Ambrose-Kakutani representation theorems for Borel semiflows

David McClendon
Northwestern University

dmm@math.northwestern.edu
http://www.math.northwestern.edu/~dmm
**Ambrose-Kakutani Theorem**

**Theorem** (Amb 1940, Amb-Kak 1942) *Any aperiodic measure-preserving flow* $T_t$ *on a standard probability space* $(X, \mathcal{X}, \mu)$ *is isomorphic to a suspension flow.*

A *suspension flow* $(G, \mathcal{G}, \nu, S_t)$, also called a *flow under a function*, looks like this:

![Diagram of suspension flow](image)
Suspension semiflows

If the return-time transformation $\hat{S}$ in the previous picture is not injective, then we obtain a suspension semiflow:
Our problem

Let

- $X$ be an uncountable Polish space, with
- $\mathcal{B}(X)$ its $\sigma$–algebra of Borel sets,
- $\mu$ a probability measure on $(X, \mathcal{B}(X))$ and
- $T_t : X \times \mathbb{R}^+ \to X$ an aperiodic, jointly Borel action by surjective maps preserving $\mu$.

Call $(X, \mathcal{B}(X), \mu, T_t)$ a Borel semiflow.

Question: What Borel semiflows are isomorphic to suspension semiflows?
A restriction: discrete orbit branchings

For any point $z$ not in the base of a suspension semiflow $(G, S_t)$, $\#(S_{-t}(z)) = 1$ for $t$ small enough. So if we let

$$B = \{ z \in G : \#(S_{-t}(z)) > 1 \forall t > 0 \},$$

every point $z \in G$ must satisfy:

The set of times $t \geq 0$ where $S_t(z) \in B$ is a discrete subset of $\mathbb{R}^+$. 

More generally, we have the following for any suspension semiflow $(G, S_t)$:

Given any $z$, the set of times $t_0 \geq 0$ where

$$\bigcup_{t < t_0} S_{-t}S_t(z) \neq \bigcap_{t > t_0} S_{-t}S_t(z)$$

is a discrete subset of $\mathbb{R}^+$. 

Any Borel semiflow for which the preceding sentence holds is said to have discrete orbit branchings.
Another issue: instantaneous discontinuous identifications

Suppose \((X, T_t)\) is a Borel semiflow and that \(x\) and \(y\) are two distinct points in \(X\) \((x \neq y)\) with

\[
T_t(x) = T_t(y) \forall t > 0.
\]

We say that \(x\) and \(y\) are \textit{instantaneously and discontinuously identified (IDI)} by \(T_t\).

Define the (Borel) equivalence relation:

\[
\text{IDI} = \{(x, y) \in X^2 : T_t(x) = T_t(y) \forall t > 0\}.
\]

This relation must contain the diagonal \(\Delta\). If \(\text{IDI} = \Delta\), we say that \(T_t\) has no IDIs.

\(T_t\) has no IDIs if and only if \(T_{(0,\infty)}(x)\) determines \(x\) uniquely for every \(x \in X\).

Suspension semiflows (as defined thus far) have no IDIs.
A conjecture

We conjecture that the previously described issues are the only restrictions to isomorphism with a suspension semiflow, i.e.

**Conjecture** Any Borel semiflow with the discrete orbit branching property that has no IDIs is isomorphic to a suspension semiflow.
A partial result

Theorem 1 (M) If a countable-to-1 Borel semi-flow \((X, T_t)\) is such that

1. \(T_t\) has discrete orbit branchings, and
2. \(T_t\) has no IDIs,

then \((X, T_t)\) is isomorphic to a suspension semi-flow \((G, S_t)\), with the caveat that the measure \(\hat{\nu}\) on the base may be \(\sigma\)-finite.

Note: The measure \(\nu = \hat{\nu} \times \lambda\) on \(G\) is a probability measure.

Note: Asking that \(T_t\) being countable-to-1 is virtually equivalent to asking that \(T_t\) be \textit{bimeasurable}, that is, that \(T_t(A)\) is Borel for every \(t \geq 0\) and every Borel \(A \subseteq X\).
An example with infinite base measure

Consider the map $\hat{S} : \mathbb{R} \to \mathbb{R}$ defined by $\hat{S}(x) = x - \frac{1}{x}$.

Let $\hat{X} = \mathbb{R} - \bigcup_n \hat{S}^{-n}(0)$. (This will be the base of the suspension semiflow.)

$\hat{S} : \hat{X} \to \hat{X}$ preserves Lebesgue measure, is ergodic, and is everywhere 2-to-1.
**An example with infinite base measure**

Construct a suspension semiflow with base $\hat{X}$, return map $\hat{S}$ with height function $f$:

This suspension semiflow has the discrete orbit branching property but is not isomorphic to any suspension semiflow where the measure on the base is finite.

**Question:** What conditions ensure isomorphism with a suspension semiflow where the measure on the base is finite?
Some ingredients of the proof

Lemma 1 (Krengel 1976, Lin & Rudolph 2002)
Every Borel semiflow has a measurable cross-section $F$ with measurable return-time function $r_F$ bounded away from zero.

Consequence: Every $x \in X$ can be written $x = T_t(y)$ where $y \in F$ and $0 \leq t < r_F(y)$. 
Another lemma

Lemma 2 There is a countable list of Borel functions \( j_i \) taking values in \( \mathbb{R}^+ \) whose domains \( J(i) \) are Borel subsets of \( X \) so that \( x \) has an orbit branching at time \( t_0 \), i.e.

\[
\bigcup_{t < t_0} T_{-t}T_t(z) \neq \bigcap_{t > t_0} T_{-t}T_t(z),
\]

if and only if \( j_i(x) = t_0 \) for some \( i \).
More on Lemma 2

To establish Lemma 2, consider the set

\[ B^* = \{(x, t) \in X \times \mathbb{R}^+ : x \text{ has orbit branching at time } t\} \]

Since each \( T_t \) is countable-to-1, for any Borel \( A \subseteq X \), \( T_t(A) \) is Borel for each \( t \geq 0 \). Using this, one can show that \( B^* \) is a Borel set.

Since \( B^* \) must have countable sections by assumption, the Lusin-Novikov theorem applies.
Combining the two lemmas

Superimpose the pictures from the previous two lemmas:
Cutting and rearranging

Make a new section $G_1$ (with return time function $g$) consisting of $F$ together with all orbit branchings of $T_t$: 
Obtaining an isomorphism

With respect to this new section, every $x \in X$ can be written uniquely as $x = T_t(y)$ where $y \in G_1$ and $0 \leq t < g(y)$. This allows for an isomorphism between $(X, T_t)$ and the suspension semiflow over $G_1$.
Finite measures on the base

**Theorem 2** (M) *If a countable-to-1 Borel semi-flow* \((X, T_t)\) *is such that*

1. \(T_t\) has no IDIs, and
2. there is some \(c > 0\) such that if \(x\) has orbit branchings at times \(t\) and \(t'\), then \(|t - t'| > c\),

then \((X, T_t)\) *is isomorphic to a suspension semi-flow* \((G, S_t)\) *where the measure on the base is finite.*

**Proof:** Adapt the preceding argument to construct a section \(G_1\) with return-time function bounded away from zero.
What if the semiflow has IDIs?

**Definition:** Start with the following:
1. Two standard Polish spaces $G_1$ and $G_2$.
2. A $\sigma$–finite Borel measure $\hat{\nu}$ on $G_1 \cup G_2$.
3. A measurable function $g : G_1 \to \mathbb{R}^+$ with $\int g \, d\hat{\nu} = 1$.
4. A measurable map $\sigma : G_1 \cup G_2 \to G_1$ such that $\sigma|_{G_1} = id$.
5. A measurable map $\hat{S} : G_1 \to G_1 \cup G_2$.

Now let $G$ be the set
\[
\{(z, t) \in G_1 \times \mathbb{R}^+ : 0 \leq t < g(z)\} \cup (G_2 \times \{0\})
\]
(endowed with subspace product topology) and define the Borel semiflow $S_t$ on $G$ as indicated in the picture on the next slide:
(G, S_t) is called a suspension semiflow with IDIs. Notice that for any \( x \in G_2 \), \((x, \sigma(x)) \in \text{IDI}\).
An Ambrose-Kakutani type theorem with IDIs

Theorem 3 \((M)\) A countable-to-1 Borel semi-flow \((X, T_t)\) is isomorphic to a suspension semi-flow with IDIs if and only if \(T_t\) has discrete orbit branchings.
Questions

Suppose one considered a Borel semiflow that is not necessarily countable-to-1.

**Q1.** Is the discrete orbit branching property sufficient to guarantee isomorphism with a suspension semiflow with IDI?

**Q2.** How complicated can the IDI relation be? In particular, when does the relation IDI have a Borel selector?

- Always?

- If the semiflow has discrete orbit branchings?
More questions

Q3. Given a Borel semiflow \((X, T_t)\), can one choose a Polish topology on \(X\) with the same Borel sets as the original topology such that the action \(T_t\) is jointly continuous?

A3. No, if IDI \(\neq \Delta\).

Conjecture If \(T_t\) has no IDIs, then Q3 has an affirmative answer.

Theorem 4 (M) For countable-to-1 Borel semiflows with discrete orbit branchings and no IDIs, the conjecture holds.