SDEs: A first example

- We see in this example
  i. a deep connection between diffusion of probability described by FPE and pathwise evolution described by SDE
  ii. a connection between the “drift” and the “deterministic” part of the dynamics
  iii. a connection between the “diffusion” and the “stochastic” part of the dynamics
  iv. the stationary pdf includes contributions from both deterministic & stochastic terms. Fluctuations drive system away from deterministic attractor, dynamics pushes it back: stationary pdf balances these tendencies.
- These ideas important enough to help get Albert Einstein his Nobel Prize (for early work on the subject) ...

Frankignoul & Hasselmann 1977

- These ideas also at heart of first stochastic climate model (Frankignoul & Hasselmann 1977), used to explain why SST spectra are generally red in character
- Simple model: SST anomalies driven by “fast” atmospheric forcing (assumed to be white in time) and “slow” relaxation to climatology through ocean mixed layer physics
- With $T = \text{SST anomaly}$, have model
  \[
  \frac{dT}{dt} = -\lambda T + \gamma \dot{W}
  \]
  which predicts red SST response to fast, rapidly-decorrelating atmospheric forcing

Frankignoul & Hasselmann 1977

- Observationally, SST variability “red” and fluxes are “white”

Multivariate Ornstein-Uhlenbeck processes

- Have so far focused on scalar stochastic processes; can generalise results so far for $x \in \mathbb{R}^N$
- Vector white noise $\dot{W}(t)$ is an array of independent white noise processes
  \[
  \mathbb{E}\{\dot{W}_i(t)\dot{W}_j(t')\} = \delta_{ij}\delta(t-t')
  \]
- For constant matrices $A$ and $B$, SDE
  \[
  \frac{d}{dt} x = Ax + B\dot{W}
  \]
  has solution (with $P(t) = e^{At}$)
  \[
  x(t) = P(t)x(0) + \int_0^t P(t-s)B\dot{W}(s) \, ds
  \]
Multivariate Ornstein-Uhlenbeck processes

- Solution is Gaussian with mean and stationary autocovariance

\[
E\{x(t)\} = P(t)x(0)
\]
\[
C_{xx}(\tau) = E\{x(t+\tau)x^T(t)\} = P(\tau)C_{xx}(0)
\]

where stationary covariance satisfies Lyapunov equation

\[
AC_{xx}(0) + C_{xx}(0)A^T = -BB^T
\]

⇒ balance between deterministic dynamics (A), stochastic forcing (B) and statistics of response (C_{xx}(0)); an example of a fluctuation-dissipation relationship

- pdf \(p(x)\) satisfies Fokker-Planck equation

\[
\partial_t p = -\sum_i A_i \partial_i p + \sum_{ij} (BB^T)_{ij} \partial^2_{ij} p
\]

Optimal Perturbations

- If dynamical matrix A is non-normal, i.e.

\[
AA^T \neq A^T A
\]

so eigenvectors of A are not orthogonal, then:

- EOFs (eigenvectors of covariance \(C_{xx}(0)\)) do not coincide with eigenvectors of A (dynamical modes)

- perturbation norm

\[
N(t) = x(t)^T M x(t) = x(0)^T P(t)^T M P(t) x(0)
\]

may grow (by potentially large amount) over finite times even though asymptotically stable:

\[
\lim_{t \to \infty} N(t) = 0
\]

Linear inverse modelling

- How are A and B determined in practice?

1. **Empirical: Linear Inverse Modelling**

- estimate covariances from observations and compute

\[
A = \frac{1}{\tau} \ln \left( C_{xx}(\tau)C_{xx}(0)^{-1} \right)
\]

- Compute B from Lyapunov equation

\[
AC_{xx}(0) + C_{xx}(0)A^T = -BB^T
\]

- Issues:

  i. if \(x(t)\) not truly Markov, estimates will depend on lag \(\tau\)

  ii. must enforce positive-definiteness of \(BB^T\)

- Defining amplification factor

\[
n(t) = \frac{x(0)^T P(t)^T M P(t) x(0)}{x(0)^T M x(0)}
\]

optimal perturbation \(e\) maximises \(n(t)\) subject to constraint \(x(0)^T M x(0) = 1\) ⇒ generalised eigenvalue problem:

\[
P(t)^T M P(t) e = \lambda M e
\]

- Response to fluctuating forcing:

\[
\text{var}(x^T(t) M x(t)) = B^T \left( \int_0^t P(s)^T M P(s) ds \right) B
\]

⇒ importance of projection of noise structure on “average” optimals for maintaining variance; “stochastic optimals”
Stochastic ENSO Models

2. Mechanistic: Stochastic Reduction of Tangent Linear Model
   - Partition dynamics into “slow” and “fast” variables $x$ and $y$:
     \[
     \frac{dx}{dt} = f(x, y) \\
     \frac{dy}{dt} = \frac{1}{\epsilon} g(x, y)
     \]
   - Reduce coupled system to effective stochastic dynamics for $x$:
     \[
     \frac{dx}{dt} = Lx + N(x, x) + S(x) \circ \dot{W}
     \]
     (many ways of doing this; some formal, some ad hoc)
   - Linearise model around appropriate state (e.g. climatological mean)

Mechanistic Model: 6-Month Stochastic Optimals

SDEs: Multiplicative vs. additive noise

- General 1d SDE
  \[
  \frac{dx}{dt} = a(x, t) + b(x, t) \dot{W}
  \]
- Have so far focused on case of constant noise strength $b(x, t)$ (that is, noise independent of state of system)
- In such a case, noise said to be additive
- In general, however, $b(x, t)$ depends on $x$ and noise is multiplicative
- Things become much more complicated ...
- We can formally integrate the SDE:
  \[
  x(t) = x(0) + \int_0^t a(x, t) dt + \int_0^t b(x, t) dW(t)
  \]
  but we need to make clear what is meant by stochastic integral