Markov chains

- By properties of joint and conditional pdfs:

$$p(x_n) = \int p(x_n, x_{n-1})\, dx_{n-1} = \int p(x_n|x_{n-1})p(x_{n-1})\, dx_{n-1} = \mathcal{L}_n[p(x_{n-1})]$$

- Evolution of probabilities governed by operator $\mathcal{L}$ (which will be independent of time for a stationary process)
- If space of x_n is finite integers, $\mathcal{L}$ is a matrix L and

$$\mathbf{p}_n = L\mathbf{p}_{n-1} = L^n \mathbf{p}_0$$

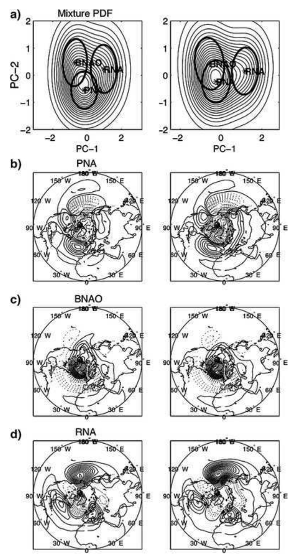
  [where $\mathbf{p}_n = (p(x_n = 1), p(x_n = 2), ..., p(x_n = K))$]
- For processes which conserve probability (e.g. $\sum_i L_{ij} = 1$) the maximum eigenvalue of L is 1 with eigenvector $\mathbf{p}_s$ (the **stationary pdf**). As $n \to \infty$, $\mathbf{p}_n$ approaches $\mathbf{p}_s$

Markov chains: An example

Transition probabilities between "preferred regimes"

		from		
		PNA	BNAO	RNA
	PNA	0.30	0.34	0.37
to	BNAO	0.49	0.26	0.25
	RNA	0.36	0.30	0.34

(from Kondrashov et al., *Clim Dyn.*, 2007.



Chapman-Kolmogorov equation

- Again, by properties of joint and conditional pdfs:

$$p(x_n|x_{n-2}) = \int p(x_n|x_{n-1})p(x_{n-1}|x_{n-2})\, dx_{n-1}$$

$\Rightarrow$ an equation for the evolution of the "transfer matrix" $p(x_i|x_j)$; as a nonlinear integral equation it's a bit tricky to solve

- Now focus on "smooth" processes with continuous paths and with finite mean $a(x,t)$ (the **drift**) and "variance" $b(x,t)$ (the **diffusion**) of the local tendency such that for (for all $\epsilon > 0$)

$$\lim_{\delta\to 0} \int_{|x(t+\delta)-x(t)|<\epsilon} \left(\frac{x(t+\delta)-x(t)}{\delta}\right) p(x(t+\delta)|x(t))\, dx = a(x,t) + O(\epsilon)$$

$$\lim_{\delta\to 0} \int_{|x(t+\delta)-x(t)|<\epsilon} \left(\frac{[x(t+\delta)-x(t)]^2}{\delta}\right) p(x(t+\delta)|x(t))\, dx = b^2(x,t) + O(\epsilon)$$

Chapman-Kolmogorov equation

- For such a process the Chapman-Kolmogorov equation can be transformed into the PDE (with $p = p(x(t)|x(t'))$)

$$\begin{aligned} \partial_t p &= -\partial_x[a(x,t)p] + \frac{1}{2}\partial^2_{xx}[b^2(x,t)p] \\ &= -\partial_x\left[\left(a - \frac{1}{2}b\partial_x b\right)p\right] + \frac{1}{2}\partial_x[b\partial_x(bp)] \end{aligned}$$

  for the pdf at time t conditioned on the state of the system at time t'
- This equation, known as the **Fokker-Planck Equation** (FPE), describes probability diffusing conservatively through state space from the original distribution
- If a, b independent of time pdf will approach **stationary pdf** p_s:

$$0 = -\partial_x[a(x)p_s(x)] + \frac{1}{2}\partial^2_{xx}[b^2(x)p_s(x)]$$

The Wiener process $W(t)$

- Consider the case $a(W,t) = 0$ (no mean tendency) and $b(W,t) = 1$:

$$\partial_t p = \frac{1}{2}\partial_{ww} p$$

- This is a classical diffusion equation; with initial condition $p(W(t_0)) = \delta(W - W_0)$ the solution is

$$p(W(t)) = \frac{1}{\sqrt{2\pi(t-t_0)}} \exp\left(-\frac{(W-W_0)^2}{2(t-t_0)}\right)$$

- Gaussian with mean$(W) = 0$ and std$(W) = \sqrt{t-t_0}$ (sound familiar?)
- The **Wiener process** (or Brownian motion) is continuous generalisation of random walk; as std(W) is time-dependent, this is a
  non-stationary process

The Wiener process $W(t)$: curious features

1. Paths $W(t)$ are continuous everywhere but differentiable nowhere

$$\text{std}\left(\frac{W(t+\delta) - W(t)}{\delta}\right) = \frac{\sqrt{\delta}}{\delta} \to \infty \text{ as } \delta \to 0$$

2. Increments $\Delta_{tt'}W = W(t) - W(t')$ are Gaussian, stationary

$$p(\Delta_{tt'}W) = \frac{1}{\sqrt{2\pi(t-t')^2}} \exp\left(-\frac{(\Delta_{tt'}W)^2}{2(t-t')}\right)$$

   and independent

$$p(\Delta_{t_3t_2}W, \Delta_{t_2t_1}W) = \frac{\exp\left(-\frac{(\Delta_{t_3t_2}W)^2}{2(t_3-t_2)}\right)}{\sqrt{2\pi(t_3-t_2)}} \frac{\exp\left(-\frac{(\Delta_{t_2t_1}W)^2}{2(t_2-t_1)}\right)}{\sqrt{2\pi(t_2-t_1)}}$$

$\to$ in particular, $\text{cov}(\Delta_{t_3t_2}W, \Delta_{t_2t_1}W) = 0$

White noise

- $W(t)$ not strictly differentiable, but can formally define **white noise**

$$\dot{W}(t) = \lim_{\delta \to 0} \frac{W(t+\delta) - W(t)}{\delta}$$

  such that std$(\dot{W}(t)) = \infty$ and

$$\begin{aligned} \text{cov}(\dot{W}(t), \dot{W}(t')) &= \lim_{\delta,\delta' \to 0} \mathsf{E}\left\{\left(\frac{W(t+\delta) - W(t)}{\delta}\right)\left(\frac{W(t'+\delta') - W(t')}{\delta'}\right)\right\} \\ &= \delta(t-t') \end{aligned}$$

  (because if $t \neq t'$ sufficiently small increments don't overlap)

- White noise infinitely fast, infinitely strong with

$$\mathsf{E}\left\{\dot{W}(t)\dot{W}(t')\right\} = \delta(t-t')$$

  (note: mathematicians usually write $dW(t') = \dot{W}(t')dt$)

An aside: units of white noise

- What are the units of white noise? We have:

$$\mathsf{E}\left\{\dot{W}(t)\dot{W}(t')\right\} = \delta(t-t')$$

  so $[W] \sim [\delta]^{1/2}$

- What are the units of a delta function? We have

$$\int \delta(t-t')\, dt = 1 \text{ (dimensionless)}$$

  so $[\delta] \sim [t]^{-1}$

- We conclude: $\dot{W}(t)$ has the dimensions of *one over the square root of time*; has important implications for estimating the coefficient of white noise in stochastic differential equations

Ornstein-Uhlenbeck process (red noise)

- Consider FPE with $a(x,t) = -1/\tau$ and $b^2(x,t) = \gamma^2$ (both constant) and initial condition $p(x(t_0)) = \delta(x - x_0)$; known as an **Ornstein-Uhlenbeck (O-U) process**, or **red noise**
- Can show that solution is Gaussian with

$$\begin{aligned} \mathrm{mean}(x(t)) &= x_0 e^{-t/\tau} \\ \mathrm{var}(x(t)) &= \frac{\gamma^2\tau}{2}\left(1 - e^{-2t/\tau}\right) \end{aligned}$$

- As initial transients die out ($t \to \infty$) $p(x)$ approaches stationary pdf with mean zero and stationary autocovariance

$$\mathrm{cov}(x(t), x(t')) = \frac{\gamma^2\tau}{2}\exp(-|t - t'|/\tau)$$

$\Rightarrow \tau$ is the "memory" of the system

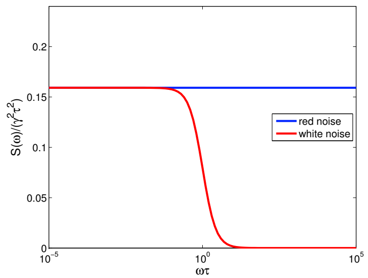
Red and white noise

- With $\gamma = \sigma/\tau$,

$$\mathrm{cov}(x(t), x(t')) = \sigma^2\left(\frac{1}{2\tau}\exp(-|t - t'|/\tau)\right) \to \sigma^2\delta(t-t') \text{ as } \tau \to 0$$

- In this limit, red noise becomes white noise scaled by factor σ
- Power spectra are Fourier transforms of autocovariances:

$$\begin{aligned} S_W(\omega) &= \mathsf{E}\{W(\omega)^2\} = \frac{1}{2\pi} \\ S_x(\omega) &= \mathsf{E}\{x(\omega)^2\} = \frac{\gamma^2\tau^2}{2\pi(1 + \omega^2\tau^2)} \end{aligned}$$



SDEs: A first example

- If we can (formally) differentiate $W(t)$ to get $\dot{W}(t)$, can (formally) integrate $\dot{W}(t)$ to get $W(t)$ back
- Consider forced linear differential equation with $x(0) = x_0$:

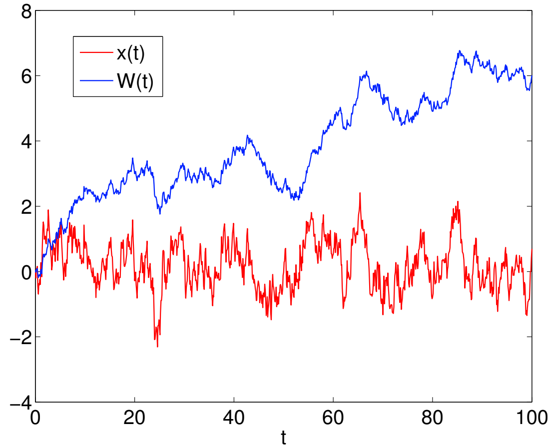
$$\frac{d}{dt}x = -\frac{1}{\tau}x + \gamma\,\dot{W}$$

- This **stochastic differential equation** (SDE) has solution

$$x(t) = e^{-t/\tau}x_0 + \gamma\int_0^t e^{-(t-t')/\tau}\dot{W}(t')\,dt'$$

- $x(t)$ is a Gaussian stochastic process with same mean and autocovariance as O-U process: in fact, this is a *pathwise* description of red noise as a weighted integral of white noise

SDEs: A first example



$$\frac{dx}{dt} = -x + \dot{W}$$