SDEs: A first example

- We see in this example
  1. a deep connection between diffusion of probability described by FPE and pathwise evolution described by SDE
  2. a connection between the “drift” and the “deterministic” part of the dynamics
  3. a connection between the “diffusion” and the “stochastic” part of the dynamics
  4. the stationary pdf includes contributions from both deterministic & stochastic terms. Fluctuations drive system away from deterministic attractor, dynamics pushes it back: stationary pdf balances these tendencies.
- These ideas important enough to help get Albert Einstein his Nobel Prize (for early work on the subject) ...

Frankignoul & Hasselmann 1977

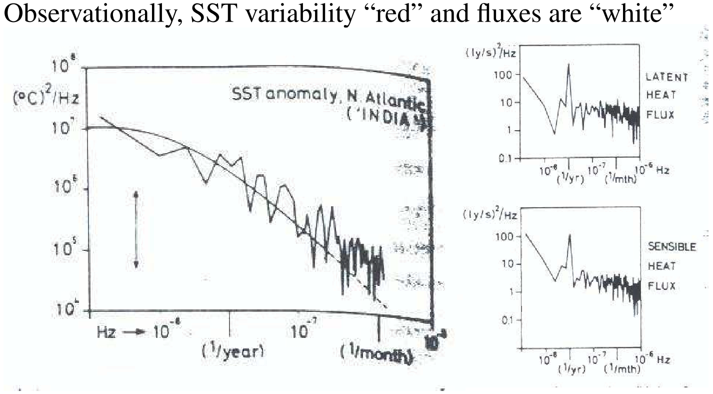
- These ideas also at heart of first **stochastic climate model** (Frankignoul & Hasselmann 1977) , used to explain why SST spectra are generally red in character
- Simple model: SST anomalies driven by “fast” atmospheric forcing (assumed to be white in time) and “slow” relaxation to climatology through ocean mixed layer physics
- With T = SST anomaly, have model

$$\frac{dT}{dt} = -\lambda T + \gamma \dot{W}$$

which predicts red SST response to fast, rapidly-decorrelating atmospheric forcing

Frankignoul & Hasselmann 1977

- Observationally, SST variability “red” and fluxes are “white”



From Frankignoul & Hasselmann *Tellus* 1977

- Linear stochastic model ⇒ simple null hypothesis for observed variability

Multivariate Ornstein-Uhlenbeck processes

- Have so far focused on scalar stochastic processes; can generalise results so far for $\mathbf{x} \in \mathsf{R}^N$
- Vector white noise $\dot{\mathbf{W}}(t)$ is an array of independent white noise processes

$$\mathsf{E}\{\dot{W}_i(t)\dot{W}_j(t')\} = \delta_{ij}\delta(t-t')$$

- For constant matrices A and B, SDE

$$\frac{d}{dt}\mathbf{x} = A\mathbf{x} + B\dot{\mathbf{W}}$$

has solution (with $P(t) = e^{At}$)

$$\mathbf{x}(t) = P(t)\mathbf{x}(0) + \int_0^t P(t-t')B\dot{\mathbf{W}}(t')\,dt'$$

Multivariate Ornstein-Uhlenbeck processes

- Solution is Gaussian with mean and stationary autocovariance

$$\begin{aligned} \mathsf{E}\{\mathbf{x}(t)\} &= P(t)\mathbf{x}(0) \\ C_{xx}(\tau) &= \mathsf{E}\{\mathbf{x}(t+\tau)\mathbf{x}^T(t)\} = P(\tau)C_{xx}(0) \end{aligned}$$

where stationary covariance satisfies **Lyapunov equation**

$$AC_{xx}(0) + C_{xx}(0)A^T = -BB^T$$

$\Rightarrow$ balance between deterministic dynamics (A), stochastic forcing (B) and statistics of response ($C_{xx}(0)$); an example of a **fluctuation-dissipation relationship**

- pdf $p(\mathbf{x})$ satisfies Fokker-Planck equation

$$\partial_t p = -\sum_i A_i \partial_i p + \sum_{ij} (BB^T)_{ij} \partial^2_{ij} p$$

Optimal Perturbations

- If dynamical matrix A is *non-normal*, i.e.

$$AA^T \neq A^T A$$

so eigenvectors of A are not orthogonal, then:

  - EOFs (eigenvectors of covariance $C_{xx}(0)$) do not coincide with eigenvectors of A (dynamical modes)
  - perturbation norm

$$N(t) = \mathbf{x}(t)^T M \mathbf{x}(t) = \mathbf{x}(0)^T P(t)^T M P(t) \mathbf{x}(0)$$

may grow (by potentially large amount) over finite times even though asymptotically stable:

$$\lim_{t\to\infty} N(t) = 0$$

Optimal Perturbations

- Defining amplification factor

$$n(t) = \frac{\mathbf{x}(0)^T P(t)^T M P(t) \mathbf{x}(0)}{\mathbf{x}(0)^T M \mathbf{x}(0)}$$

optimal perturbation $\mathbf{e}$ maximises $n(t)$ subject to constraint $\mathbf{x}(0)^T M \mathbf{x}(0) = 1 \Rightarrow$ generalised eigenvalue problem:

$$P(t)^T M P(t) \mathbf{e} = \lambda M \mathbf{e}$$

- Response to fluctuating forcing:

$$\mathrm{var}(\mathbf{x}^T(t) M \mathbf{x}(t)) = B^T \left(\int_0^t P(s)^T M P(s) ds \right) B$$

$\Rightarrow$ importance of projection of noise structure on “average” optimals for maintaining variance; “stochastic optimals”

Linear inverse modelling

- How are A and B determined in practice?

1. **Empirical: Linear Inverse Modelling**
   - estimate covariances from observations and compute

$$A = \frac{1}{\tau} \ln \left(C_{xx}(\tau) C_{xx}(0)^{-1} \right)$$

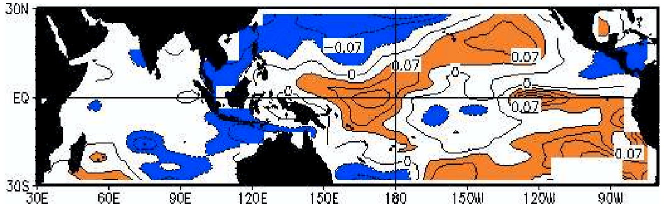
   - Compute B from Lyapunov equation

$$AC_{xx}(0) + C_{xx}(0)A^T = -BB^T$$

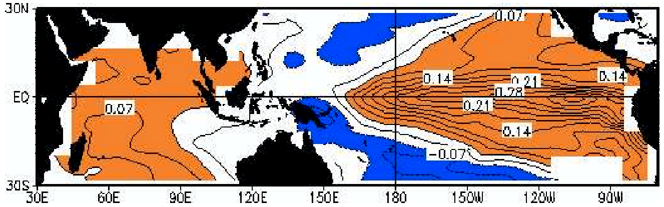
   - Issues:
     i. if $\mathbf{x}(t)$ not truly Markov, estimates will depend on lag τ
     ii. must enforce positive-definiteness of BB^T

LIM: ENSO SST Optimal Perturbations

Optimal perturbation $\mathbf{e}$



Perturbation $P(t)e$ at $t = 7$ months



From Penland and Sardeshmukh (1995)

Stochastic ENSO Models

2. **Mechanistic: Stochastic Reduction of Tangent Linear Model**
   - Partition dynamics into "slow" and "fast" variables $\mathbf{x}$ and $\mathbf{y}$:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{y})$$
$$\frac{d\mathbf{y}}{dt} = \frac{1}{\epsilon}\mathbf{g}(\mathbf{x}, \mathbf{y})$$

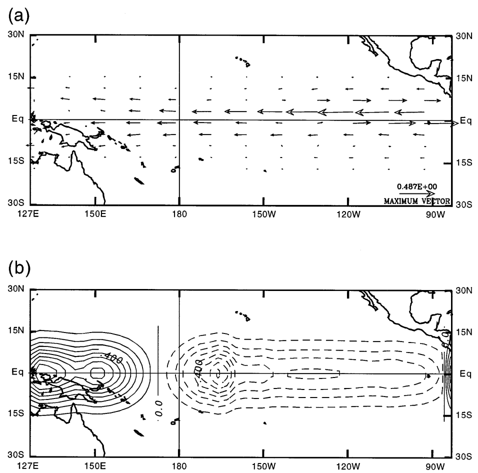
   - Reduce coupled system to effective stochastic dynamics for $\mathbf{x}$:

$$\frac{d\mathbf{x}}{dt} = L\mathbf{x} + N(\mathbf{x}, \mathbf{x}) + S(\mathbf{x}) \circ \dot{\mathbf{W}}$$

   (many ways of doing this; some formal, some ad hoc)

   - Linearise model around appropriate state (e.g. climatological mean)

Mechanistic Model: 6-Month Stochastic Optimals



From Kleeman and Moore, *J. Atmos. Sci.*, 1997.

SDEs: Multiplicative vs. additive noise

- General 1d SDE

$$\frac{dx}{dt} = a(x,t) + b(x,t)\dot{W}$$

- Have so far focused on case of constant noise strength $b(x,t)$ (that is, noise independent of state of system)
- In such a case, noise said to be **additive**
- In general, however, $b(x,t)$ depends on x and noise is **multiplicative**
- Things become much more complicated ...
- We can formally integrate the SDE:

$$x(t) = x(0) + \int_0^t a(x,t)dt + \int_0^t b(x,t)dW(t)$$

but we need to make clear what is meant by stochastic integral