Kinetic Models in Econophysics

Joint work with G. Toscani and B. Düring

Daniel Matthes
Vienna University of Technology, Austria

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The Kinetic Approach to Modelling Simple Markets

General Idea: In an extremely simple market model [1] trading agents behave like colliding molecules in a homogeneous gas, according to the following dictionary:

econophysics	particle dynamics
agents	molecules
wealth	momentum
mean wealth	total momentum
trade event	binary collision

Here the mean wealth (**1st** momentum) plays the same pivotal role as the total energy (**2nd** momentum) for Maxwell molecules.

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What makes these models special is an intrinsic randomness:
The risky assets that is exchanged in trades have a stochastic value.

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Modelling Trades with the Boltzmann Equation

$f(t;w)$ is the density of agents with wealth $w \in \mathbb{R}$ at time $t > 0$.

Paradigm (around 1985)

f satisfies a homogeneous one-dim. Boltzmann equation,

$$\partial_t f = Q_+(f,f) - f,$$

with collisional gain operator Q_+,

$$\int_{\mathbb{R}} \phi(w) Q_+(f,f)\,dw = \frac{1}{2} \iint_{\mathbb{R}\times\mathbb{R}} \mathbb{E}\left[\phi(w') + \phi(w'_*)\right] f(w)dw\, f(w_*)dw_*,$$

realizing the "trade rules"

$$w' = Lw + Rw_*, \quad w'_* = L_* w_* + R_* w$$

with random variables L, L_*, R, $R_* \geq 0$.

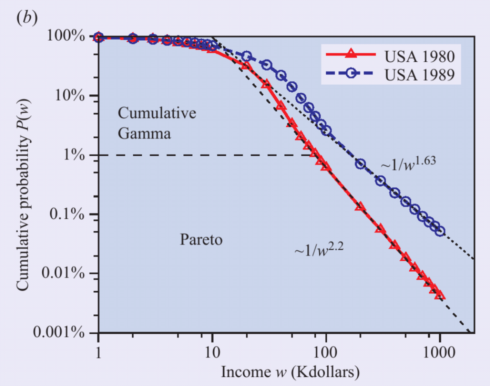
Particle interpretation:
Pre-trade wealths w, w_* change into post-trade wealths w', w'_*.

Pareto Tails

Let $f_\infty(w)$ denote the stationary wealth density,
and $F_\infty(w) = \int_w^\infty f_\infty(w')\,dw'$ the associated distribution function.

Pareto's Law (V. Pareto in "Cours d'Économie Politique" 1897)

$F_\infty(w) \approx w^{-\nu}$ *for* $w \gg 1$ *with a Pareto-index* $\nu \in (1.5, 2.5)$.



Classes of Models

- **Exponential growth of wealth**, $\mathbb{E}[w' + w'_*] \approx (1+\epsilon)(w + w_*)$.
  Pareto tails appear in self-similar solutions.
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- **Conservation in the mean**, $\mathbb{E}[w' + w'_*] = w + w_*$.
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- **Strict conservation**, $w' + w'_* = w + w_*$.
  For Pareto tails, mix species of different trading preferences.
  Angle, Social Forces **65** (1986), *Dragulescu, Yakovenko,* Eur.Phys.J.B **17** (2000), *Bouchaud, Mezard*, Physica A **282** (2000), *Chatterjee, Chakrabarti, Stinchcombe,* Phys.Rev.E **72** (2005), *Reptowicz, Hutzler, Richmond,* Physica A **356** (2005), *Mohanty,* Phys.Rev.E **74** (2006), *D.M., Toscani*, Kin.Rel.Models **1** (2008), ...

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Lots of closely related work on inelastic Maxwell molecules exist, like *Carlen, Gabetta, Toscani*, Comm.Math.Phys. **199** (1999), *Carlen, Carvalho, Gabetta*, Comm.Pure Appl.Math. **53** (2000), *Bobylev, Carrillo, Gamba*, J.Stat.Phys. **98** (2000), *Bobylev, Cercignani*, J.Stat.Phys. **110** (2003), *Carrillo, Cordier, Toscani*, Disc.Cont.Dyn.Syst.A **24** (2009), ...

The CPT-Model — Trade Rules

Trade Rules (S.Cordier&L.Pareschi&G.Toscani, `J.Stat.Phys.` (2005))

$$w' = \underbrace{\lambda \qquad}_{L} w + \underbrace{(1-\lambda)}_{R} w_*, \quad w'_* = \underbrace{\lambda \qquad}_{L_*} w_* + \underbrace{(1-\lambda)}_{R_*} w,$$

1. *the number $\lambda \in (0,1)$ is the saving propensity (i.e. the fraction of wealth not available for trading),*

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1. *the number $\lambda \in (0,1)$ is the saving propensity (i.e. the fraction of wealth not available for trading),*
2. *the centered i.i.d. random varibles η, $\eta_* \in (-\lambda, +\infty)$ define the risk (i.e. gains/losses due to stochastic value of the traded assets).*

The CPT model conserves the mean wealth,

$$\mathbb{E}[w' + w'_*] = \mathbb{E}[1+\eta]w + \mathbb{E}[1+\eta_*]w_* = w + w_*,$$

but is not strictly conservative unless $\eta \equiv \eta_* \equiv 0$ a.s.

Existence of Pareto tails

More generally, one can consider rules of the form

$$w' = Lw + Rw_*, \quad w'_* = L_* w_* + R_* w,$$

with random variables $L, L_*, R, R_* \geq 0$ satisfying

$$\mathbb{E}[L + R_*] = \mathbb{E}[L_* + R] = 1 \quad \Longrightarrow \quad \mathbb{E}[w' + w'_*] = w + w_*.$$

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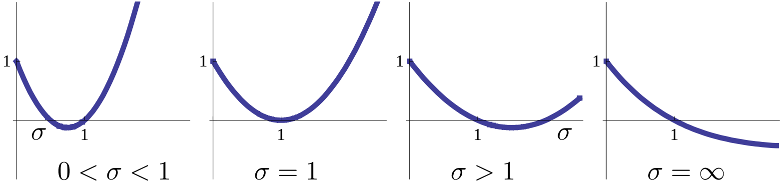
Tail properties of the corresponding steady state f_∞ are read off from

$$\mathbf{S}(s) = \frac{1}{2}\mathbb{E}[L^s + L_*^s + R^s + R_*^s] - 1.$$

Convexity of $\mathbf{S}$ and $\mathbf{S}(0) = 1$, $\mathbf{S}(1) = 0$ admit these possibilites:



Main Result for Models with Conservation of Mean Wealth

Theorem (D.M&G.Toscani `J.Stat.Phys.` (2008))

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The unique transient solution $f(t;w)$ to the Boltzmann equation converges weakly-$\star$ to the unique steady state $f_\infty(w)$.

1. *If $0 < \sigma < 1$, then f_∞ is a* **Dirac distribution** *at $w = 0$.*
2. *If $1 < \sigma < +\infty$, then f_∞ posseses a* **Pareto tail** *of index $\nu = \sigma$.*
3. *If $\sigma = +\infty$, then f_∞ possesses a* **slim tail***.*

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Remark 1: The steady state is always supported on $\mathbb{R}_+$.

Remark 2: Under additional moment and regularity hypotheses on f_0, the convergence $f(t) \to f_\infty$ is strong in L^1 and at exponential rate in t.

Elements of the Proof

First: Show contractivity of the evolution in Fourier distance, [2]

$$d_s\big(f(t), g(t)\big) \leq \exp\big(\mathbf{S}(s)\cdot t\big)\, d_s(f_0, g_0).$$

Second: Study evolution of the momentum hierarchy,

$$\frac{d}{dt}\int w^s\, f(t;w)dw = C(t) + \mathbf{S}(s)\cdot\int_{\mathbb{R}} w^s\, f(t;w)dw.$$

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For $\mathbf{S}(s) < 0$, one has

- weak-$\star$ convergence $f(t) \rightharpoonup f_\infty$ at exponential rate in d_s,
- t-uniform boundedness of the sth moment.

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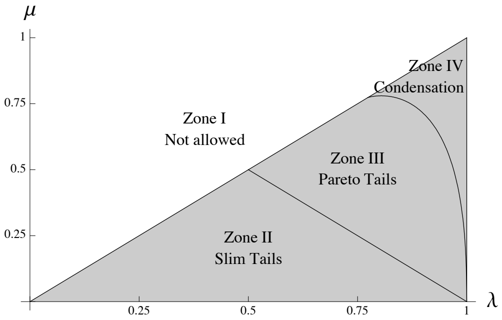
Regimes for High Societies

In the CPT model with $\eta = \pm\mu$, i.e.,

$$w' = (\lambda \pm \mu)w + (1-\lambda)w_*, \quad w'_* = (\lambda \pm \mu)w_* + (1-\lambda)w,$$

one obtains the following classification for f_∞:



- Region II: Socialism (Slim tails)
- Region III: Capitalism (Pareto tails)
- Region IV: Plutocracy (Dirac distribution)

Special Cases

Trade Rules (Winner-takes-all–model)

$$w' = w + w_*, \qquad w'_* = 0.$$

Wealth condensation occurs. One explicit solution is given by

$$f(t;w) = \Big(\frac{2}{2+t}\Big)^2 \exp\Big(-\frac{2w}{2+t}\Big)\mathbf{1}_{w>0} + \frac{t}{2+t}\delta_0(w).$$

More and more wealth is accumulated by fewer and fewer people.

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Trade Rules (Pure Exchange model)

$$w' = \lambda w + (1-\lambda)w_*, \quad w'_* = \lambda w_* + (1-\lambda)w.$$

The unique steady state $f_\infty = \delta_1$ is concentrated in the mean wealth. All agents are equally rich eventually.

The CCM-Model — Trade Rules

Trade Rules (Chakrabarti&Chatterjee&Manna `Physica A` (2004))

$$w' = \lambda w + (1-\lambda_*) w_*, \qquad w'_* = \lambda_* w_* + (1-\lambda) w,$$

with *agent-specific* *time-independent saving propensities* $\lambda,\ \lambda_* \in (0,1)$.

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- With the density ρ of λ on $(0,1)$,

$$f_\infty(w) = \int_0^1 \tilde{f}_\infty(\lambda,w)d\lambda = \frac{\gamma}{w^2}\rho\Big(1-\frac{\gamma}{w}\Big),$$

Calculate moments of f_∞,

$$\int_0^\infty w^s f_\infty(w)dw = \int_{1/\gamma}^\infty w^s \frac{\gamma}{w^2}\rho\Big(1-\frac{\gamma}{w}\Big)dw = \gamma^s \underbrace{\int_0^1 (1-\lambda)^{-s}\rho(\lambda)d\lambda}_{=:\mathbf{Q}(s)}.$$

CCM–model — Existence of Pareto Tails

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Theorem (D.M.&G.Toscani `Kinet.Rel.Models` (2008))

Assume $\mathbf{Q}(1) < \infty$. Let f_0 have unit first and finite second momentum.

Then the transient wealth distribution $f(t;w)$ converges weakly-$\star$ to the unique steady distribution

$$f_\infty(w) = \frac{1}{\mathbf{Q}(1)w^2}\,\rho\Big(1-\frac{1}{\mathbf{Q}(1)w}\Big),$$

and f_∞ has a Pareto tail of index $\nu = \inf\{s \,|\, \mathbf{Q}(s) = +\infty\}$.

Time Scales in Relaxation

Two processes:

1. Agents of the same λ accumulate at "local" mean wealth $W(t;\lambda)$
2. $W(t;\lambda)$ tends to limit $\mathbf{Q}(1)^{-1}(1-\lambda)^{-1}$

Result from direct simulation Monte Carlo for 30 families of agents with 30 members each; vertical axis shows $\mathbf{Q}(1)(1-\lambda)w$.

Creating of a Pareto Tail

To show: $\tilde{f}(t;\lambda,w)$ concentrates on

$$\mathcal{M} := \left\{ (\lambda,w) \middle| \; (1-\lambda)w = \mathbf{Q}(1)^{-1} \right\} \subset (0,1) \times \mathbb{R}_+.$$

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1. **Fast dynamics:** Concentration of the λ-species at $w = W(t;\lambda)$,

$$\rho(\lambda)^{-1} \int_0^\infty \big(w - W(t;\lambda)\big)^2 \tilde{f}(t;\lambda,w)dw \to 0$$

in L^1_ρ at rate $t^{-\nu}$.

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Optimality: Lower bound on Wasserstein distance W_1 follows from

- $W(t;\lambda) \le W(0;\lambda) + t$, and
- "unfilled" Pareto tail $t < w < \infty$ has first momentum $\approx t^{-(\nu-1)}$.

Summary

- **Mandelbrot's Paradigma:**
  Simple markets behave like homogeneous Boltzmann gases.
- **Benchmark:**
  Steady states should exhibit Pareto tails.
- **Equilibration:**
  Pareto tails...
  - ...are exponentially stable in the CPT-model,
    which conserves wealth in the statistical mean only.
  - ...are only algebraically stable in the CCM-model,
    which is strictly conservative.

Thank you!