Optimal transportation and optimal control in a finite horizon framework

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MOTIVATIONS - A commitment problem (1)

- Micro labor market (e.g firm, university, hospital...), employees and tasks.

- Time period: $[0, T]$

- Employees and positions are described by a fixed number d of characteristics that evolve during the time period.

- The pair employee-position that results from the initial commitment $(x, y) \in \mathbb{R}^{2d}$ is then denoted by $(X_t^{x,y}, Y_t^{x,y}) \in \mathbb{R}^{2d}$ (stochastic OR deterministic evolutions)

MOTIVATIONS - A commitment problem (2)

- Assume that the initial densities $\mu_0(dx)$ and $\nu_0(dy)$ are data of the model (the company can evaluate the employees personnality and the jobs requirement)

- Define a transportation cost between a target employee and position (for instance: **the more similar are the characteristics** of the job and the worker, **the smaller is the cost** for the firm)

- Optimal commitment $=$ optimal initial coupling of a job to an employee (for a specific cost), not negociable during $[0, T]$.

MOTIVATIONS - A commitment problem (3)

- Given the two initial densities, the minimization problem consists in finding today an optimal commitment, it takes the general form:

$$\inf_{\pi_0 \in \Pi(\mu_0,\nu_0)} \inf_u \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbb{E}\left\{c\left(X_t^{x,y,u}, Y_t^{x,y,u}, u\right)\right\} d\pi_0(x,y)dt$$

$$\text{s.t. } d(X_t, Y_t) = b(X_t, Y_t, u_t)dt + \sigma(X_t, Y_t, u_t)dW_t$$

We will study two cases:

- The case of pure diffusion: $b \equiv 0, \sigma \equiv \text{constant} > 0$
- The case where the firm has the possibility to educate its employees: $b = (u, 0), \sigma \equiv 0$

Brownian case - The minimization problem (1)

- For the sake of simplicity, we look at the one dimensional case $d = 1$.

- Assume that the diffusion of the characteristics of the employees and positions have zero drift:

$$\begin{cases} d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \sigma d \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} \\ \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \end{cases} \tag{1}$$

Brownian case - The minimization problem (2)

- Where σ is the matrix of variance-covariance. We will focus on the uncorrelated case $\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$ (it can be generalized).

- Define $\nu := \frac{1}{2}(\sigma_1^2 + \sigma_2^2)$.

- W_t^1 and W_t^2 are two independant brownian motions.

- X_0 and Y_0 are respectively distributed according to μ_0 and ν_0, and assume that they have finite moment of order two.

Brownian case - The minimization problem (3)

- Define (π_t) (π_0 evolution) as a solution of the heat equation:

$$\begin{cases} \partial_t \pi - \nu \Delta \pi = 0 \text{ in } (0,T] \times \mathbb{R}^2 \\ \pi(0,.) = \pi_0 \in \Pi(\mu_0, \nu_0) \end{cases}$$

Let us call $\tilde{\Pi}(\mu_0, \nu_0, T)$ the set of solutions of this problem.

- By using Ito's lemma we obtain two equivalent formulations of the problem:

$$\inf_{\pi_0 \in \Pi(\mu_0,\nu_0)} \int_0^T \int_{\mathbb{R}\times\mathbb{R}} \mathbb{E}\left\{c\left(X_t^{x,y}, Y_t^{x,y}\right)\right\} d\pi_0(x,y)dt$$

$$\Leftrightarrow \inf_{(\pi_t) \in \tilde{\Pi}(\mu_0,\nu_0,T)} \int_0^T \int_{\mathbb{R}\times\mathbb{R}} c(x,y) d\pi_t(x,y)dt$$

Brownian case - Optimal transportation cost

- Let us define the heat kernel $G_t(x,y) = \frac{1}{2\sqrt{\pi t \nu}} e^{-\frac{x^2+y^2}{4t\nu}}$.

- Then it is easy to state the main result:

$$\inf_{\pi_0 \in \Pi(\mu_0,\nu_0)} \int_0^T \int_{\mathbb{R}\times\mathbb{R}} \mathbb{E}\left\{ c\left(X_t^{x,y}, Y_t^{x,y} \right) \right\} d\pi_0(x,y) dt$$

$$= \inf_{\pi_0 \in \Pi(\mu_0,\nu_0)} \int_{\mathbb{R}\times\mathbb{R}} \tilde{c}(x,y) d\pi_0(x,y)$$

- This is a **standard Monge-Kantorovich problem** with a **new transportation cost**: $\tilde{c}(x,y) \equiv \int_0^T (G_t * c)(x,y) dt$

Brownian case - Quadratic cost

- We now consider the particular case $c(x,y) = \frac{|x-y|^2}{2}$.
- A Gaussian-moment computation leads to:
  $\tilde{c}(x,y) = T \times \left(\frac{|x-y|^2}{2} + \nu\right)$
- And finally, we can write:

$$\inf_{\pi_0 \in \Pi(\mu_0,\nu_0)} \int_0^T \int_{\mathbb{R}\times\mathbb{R}} \mathbb{E}\left\{c\left(X_t^{x,y}, Y_t^{x,y}\right)\right\} d\pi_0(x,y)dt$$

$$= T\left(d_{MK}^2(\mu_0,\nu_0) + \nu\right)$$

- Where $d_{MK}^2(\mu_0,\nu_0)$ denotes the standard Monge-Kantorovitch distance.

Concluding remarks for the first case

- We turned the problem to a standard Monge-Kantorovich problem.

- If the drift is not null, then by using Girsanov theorem we can solve the problem in the same way

- In the next part, we change the evolution of employees and job requirements.

Optimal Transportation & Optimal Control (OT&OC)

- We now look at a situation in which, in one hand, the job requirements stay the same during all the time period.

- In the other hand, we consider that the firm has the possibility to educate its workers.

- We assume that (X_t, Y_t) evolve in $\mathbb{R}^d \times \mathbb{R}^d$, their dynamic is:

$$\begin{cases} \begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix} \\ \\ \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} \sim \begin{pmatrix} \mu_0(dx) \\ \nu_0(dy) \end{pmatrix} \end{cases}$$

OT&OC - The minimization problem

- We add a trainning cost (education effort, quadratic cost) and in the following, we will use the quadratic transportation cost (denoted by c, again...).

- Then the principal minimization problem (P*) reads:

$$\inf_{\pi_0 \in \Pi(\mu_0,\nu_0)} \inf_{u=\dot{X}} \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} c(X(t,x),y) d\pi_0(x,y) + \int_{\mathbb{R}^d} \frac{\left|\dot{X}(t,x)\right|^2}{2} d\mu_0(x) dt$$

OT&OC - Calculus of variations (1)

- Let us take an element $\pi_0 \in \Pi(\mu_0, \nu_0)$, and define π_0^x by the desintegration formula: $d\pi_0(x,y) := d\pi_0^x(y) d\mu_0(x)$.

- Define also $\Lambda^* : \pi \in M(\mathbb{R}^d \times \mathbb{R}^d) \mapsto (x \mapsto \int y d\pi^x(y)) \in L^2_{d\mu_0}(\mathbb{R}^d)$

- One can write the Euler equation related to the problem:

$$\inf_{u=\dot{X}} \int_0^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{|X(t,x)-y|^2}{2} d\pi_0^x(y) + \frac{\left|\dot{X}(t,x)^2\right|}{2} \right) d\mu_0(x) dt$$

OT&OC - Calculus of variations (2)

- So for a fixed π_0, we find:

$$X_{\pi_0}(t,x) = \alpha(t)x + (\alpha(t)-1)\Lambda^*\pi_0(x)$$

With $\alpha(t) = \frac{e^{t-2T}+e^{-t}}{1+e^{-2T}}$

- And finally we can **rewrite problem (**P***)**, up to a constant depending on T, μ_0 and ν_0 :

$$\inf_{\pi_0 \in \Pi(\mu_0,\nu_0)} \int_{\mathbb{R}^d} a(T)x\Big(\Lambda^*\pi_0(x)\Big) + b(T)\Big(\Lambda^*\pi_0(x)\Big)^2 d\mu_0(x)$$

OT&OC - Duality (1)

- Define $\Lambda : f \in L^2_{d\mu_0}(\mathbb{R}^d) \mapsto ((x,y) \mapsto yf(x)) \in C^0(\mathbb{R}^d \times \mathbb{R}^d)$
- and $\Phi(\Lambda f) = \inf_{\pi_0 \in \Pi(\mu_0,\nu_0)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \Lambda f(x,y) d\pi_0(x,y)$
- We intodoce the primal problem (P):

$$\sup_{f \in L^2(\mathbb{R}^d, d\mu_0)} \frac{-1}{4b(T)} \|f\|^2_{L^2_{d\mu_0}} + < \frac{Id}{2b(T)}, f > + K(T,\mu_0) + \Phi(\Lambda f)$$

OT&OC - Duality (2)

- Of course (P^*) is the dual of (P)
- Then by using convex duality, we have:

$$\inf(P^*) = \max(P)$$

- Before giving some optimality conditions let us denote:

$$F(\Lambda^*\pi_0) := \int_{\mathbb{R}^d} a(T)x\Big(\Lambda^*\pi_0(x)\Big) + b(T)\Big(\Lambda^*\pi_0(x)\Big)^2 d\mu_0(x)$$

$$-F^*(f) := \frac{-1}{4b(T)}\|f\|^2_{L^2_{d\mu_0}} + < \frac{Id}{2b(T)}, f > +K(T,\mu_0)$$

OT&OC - Duality (3)

In brief, we have the following equality:

$$\inf_{\pi_0 \in \Pi(\mu_0,\nu_0)} F(\Lambda^* \pi_0) = \max_{f \in L^2(\mathbb{R}^d, d\mu_0)} -F^*(f) + \Phi(\Lambda f)$$

where

$$\Phi(\Lambda f) = \inf_{\pi_0 \in \Pi(\mu_0,\nu_0)} \int_{\mathbb{R}^d \times \mathbb{R}^d} y f(x) d\pi_0(x,y)$$

OT&OC - Some optimality conditions

Proposition:

- $\bar{\pi_0}$ solution of (P*) and $\bar{f}$ solution of (P)
  $\Leftrightarrow F(\Lambda^* \bar{\pi_0}) + F^*(\bar{f}) = \Phi(\Lambda \bar{f})$

- If $\bar{f}$ solution of (P) and $\bar{\pi_0}$ solution of (P*), then $\bar{\pi_0}$ is a minimizer for $\Phi(\Lambda \bar{f})$

- If $\bar{f}$ solution of (P) and $\bar{\pi_0}$ minimize $\Phi(\Lambda \bar{f})$, then $\bar{\pi_0}$ is a solution of (P*)

OT&OC - A subgradient method to solve (P)

1. initialization: f_0

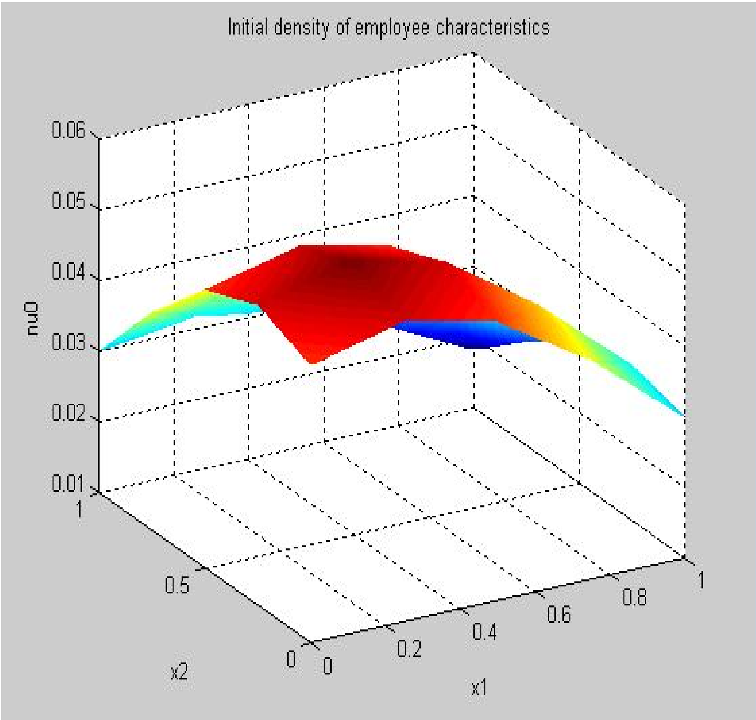
2. Define a step ρ_k s.t. $\sum \rho_k = \infty$ and $\sum \rho_k^2 < \infty$

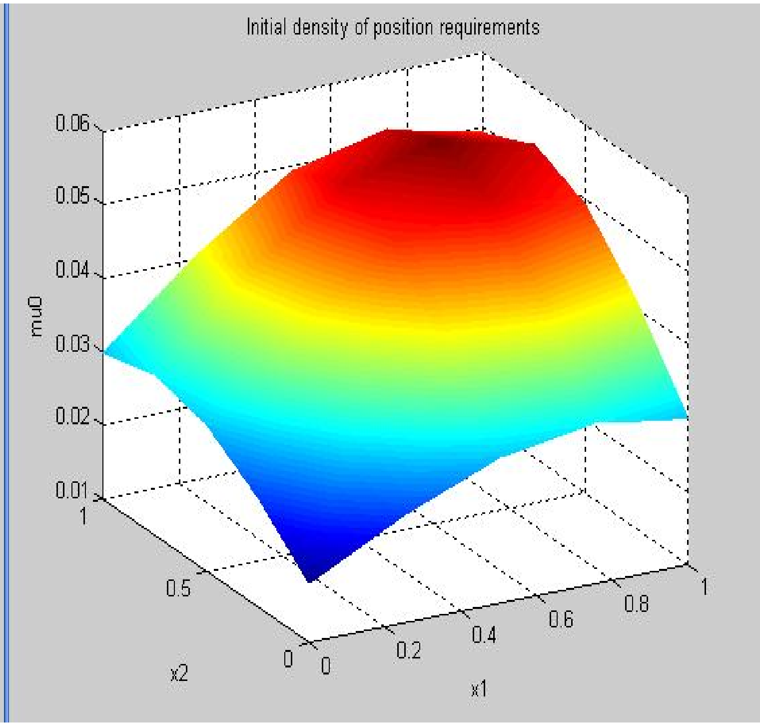
3. compute $(\Lambda^* \bar{\pi}_0)_k$ by solving $\Phi(\Lambda f_k)$ (e.g. simplex)

4. $f_{k+1} = f_k - \rho_k d_k$ where $d_k = \frac{-1}{2b(T)} f_k + \frac{x}{2b(T)} + (\Lambda^* \bar{\pi}_0)_k$

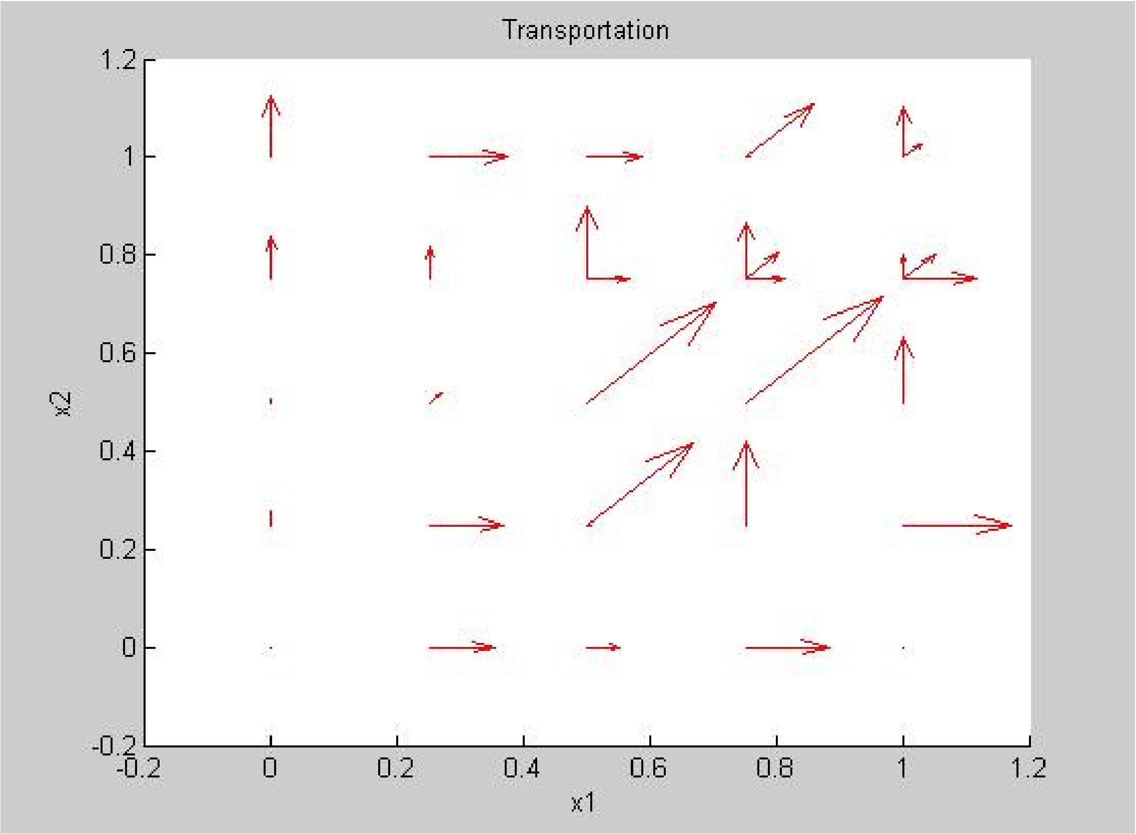
5. if: $\left\| f_{k+1} - f_k \right\| < \mathsf{TOL}$ stop
   else: $\rightarrow$ 2.

OT&OC - Simulation with d=2 - μ_0 and ν_0





OT&OC - Simulation with d=2 - Transportation for T=1



OT&OC - Concluding remarks

- We have studied optimal transportation problems applied to the economics with dynamical framework (time finite period).

- We gave closed formula for two cases:
  brownian diffusion
  optimal control

The main extensions are:

- to do a general existence proof

- to obtain interesting simulations for the model

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