Symbolic Extension Entropy, Cantor-Bendixson Rank, and Maps of the Interval

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joint work with David Burguet

Outline

The Setup

Consider

- a dynamical system (X, T), where X is a compact metrizable space and $T : X \to X$ is continuous
- $M(X, T)$ the space of all Borel probabilities on X which are invariant under T
- $h_{top}(T)$ the topological entropy of T, and $h : M(X, T) \to [0, \infty]$ the entropy function on invariant measures.

We will always assume $h_{top}(T) < \infty$.

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Recall that $M(X, T)$ is always a metrizable Choquet simplex, *i.e.* it

- is a compact convex metrizable subspace of a locally convex topological vector space and
- has the property that each point can be written *uniquely* as a convex (harmonic) combination of the extreme points (given by the ergodic decomposition).

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Symbolic Extensions

Definition

- Let (Y, S) be a subshift of a (two-sided) full shift on a finite alphabet. (Y, S) is called a *symbolic extension* of the dynamical system (X, T) if there exists a continuous surjection $\pi : Y \to X$ such that $\pi S = T\pi$.

$$\begin{tikzcd} Y \arrow[r, "S"] \arrow[d, "\pi"'] & Y \arrow[d, "\pi"'] \\ X \arrow[r, "T"'] & X \end{tikzcd}$$

- Given a symbolic extension (Y, S) of (X, T), we define the *extension entropy function* $h_{ext}^{\pi} : M(X, T) \to [0, \infty)$ by

$$h_{ext}^{\pi}(\mu) = \sup\{h(\nu) : \nu \in M(Y, S),\ \pi\nu = \mu\}.$$

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Symbolic Extension Entropy and Residual Entropy

In order to analyze how well a dynamical system (X, T) can be approximated by subshifts, we consider the *symbolic extension entropy* (abbreviated as sex entropy) and the *residual entropy* functions:

at the topological level, let

- $h_{sex}(T) = \inf\{h_{top}(S) : (Y, S) \text{ is a symbolic extension of } (X, T)\}$
- $h_{res}(T) = h_{sex}(T) - h_{top}(T)$

and at the level of measures, let

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Entropy Structure

Boyle and Downarowicz (in [BD]) exposed a remarkable functional analytic characterization of all of these quantities. The concept of entropy structure, an idea developed in great generality by Downarowicz, lies at the heart of their approach.

Definition

Given a dynamical system (X, T), an *entropy structure* $\mathcal{H} = (h_k)$ of (X, T) is an allowable, non-decreasing sequence of nonnegative functions on $M(X, T)$ such that $\lim_k h_k = h$.

The term allowable is made precise by the notion of *uniform equivalence* [D], which is an equivalence relation on non-decreasing sequences of nonnegative functions which converge to the same limit.

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- Almost all general approaches to entropy give rise to an entropy structure.
- In this way, entropy structure provides a grand unification of the theory of entropy.
- Entropy structure reflects precisely how entropy emerges at refining scales.
- Entropy structure is an invariant of topological conjugacy, and it determines almost all previously known entropy invariants.

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The Transfinite Sequence

Given an entropy structure $\mathcal{H}$ on a Choquet simplex K, we define an associated transfinite sequence of functions on K:

Definition

Let $\tau_k = h - h_k$. Then the transfinite sequence $\mathcal{U} = (u_\alpha)$ associated to $\mathcal{H}$ is given by

- $u_0 \equiv 0$
- $u_{\alpha+1} = \lim_k \widetilde{u_\alpha + \tau_k}$
- $u_\alpha = \widetilde{\sup_{\beta<\alpha}} u_\beta$ for any limit ordinal α.

A positive value at any of the these functions reflects non-uniform convergence of h_k to h, and is related to the defect of upper semi-continuity of h.

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Sex Entropy and the Transfinite Sequence

Theorem (BD)

If $\mathcal{H}$ is an entropy structure for (X, T), then

1. *there exists a countable ordinal α such that $u_\alpha = u_{\alpha+1}$, and*
2. *$u_\alpha = u_{\alpha+1}$ if and only if $h_{sex} = h + u_\alpha$.*

The least ordinal α such that $u_\alpha = u_{\alpha+1}$ is called the *order of accumulation* of the entropy structure $\mathcal{H}$ or of the dynamical system. We write $\alpha_0(\mathcal{H})$ or just α_0 for the order of accumulation.

Note that by this theorem, $h_{res} = u_{\alpha_0}$.

With this theorem, along with the rest of the work of Boyle, Downarowicz, and Serafin, questions about symbolic extensions and entropy can be translated into questions of a purely functional analytic nature.

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Downarowicz-Serafi n Realization Theorem

Question

Given a space X and a regularity class $\mathcal{C}$ of self-mappings of X, which entropy structures can be realized by a function in $\mathcal{C}$?

At the level of topological dynamics, this question was answered by Downarowicz and Serafin:

Theorem (DS)

An candidate sequence $\mathcal{H}$ on a Choquet simplex K is (up to affine homeomorphism) an entropy structure for some dynamical system if and only if it is uniformly equivalent to a non-increasing sequence of nonnegative, upper semi-continous functions with upper semi-continuous differences.

Which Possible Orders of Accumulation are Realized?

In this work we investigate the following questions:

Question

Given a Choquet simplex K, which countable ordinals can be realized as the order of accumulation of an entropy structure on K?

In [BD], Boyle and Downarowicz created explicit simplices and entropy structures realizing all finite orders of accumulation. In [D], Downarowicz expressed his firm belief that all countable ordinals should be realized.

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Which Possible Orders of Accumulation are Realized?

Question

Which countable ordinals can be realized as the order of accumulation of a continuous map of the interval?

David Burguet has produced examples of C^r interval maps with infinite orders of accumulation, but his bounds do not allow one to know exactly which orders of accumulation are achieved.

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Background on Cantor-Bendixson Rank

In order to answer the first question, we recall:

- For a topological space X, the derived set X' is the set of all accumulation points of X.
- Given a topological space X, we may use transfinite induction to define $X_0 = X$, $X_{\alpha+1} = X'_\alpha$ and $X_\alpha = \cap_{\beta<\alpha} X_\beta$.
- The *Cantor-Bendixson rank* of X, $|X|_{CB}$, is defined to be the least ordinal α such that $X_\alpha = X_{\alpha+1}$.
- Any Polish space X has countable Cantor-Bendixson rank.
- Any Polish space can be written as a disjoint union of a countable set and a perfect set.
- For a Choquet simplex K, $\mathrm{ex}(K)$ is a Borel subset of K (G_δ in fact), and E is a Polish space iff E is homeomorphic to $\mathrm{ex}(K)$ for some K (Choquet).

This concludes the introduction. We now set out to answer

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Accumulation on Bauer Simplices

Facts

- A Choquet simplex K is called *Bauer* if the set of extreme points $\mathrm{ex}(K)$ is closed in K.
- If $M = M(X, T)$ is a Bauer simplex, then all the elements of the transfinite sequence $\mathcal{U}$ are harmonic on M.
- Therefore if M is a Bauer simplex, in order to compute $\mathcal{U}$ one need only consider the entropy structure restricted to the extreme points of M, compute $\mathcal{U}$ at the extreme points, and use the harmonic extension. This suggests that only the topology of the extreme points is relevant to the accumulation of sex entropy.

Accumulation on Bauer Simplices

Theorem

Let K be any Bauer simplex.

1. *If* $\operatorname{ex}(K)$ *is countable, then*
   - $\{\alpha_0(\mathcal{H}) \text{ on } K\} = [0, |\operatorname{ex}(K)|_{CB} - 1]$ *if* $|\operatorname{ex}(K)|_{CB}$ *is finite, and*
   - $\{\alpha_0(\mathcal{H}) \text{ on } K\} = [0, |\operatorname{ex}(K)|_{CB}]$ *if* $|\operatorname{ex}(K)|_{CB}$ *is infinite.*
2. *If* $\operatorname{ex}(K)$ *is uncountable, then for every countable* β*, there exists an entropy structure* $\mathcal{H}_\beta$ *on* K *such that* $\alpha_0(\mathcal{H}_\beta) = \beta$*.*

Combining this result with the realization theorem of Downarowicz and Serafin, we obtain

Corollary

Every countable ordinal α *is realized as the order of accumulation of a dynamical system.*

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Accumulation on Choquet Simplices

In the general case, $\text{ex}(M(X, T)) = M_{erg}(X, T)$ need not be closed in $M(X, T)$. In this case the functions in the transfinite sequence are not necessarily harmonic; they need only be concave.

Notation.

- If E is a Polish space, let $\rho(E) = \sup\{|F|_{CB} : F \text{ is compact in } E\}$
- If K is a Choquet simplex, let $\beta(K) = \sup\{\gamma : \text{ there exists an } \mathcal{H} \text{ on } K \text{ such that } \alpha_0(\mathcal{H}) = \gamma\}$.

These definitions allow for the theorems...

Accumulation on Choquet Simplices: Topological Bounds

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Theorem

Let K be a Choquet simplex. Then

- *If $\mathrm{ex}(K)$ is countable, then $\rho(\mathrm{ex}(K)) \leq \beta(K)$.*
- *If $\overline{\mathrm{ex}(K)}$ is countable, then $\beta(K) \leq \rho(\overline{\mathrm{ex}(K)})$.*

Note that if $\overline{\mathrm{ex}(K)}$ is countable, then so is $\mathrm{ex}(K)$, and we obtain

$$\rho(\mathrm{ex}(K)) \leq \beta(K) \leq \rho(\overline{\mathrm{ex}(K)}).$$

Notation. Consider a pair $(E, \overline{E})$, where E is a Polish space and $\overline{E}$ is some compactification of E. For two such pairs $(E_1, \overline{E_1})$ and $(E_2, \overline{E_2})$, we write $(E_1, \overline{E_1}) \simeq (E_2, \overline{E_2})$ to mean that there is a homeomorphism $g : \overline{E_1} \to \overline{E_2}$ which restricts to a bijection from E_1 to E_2.

Accumulation on Choquet Simplices: Optimality of Bounds

Theorem

Let E be a Polish space and $\overline{E}$ a compactification of E. Then

1. *If both E and $\overline{E}$ are countable, then for all β in $[\rho(E), \rho(\overline{E})]$, there exists a Choquet simplex K with $(\mathrm{ex}(K), \overline{\mathrm{ex}(K)}) \simeq (E, \overline{E})$ and $\beta(K) = \beta$.*
2. *If E is countable and $\overline{E}$ is uncountable, then for all $\beta \geq \rho(E)$, there exists a Choquet simplex K with $(\mathrm{ex}(K), \overline{\mathrm{ex}(K)}) \simeq (E, \overline{E})$ and $\beta(K) = \beta$.*
3. *If E is uncountable, then for all Choquet simplices K with $(\mathrm{ex}(K), \overline{\mathrm{ex}(K)}) \simeq (E, \overline{E})$, every countable ordinal is realized as the order of accumulation of an entropy structure on K.*

Now we turn to a related but much more concrete question:

Question

Which countable ordinals can be realized as the order of accumulation of a continuous map of the interval?

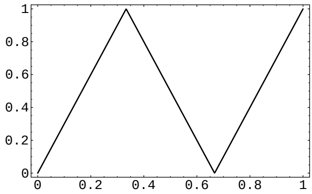
Accumulation on the Unit Interval

Theorem

For every countable ordinal α, there exists a continuous map $f_\alpha : [0,1] \to [0,1]$ such that the order of accumulation of $([0,1], f_\alpha)$ is α.

Just for illustration, how can we even get $\alpha_0 = 1$?

To get a map $f : [0,1] \to [0,1]$ with $\alpha_0 = 1$, recall the map T_3:



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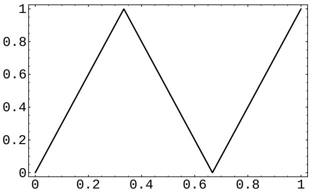
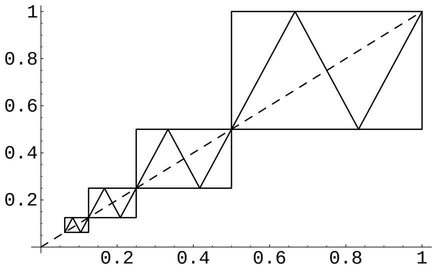


Accumulation on the Unit Interval: $\alpha_0 = 1$

- Let f be the continuous map with a copy of T_3 on each interval $I_n = [2^{-(n+1)}, 2^{-n}]$.
- On each I_n, there is a measure μ_n of maximal entropy $h(\mu_n) = 3$.
- The measures μ_n converge in the weak* topology to the point mass at 0, δ_0.



Accumulation on the Unit Interval: $\alpha_0 = 1$

Then

- Choose an entropy structure (h_k) corresponding to the decreasing sequence of scales $\epsilon_k > 0$.
- For each k, there are infinitely many intervals I_n with $\ell(I_n) < \epsilon_k$.
- Thus, for each k, there are infinitely many n such that $h(\mu_n) - h_k(\mu_n) = 3$.
- Taking the upper semi-continuous envelope at δ_0, we see that $(h - h_k)\widetilde{}(\delta_0) = 3$.
- Letting k tend to infinity gives that $u_1(\delta_0) = 3$.
- Another argument shows that $u_1 \equiv u_2$, and so $\alpha_0 = 1$.
- The key point is that for every finite scale ϵ, there are measures arbitrarily close to δ_0 whose entropy cannot be observed with precision ϵ.
- To get arbitrary orders of accumulation, one must build nested versions of this scenario, which is done using a more complicated iterative construction.

Thank You

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