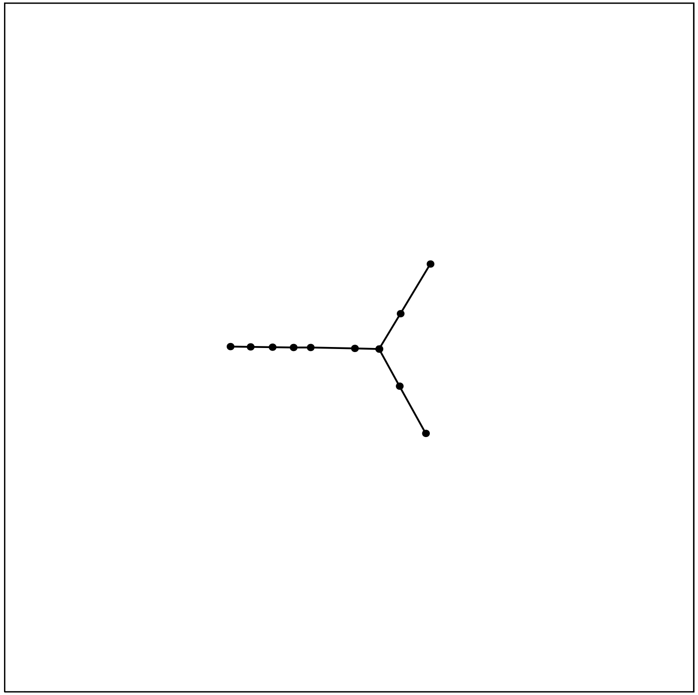
Theorem *For every $\ell > 0$ there exists an optimal network Σ_ℓ for the optimization problem above.*

Some necessary conditions of optimality on Σ_ℓ have been derived:

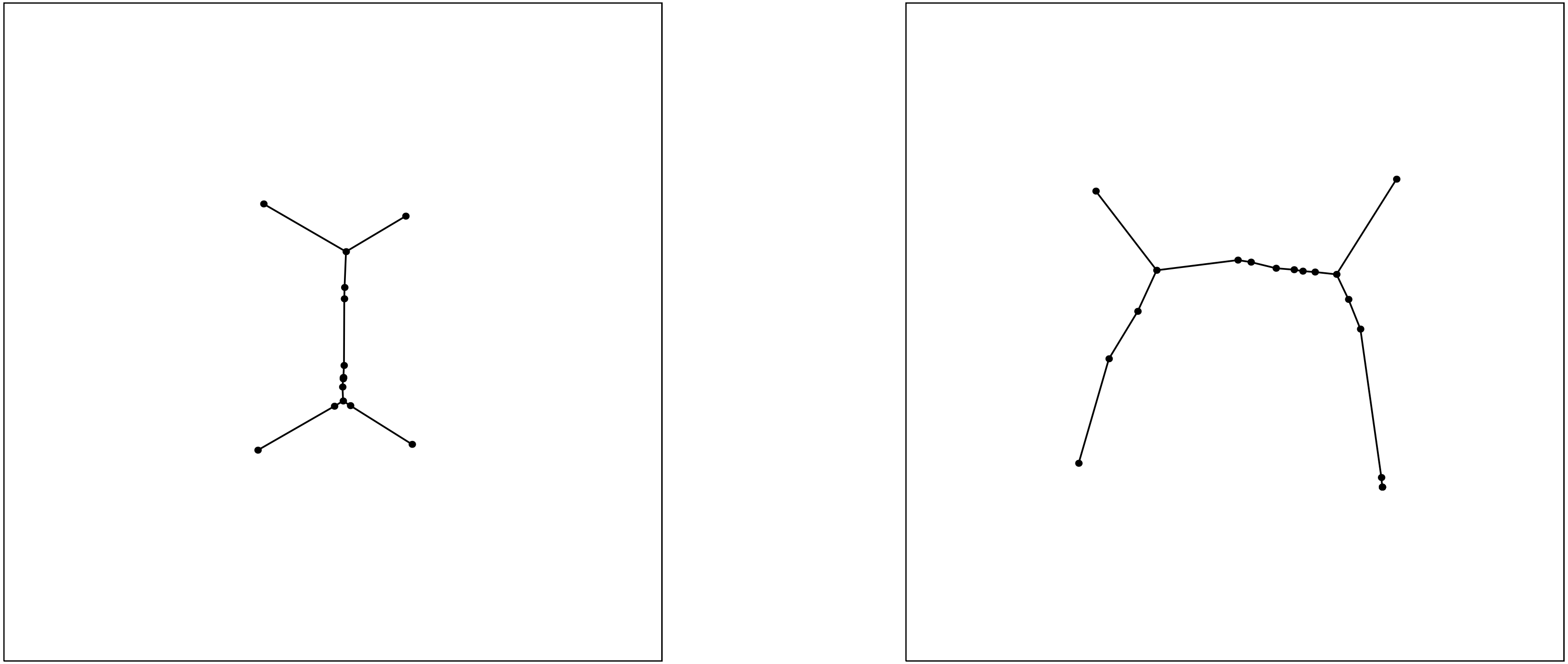
Buttazzo-Oudet-Stepanov 2002,
Buttazzo-Stepanov 2003,
Santambrogio-Tilli 2005
Mosconi-Tilli 2005
....

For instance the following facts have been proved:

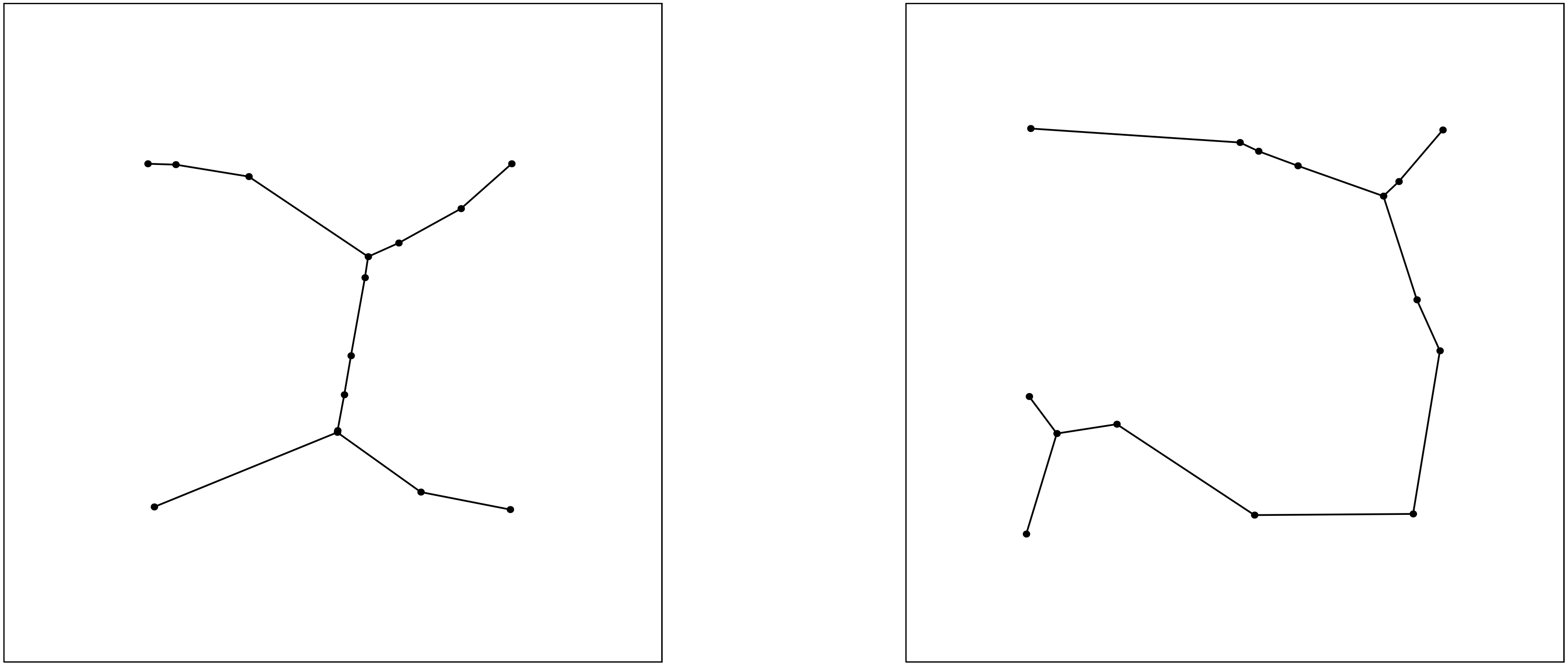
- no closed loops;
- at most triple point junctions;
- 120° at triple junctions;
- no triple junctions for small ℓ;
- asymptotic behavior of Σ_ℓ as $\ell \to +\infty$ (Mosconi-Tilli JCA 2005);
- regularity of Σ_ℓ is an open problem.

Optimal sets of length 0.5 and 1 in a unit square with $f = 1$.

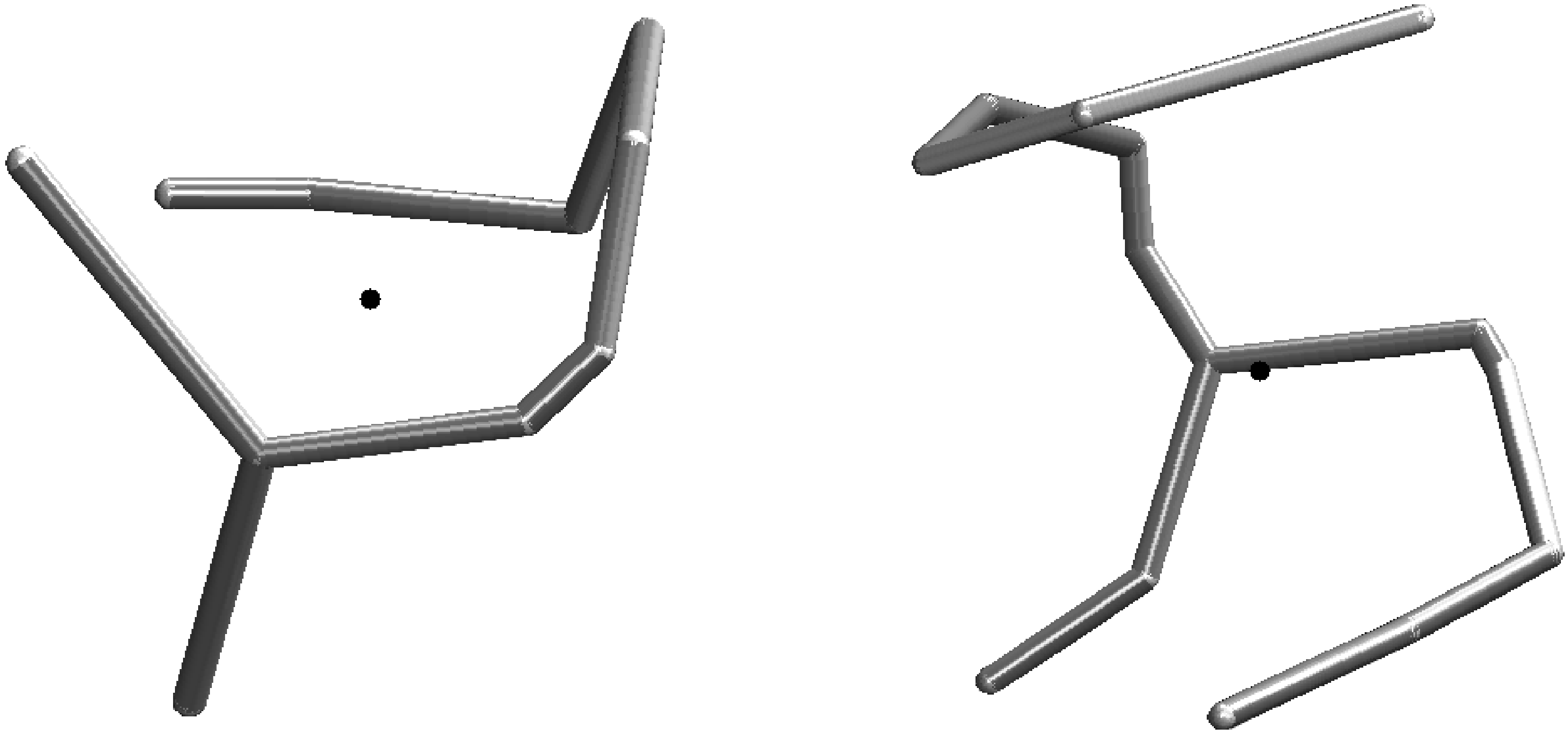


Optimal sets of length 1.5 and 2.5 in a unit square with $f = 1$.



Optimal sets of length 3 and 4 in a unit square with $f = 1$.

Optimal sets of length 1 and 2 in the unit ball of $\mathbf{R}^3$.



Optimal sets of length 3 and 4 in the unit ball of $\mathbf{R}^3$.

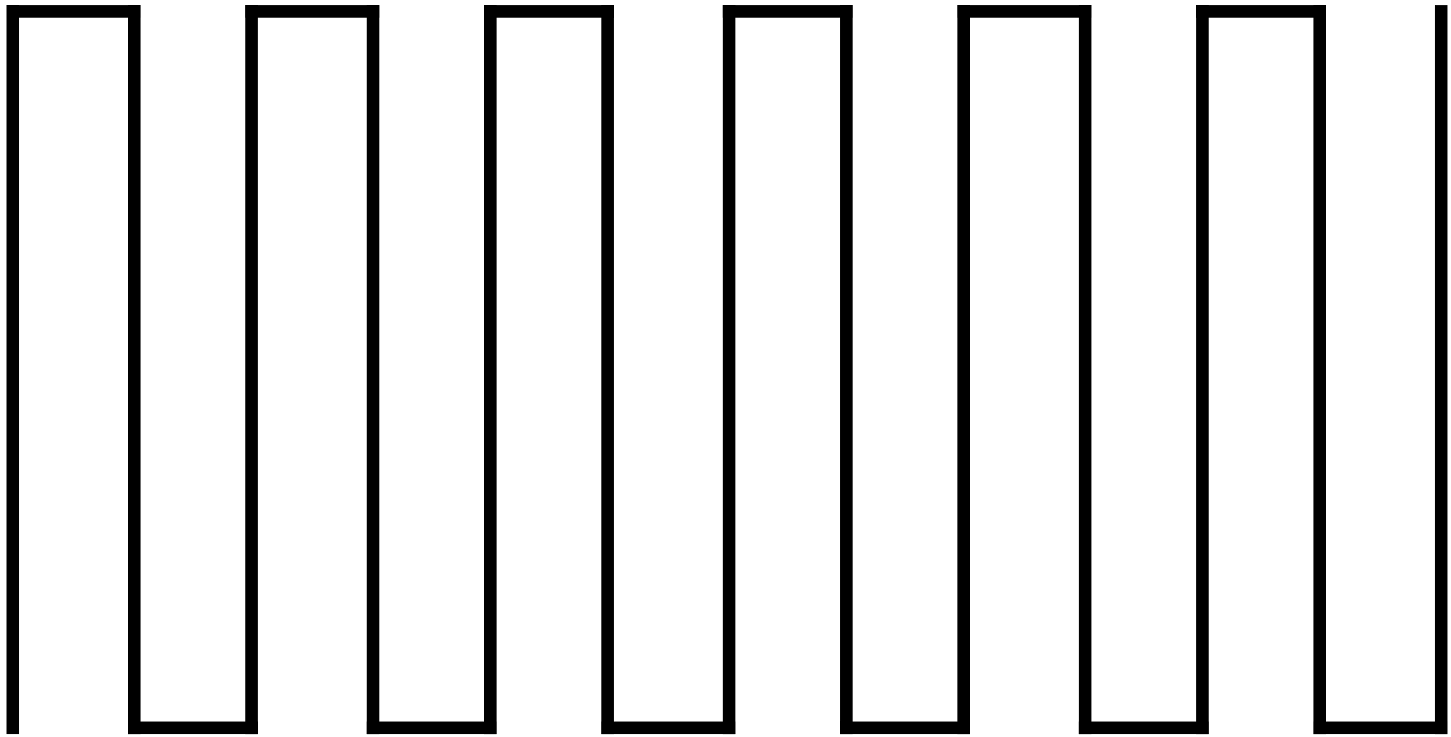
Analogously to what done for the location problem (with points) we can study the asymptotics as $\ell \to +\infty$ for the irrigation problem. This has been made by S.Mosconi and P.Tilli who proved the following facts.

- $L_\ell \approx \ell^{1/(1-d)}$ as $\ell \to +\infty$;

- $\ell^{1/(d-1)} F_\ell \to C_d \int_\Omega \mu^{1/(1-d)} f(x)\, dx$ as $\ell \to +\infty$, in the sense of Γ-convergence, where the limit functional is defined on probability measures;

- $\mu_{opt} = K_d f^{(d-1)/d}$ hence the optimal configurations Σ_n are asymptotically distributed in Ω as $f^{(d-1)/d}$ and not as f (for instance as $f^{1/2}$ in dimension two).

- in dimension two the optimal configuration approaches the one given by many parallel segments (at the same distance) connected by one segment.



Asymptotic optimal irrigation network in dimension two.

The case when irrigation networks are not a priori assumed connected is much more involved and requires a different setting up for the optimization problem, considering the transportation costs for distances of the form

$$d_\Sigma(x,y) = \inf\left\{A\Big(H^1(\theta \setminus \Sigma)\Big) + B\Big(\theta \cap \Sigma\Big)\right\}$$

being the infimum on paths θ joining x to y. On the subject we refer to the monograph:

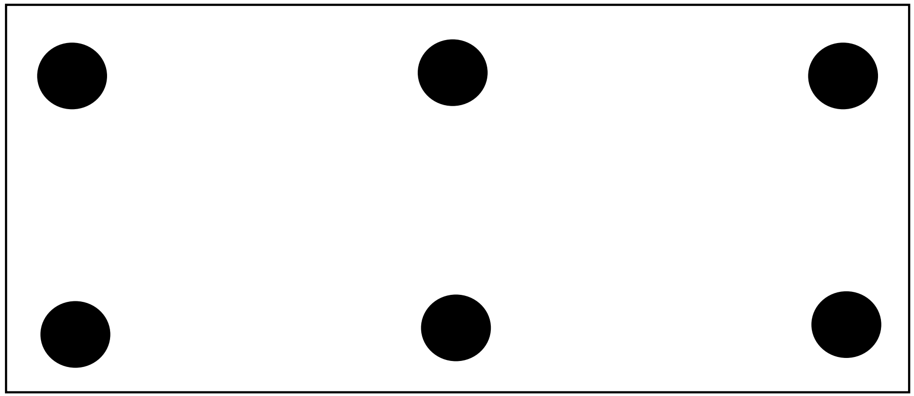
G. BUTTAZZO, A. PRATELLI, S. SOLIMINI, E. STEPANOV: *Optimal urban networks via mass transportation.* Springer Lecture Notes Math. (to appear).

The case of elastic compliance

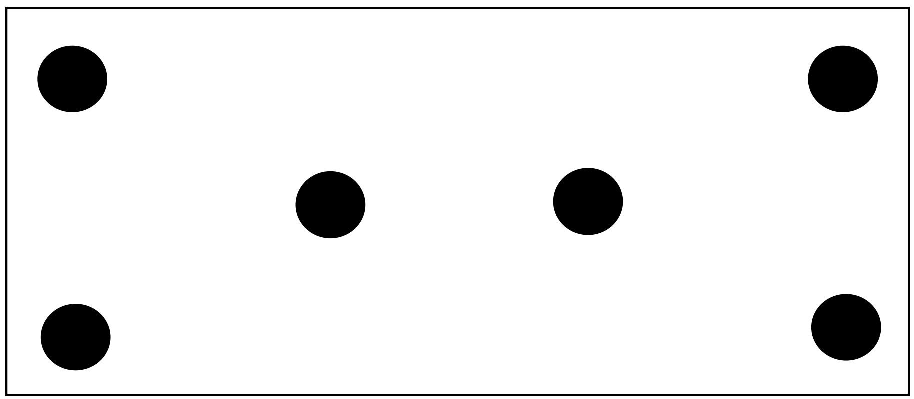
The goal is to study the configurations that provide the minimal compliance of a structure. We want to find the optimal region where to clamp a structure in order to obtain the highest rigidity.

The class of admissible choices may be, as in the case of mass transportation, a set of points or a one-dimensional connected set.

Think for instance to the problem of locating in an optimal way (for the elastic compliance) the six legs of a table, as below.



An admissible configuration for the six legs.



Another admissible configuration.

The precise definition of the cost functional can be given by introducing the elastic compliance

$$\mathcal{C}(\Sigma) = \int_\Omega f(x) u_\Sigma(x)\, dx$$

where Ω is the entire elastic membrane, Σ the region (we are looking for) where the membrane is fixed to zero, f is the exterior load, and u_Σ is the vertical displacement that solves the PDE

$$\begin{cases} -\Delta u = f & \text{in } \Omega \setminus \Sigma \\ u = 0 & \text{in } \Sigma \cup \partial\Omega \end{cases}$$

The optimization problem is then

$$\min \Big\{ \mathcal{C}(\Sigma) \ : \ \Sigma \text{ admissible} \Big\}$$

where again the set of admissible configurations is given by any array of a fixed number n of balls with total volume V prescribed.

As before, the goal is to study the optimal configurations and to make an asymptotic analysis of the density of optimal locations.

Theorem. *For every $V > 0$ there exists a convex function g_V such that the sequence of functional $(F_n)_n$ above Γ-converges, for the weak* topology on $\mathcal{P}(\overline{\Omega})$, to the functional*

$$F(\mu) = \int_\Omega f^2(x)\, g_V(\mu^a)\, dx$$

where μ^a denotes the absolutely continuous part of μ.

The Euler-Lagrange equation of the limit functional F is very simple: μ is absolutely continuous and for a suitable constant c

$$g'_V(\mu) = \frac{c}{f^2(x)}\,.$$

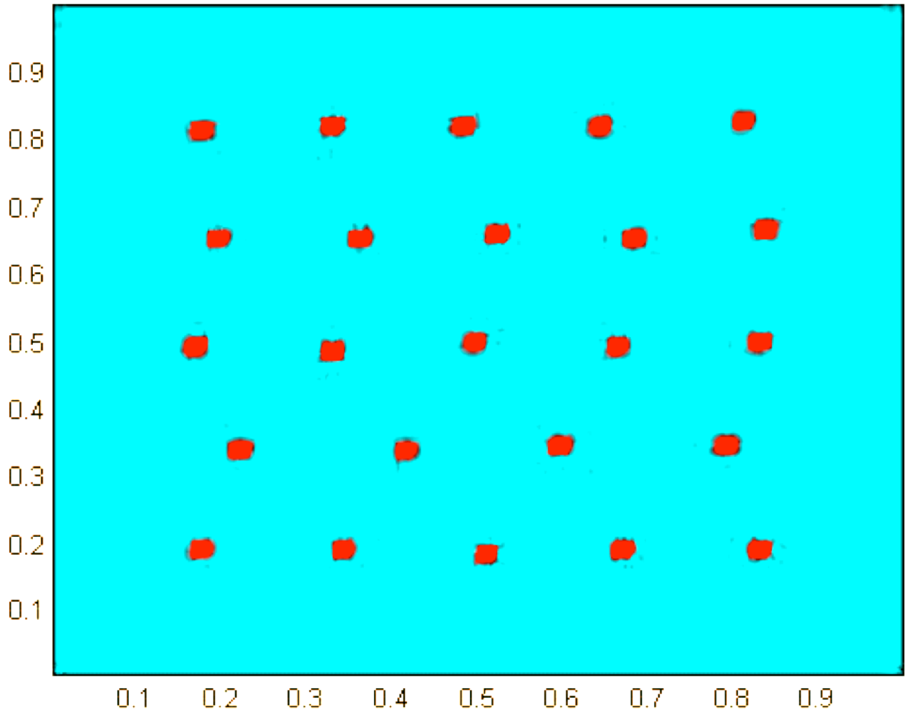
Open problems

- Exagonal tiling for $f = 1$?

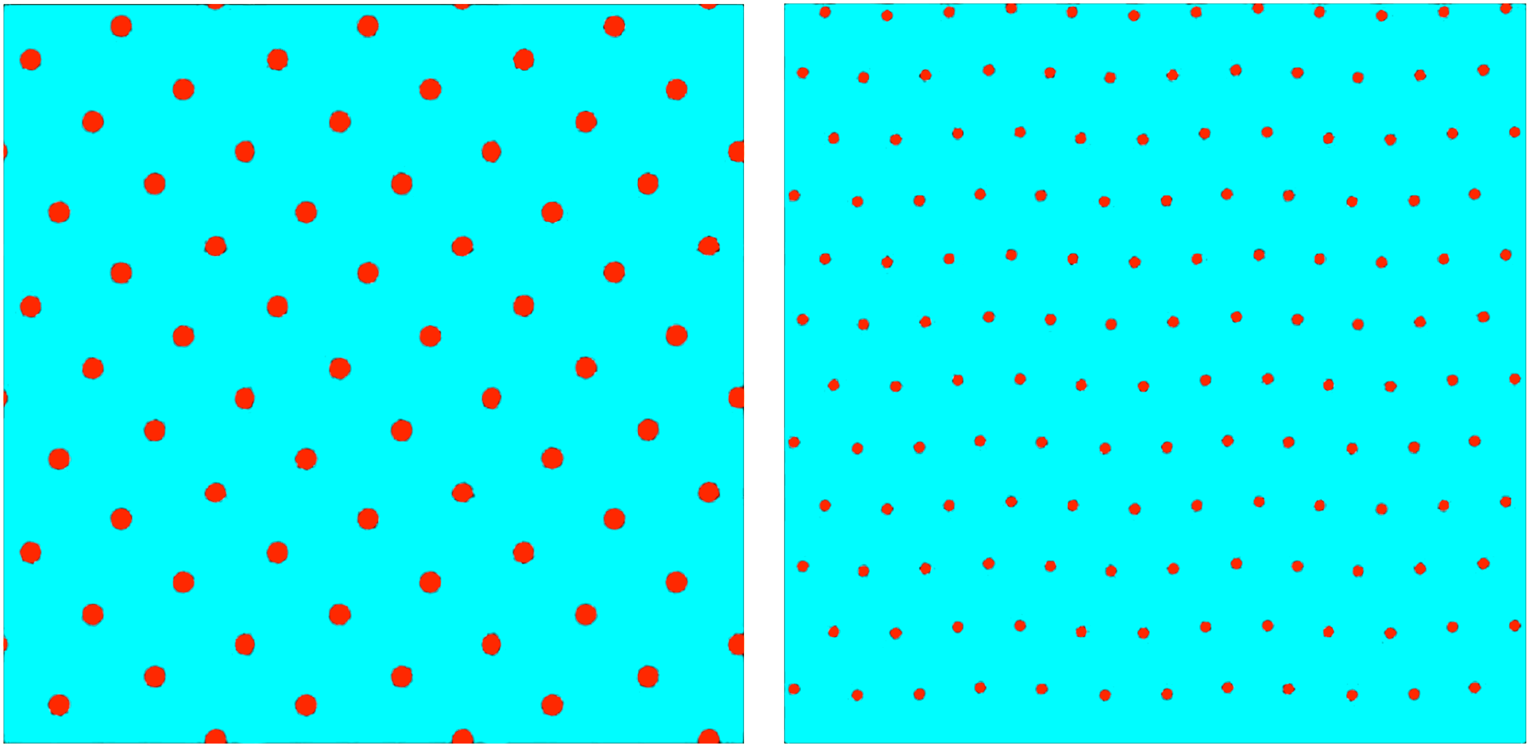
- Non-circular regions Σ, where also the orientation should appear in the limit.

- Computation of the limit function g_V.

- Quasistatic evolution, when the points are added one by one, without modifying the ones that are already located.



Optimal location of 24 small discs for the compliance, with $f = 1$ and Dirichlet conditions at the boundary.

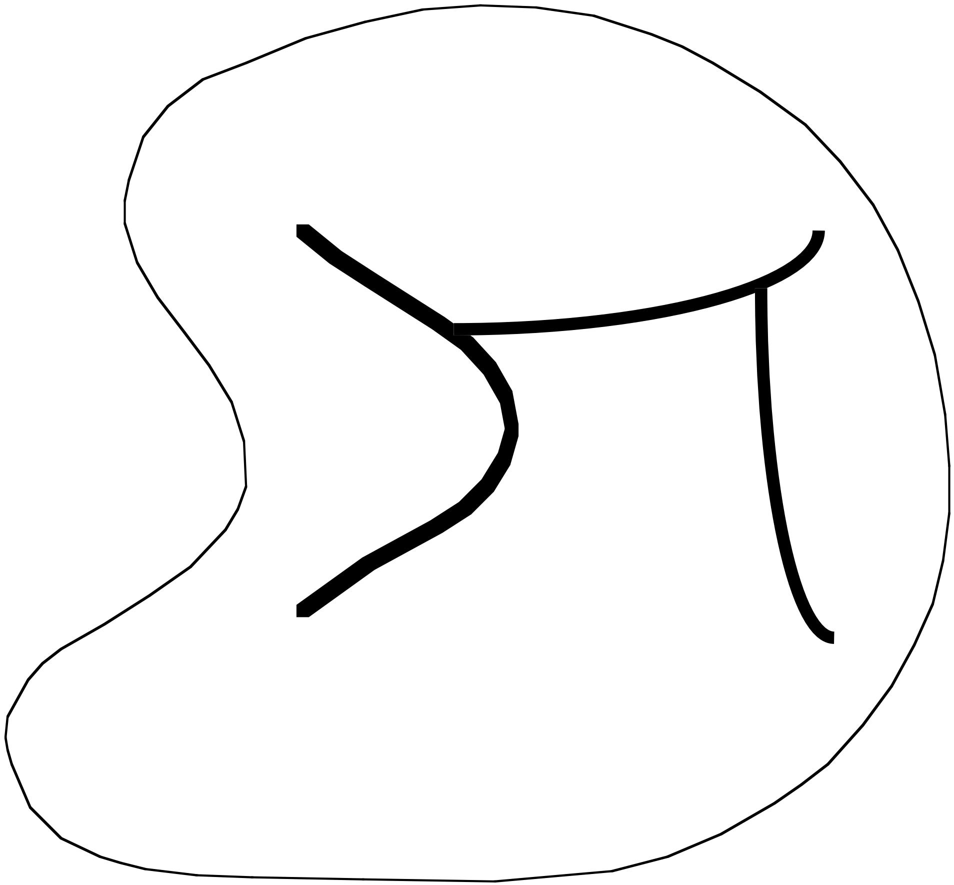


Optimal location of many small discs for the compliance, with $f = 1$ and periodic conditions at the boundary.

Optimal compliance networks

We consider the problem of finding the best location of a Dirichlet region Σ for a two-dimensional membrane Ω subjected to a given vertical force f. The admissible Σ belong to the class of all closed connected subsets of Ω with $\mathcal{H}^1(\Sigma) \leq L$.

The existence of an optimal configuration Σ_L for the optimization problem described above is well known; for instance it can be seen as a consequence of the Sverák compactness result.



An admissible compliance network.

As in the previous situations we are interested in the asymptotic behaviour of Σ_L as $L \to +\infty$; more precisely our goal is to obtain the limit distribution (density of lenght per unit area) of Σ_L as a limit probability measure that minimize the Γ-limit functional of the suitably rescaled compliances.

To do this it is convenient to associate to every Σ the probability measure

$$\mu_\Sigma = \frac{\mathcal{H}^1 \llcorner \Sigma}{\mathcal{H}^1(\Sigma)}$$

and to define the rescaled compliance functional $F_L : \mathcal{P}(\overline{\Omega}) \to [0, +\infty]$

$$F_L(\mu) = \begin{cases} L^2 \int_\Omega f u_\Sigma \, dx & \text{if } \mu = \mu_\Sigma, \ \mathcal{H}^1(\Sigma) \le L \\ +\infty & \text{otherwise} \end{cases}$$

where u_Σ is the solution of the state equation with Dirichlet condition on Σ. The scaling factor L^2 is the right one in order to avoid the functionals to degenerate to the trivial limit functional which vanishes everywhere.

Theorem. *The family of functionals* (F_L) *above* Γ*-converges, as* $L \to +\infty$ *with respect to the weak* topology on* $\mathcal{P}(\overline{\Omega})$*, to the functional*

$$F(\mu) = C \int_{\Omega} \frac{f^2}{\mu_a^2} \, dx$$

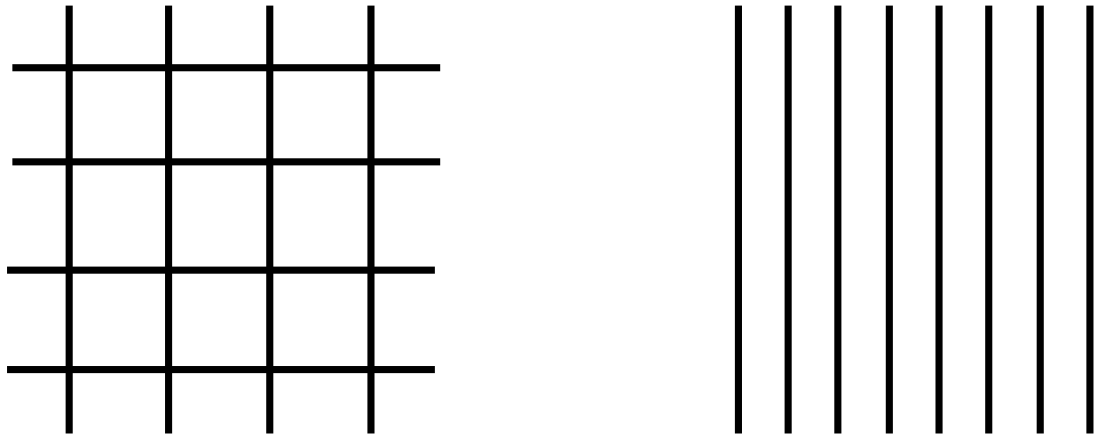
where C *is a constant.*

In particular, the optimal compliance networks Σ_L are such that μ_{Σ_L} converge weakly* to the minimizer of the limit functional, given by

$$\mu = c f^{2/3} \, dx.$$

Computing the constant C is a delicate issue.
If $Y = (0,1)^2$, taking $f = 1$, it comes from the formula

$$C = \inf \left\{ \liminf_{L \to +\infty} L^2 \int_Y u_{\Sigma_L} \, dx \ : \ \Sigma_L \text{ admissible} \right\}.$$



A grid is less performant than a comb structure, that we conjecture to be the optimal one.

Open problems

- Optimal periodic network for $f = 1$? This would give the value of the constant C.

- Numerical computation of the optimal networks Σ_L.

- Quasistatic evolution, when the length increases with the time and Σ_L also increases with respect to the inclusion (irreversibility).

- Same analysis with $-\Delta_p$, and limit behaviour as $p \to +\infty$, to see if the geometric problem of average distance can be recovered.

References

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