An Introduction to Probability and Stochastic Processes for Ocean, Atmosphere, and Climate Dynamics

3: EOFs - A Cautionary Tale

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Introduction

- We have seen that Empirical Orthogonal Functions (EOFs) are defined statistically, as the eigenvectors of the covariance matrix of a system
- These eigenvectors provide a convenient basis for describing variability. In particular, projection on subspace of leading EOFs provides useful dimensionally-reduced description
- EOFs are often interpreted physically as **independent**, **dynamical** structures (despite being statistical by construction)
- EOFs follow from dynamics, but are not necessarily themselves of individual dynamical significance
- That is, EOFs efficiently characterise variance, but generally don't tell you anything about how the distribution of variance is produced by the dynamics

Notation

- Consider vector process $\mathbf{x}(t)$
- Can assume without loss of generality that $\text{mean}(\mathbf{x}) = \mathbf{0}$
- EOFs are eigenvectors of covariance matrix $C_{xx} = \mathsf{E}\{\mathbf{x}\mathbf{x}^T\}$:

$$C_{xx}\mathbf{E}_j = \mu_j \mathbf{E}_j$$

- Principal component (PC) time series are projection coefficients of $\mathbf{x}(t)$ on EOFs:

$$\alpha_j(t) = \mathbf{x}(t) \cdot \mathbf{E}_j$$

- When $\mathbf{E}_j$ are normalised so $\mathbf{E}_j \cdot \mathbf{E}_j = 1$,

$$\mu_j = \text{var}(\alpha_j)$$

EOFs: Basic Important Facts

1. C_{xx} symmetric matrix so EOFs $\mathbf{E}_j$ are mutually orthogonal:

$$\mathbf{E}_j \cdot \mathbf{E}_k = \delta_{jk}$$

2. Principal component time series $\alpha_j(t)$ are uncorrelated:

$$\mathsf{E}\{\alpha_j \alpha_k\} = \mu_j \delta_{jk}$$

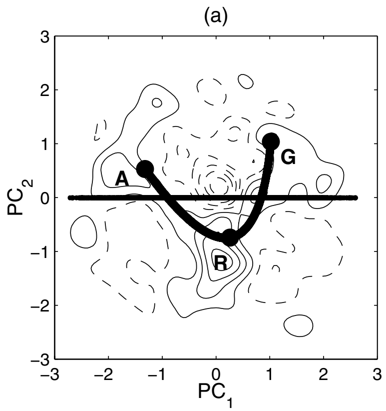
3. EOFs can be obtained variationally as basis vectors providing optimal low-dimensional approximations to full dataset: e.g. $\mathbf{E}_1$ defined as vector minimising

$$J = \mathsf{E}\left\{ ||\mathbf{x} - (\mathbf{x} \cdot \mathbf{E_1})\mathbf{E_1}||^2 \right\}$$

subject to the constraint that $||\mathbf{E}_1|| = 1$. EOFs maximise **global** variance; do not generally key into local features.

PC time series: uncorrelated vs. independent

- PC time series *uncorrelated*, but not necessarily *independent*



- EOFs of NH 500 hPa extratropical geopotential height
- Contours:

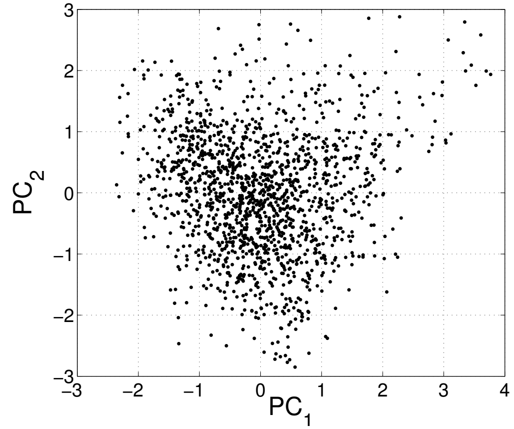
$$p(PC_1, PC_2) - p(PC_1)p(PC_2)$$

(should vanish for independent PCs)

- Thick solid line: optimal 1D *nonlinear* approximation

PC time series: uncorrelated vs. independent

- PC time series *uncorrelated*, but not necessarily *independent*
- EOFs of tropical Pacific SST



Mixing localised structures (Ambaum et al. 2001)

- EOFs maximise **global** variance over analysis domain, so will mix localised but unrelated structures
- An example: variables x, y, z such that x, z are independent with unit variance and $y = -x - z$ have covariance matrix:

$$C = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

and EOFs

$$\begin{aligned} E_1 &= (1, -2, 1) \text{ (75\% variance)} \\ E_2 &= (-1, 0, 1) \text{ (25\% variance)} \end{aligned}$$

- Unrelated variables x, z mixed together in E_1

EOFs of Multivariate O-U Processes (North 1984)

- Consider linear SDE

$$\frac{d}{dt}\mathbf{x} = A\mathbf{x} + B\dot{\mathbf{W}}$$

- Stationary covariance C_{xx} satisfies

$$AC_{xx} + C_{xx}A^T = -BB^T$$

- Suppose A is normal (i.e. $AA^T - A^TA = 0$), so it has a complete set of orthogonal eigenvectors
- Orthogonal matrix S diagonalises A: $\lambda_i \delta_{ij} = (SAS^T)_{ij}$
- Can solve Lyapunov equation for covariance: $C_{xx} = S^T G S$ where

$$G_{ij} = -\frac{(BB^T)_{ij}}{\lambda_i + \lambda_j}$$

EOFs of multivariate O-U processes (North 1984)

- If noise forcing is spatially unstructured so $BB^T = \beta^2 I$,

$$G_{ij} = -\frac{\beta^2 \delta_{ij}}{2\lambda_i}$$

so C_{xx} and A have same eigenvectors (both diagonalised by S)

- Thus, for a *linear, normal* system forced by *spatially uncorrelated* noise, the EOFs and the dynamical modes will coincide
- If the noise is spatially correlated, EOFs and dynamical modes will differ
- If the linear dynamics are non-normal, dynamical modes will be non-orthogonal $\Rightarrow$ cannot coincide with EOFs
- Even in the simplest physical systems, EOFs do not generally correspond to dynamical modes

EOFs of highly nonlinear systems (Mo&Ghil 1987)

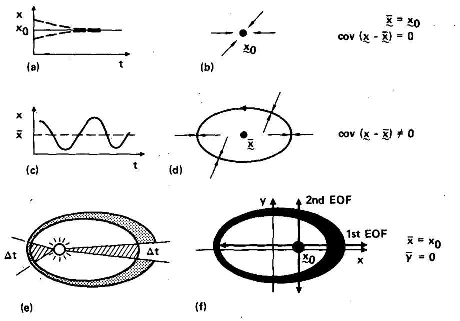
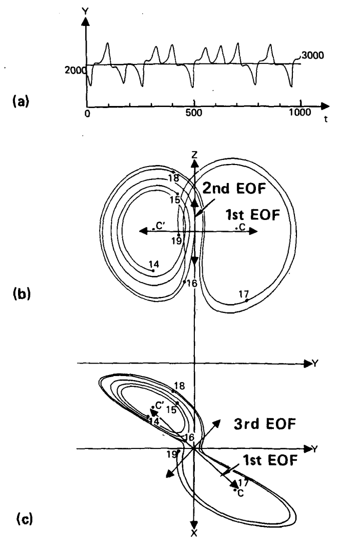
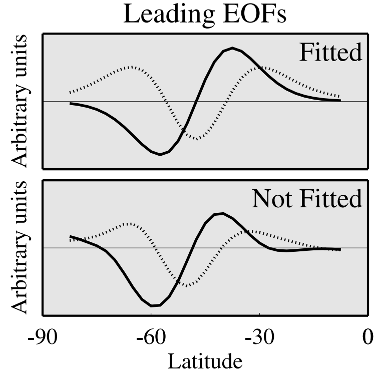


FIG. 1. Long-time behavior of solutions to forced dissipative models of nonlinear dynamics. Panels (a) and (c) indicate how one variable, x, behaves in time t, with $\bar{x}$ indicating the long-time mean. Panels (b) and (d) indicate the phase portrait of the attractor set: fixed point (b) or limit cycle (d), and the directions of stability and motion. Panels (e) and (f) show in further detail the illustrative case of elliptic motion of a planet around the Sun, with time spent along different portions of the orbit (e), and the EOFs which obtain (f). The dark shading in (e) and (f) is proportional in thickness to the local density ρ of successive points equally spaced in time, with $\rho \sim 1/v$, v being the local tangent velocity (from Ghil, 1987).



EOFs and kinematics of a zonal jet

- EOFs not dynamical modes, but are they at least related in a simple way to "natural kinematic degrees of freedom"?
- Address this question in terms of EOFs of extratropical zonal-mean zonal-wind



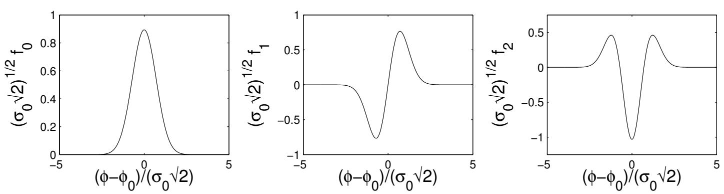
- Leading EOF of zonal-mean zonal wind is a dipole
- Model winds as Gaussian jet fluctuating in strength, position, and width

$$u(x,t) = U_0(t) \exp\left(-\frac{1}{2}\frac{(x - x_c(t))^2}{\sigma(t)^2}\right)$$

EOFs and kinematics of a zonal jet

- Observed fact: $h = \mathrm{std}(x_c)/\mathrm{mean}(\sigma) << 1$ (i.e. position fluctuations generally smaller than mean jet width)

$\Rightarrow$ can solve for EOFs analytically in terms of basis functions $f_i(x)$



Zonal Wind: Fluctuations in Single Variables

- **Strength Alone**: only one nontrivial PCA mode

$$E_u^{(1)}(x) = f_0(x) \quad \textit{monopole}$$

- **Position Alone**: if $x_c(t)$ unskewed

$$\begin{aligned} E_u^{(1)}(x) &= f_1(x) \quad \textit{dipole} \\ E_u^{(2)}(x) &= f_2(x) \quad \textit{tripole} \end{aligned}$$

but skewness in $x_c(t)$ *mixes* dipole and tripole

- **Width Alone**: leading EOF pattern

$$E_u^{(1)}(x) = x f_1(x) = f_2(x) + \frac{2}{\sqrt{3}} f_0(x)$$

Zonal Wind: Strength & Position Fluctuations

- For $U(t)$, $x_c(t)$ independent leading EOF patterns mix monopole and tripole:

$$\begin{aligned} E_u^{(1)}(x) &= f_1(x) && \textit{dipole} \\ E_u^{(2)}(x) &= \beta_0^{(+)} f_0(x) + \beta_2^{(+)} f_2(x) && \textit{mono/tripole hybrid} \\ E_u^{(3)}(x) &= \beta_0^{(-)} f_0(x) + \beta_2^{(-)} f_2(x) && \textit{mono/tripole hybrid} \end{aligned}$$

- Leading PC time series couple position & strength fluctuations:

$$\alpha_u^{(1)}(t) \sim U(t) x_c(t) + \text{h.o.t.}$$

- Even in this simple case, there is no generic relation between EOF modes and kinematic degrees of freedom

So now what?

- We have seen that EOF modes in general:
  - are not independent
  - mix localised but unrelated structures
  - do not correspond to linear or nonlinear "dynamical modes"
  - are not simply related to kinematic degrees of freedom
- So is EOF analysis rubbish? No - it's useful for characterising variance & reducing data dimensionality
- Can try to get around limitations (e.g. with rotated EOFs or nonlinear PCA); this has its uses (and its own abuses)
- In general: it's just best to not apply statistics blindly, and to select & interpret analyses based on an understanding of the system (so the medium doesn't become the message)

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