Entropy and Noncommutative Mahler Measure

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[joint work with Klaus Schmidt]

1. What is Mahler measure?

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_0 \in \mathbb{Z}[x]$$

$$m(f) := \int_0^1 \log |f(e^{2\pi i t})| \, dt$$

$$M(f) := e^{m(f)}$$

Why was Mahler interested?

$$L(f) := |c_n| + |c_{n-1}| + \dots + |c_0|$$

Clear: $L(fg) \le L(f) L(g)$

How much smaller can it be?

Quite a bit:

$$2 = L(x^n - 1) = L\big((x-1)(x^{n-1} + x^{n-2} + \dots + 1)\big)$$
$$\leq L(x-1) \cdot L(x^{n-1} + \dots + 1) = 2 \cdot n$$

How bad can this get?

Mahler introduced $M(f)$ as an "irreducible core" that is never reduced under multiplication:

$$M(fg) = M(f) \cdot M(g)$$
$$M(f) \leq L(f) \leq 2^{\deg f} M(f)$$

$\Rightarrow$

$$2^{-\deg(fg)} L(f) L(g) \leq L(fg)$$

– Mahler in Canberra –

2. Entropy and Mahler measure

α: automorphism of compact abelian group X (think $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ on $\mathbb{T}^2$)

μ: Haar measure on X

B_ε: ε-ball around 0 in X

Mathematisches Institut
der Universität

34 Göttingen, December 11, 1965

Dear Professor Mahler,

Thank you very much for your letter including the nice photography from Canberra. It seems that this is a healthy place for living, and I hope that the next year will be pleasant for you, before you go to your big trip in 1967 and retire in 1968.

My eye health is deteriorating more and more. This term I am still lecturing here, on Abelian functions, but I think that I shall definitely stop next Spring. Besides, I do not agree with the general trend of so-called „modern mathematics", so it is better to keep quiet in this period of decay.

With best wishes for the coming year I remain,

Cordially yours,

Carl L. Siegel.

$h(\alpha)$ = entropy of α, measures volume decrease:

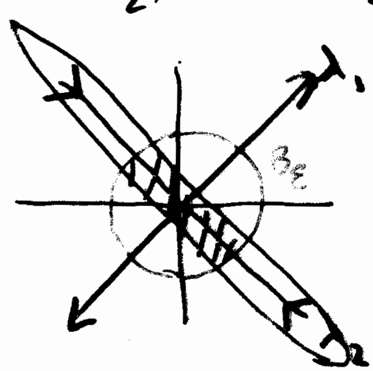
$$\mu\left(\bigcap_{j=0}^{n-1} \alpha^{-j}(B_\varepsilon)\right) \asymp e^{-h(\alpha)\cdot n} \quad \text{as } n \to \infty$$

or "officially"

$$h(\alpha) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu\left(\bigcap_{j=0}^{n-1} \alpha^{-j}(B_\varepsilon)\right)$$

<u>Ex</u>: Let $\alpha \in GL(r, \mathbb{Z})$ act on $X = \mathbb{T}^r$.

$\alpha^{-j}(B_\varepsilon)$



If α has eigenvalues $\lambda_1, \dots, \lambda_r$

$$\mu\left(\bigcap_{j=0}^{n-1} \alpha^{-j}(B_\varepsilon)\right) \asymp \prod_{|\lambda_j|>1} |\lambda_j|^{-n}$$

so $\quad h(\alpha) = \sum_{j=1}^{r} \log^+ |\lambda_j|$

$$\chi_\alpha(t) := \det(tI - \alpha) = \prod_{j=1}^{r} (t - \lambda_j)$$

Fact (Jensen): $\log^+|\lambda| = \int_0^1 \log |e^{2\pi i t} - \lambda| \, dt$

$$\therefore \; h(\alpha) = \sum_{j=1}^{r} \int_0^1 \log |e^{2\pi i t} - \lambda_j| \, dt$$

$$= \int_0^1 \log |\chi_\alpha(e^{2\pi i t})| \, dt = m(\chi_\alpha)$$

General framework:

$$f = f(x) = \sum_j f_j x^j \in \mathbb{Z}[x^{\pm 1}] \rightsquigarrow \mathbb{Z}\text{-action } \alpha_f \text{, auto of compact ab. gp. } X_f$$

$$\mathbb{T}^{\mathbb{Z}} \supset X_f := \{ t \in \mathbb{T}^{\mathbb{Z}} : \sum_i f_i t_{i+n} = 0 \text{ all } n \in \mathbb{Z} \}$$

$$\alpha_f := \leftarrow \text{left shift} \big|_{X_f} .$$

Ex: $f(x) = x^2 - x - 1$, $X_f = \{ t \in \mathbb{T}^{\mathbb{Z}} : t_{n+2} - t_{n+1} - t_n = 0, \text{ all } n \in \mathbb{Z} \}$

Actually α_f is just $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ on $\mathbb{T}^2$ in disguise:

$$\begin{tikzcd} X_f \ni \cdots \overset{t_{-1}}{\bullet} \; \overset{t_0}{\bullet} \; \overset{t_1}{\bullet} \; \overset{t_2}{\bullet} \cdots \arrow[r, "\alpha_f"] \arrow[d] & \cdots \overset{t_0}{\bullet} \; \overset{t_1}{\bullet} \; \overset{t_2}{\bullet} \; \overset{t_3}{\bullet} \cdots \arrow[d] \\ \mathbb{T}^2 \ni \begin{bmatrix} t_0 \\ t_1 \end{bmatrix} \arrow[r, "\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}"] & \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \end{tikzcd}$$

$$\checkmark \quad = \begin{bmatrix} t_1 \\ t_0 + t_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} t_0 \\ t_1 \end{bmatrix}$$

Rk: X_f is Pontryagin dual of $\mathbb{Z}[x^{\pm 1}] / \langle f(x) \rangle$

α_f is dual to mult. by x on $\uparrow$.

Thm: $h(\alpha_f) = m(f)$ all $f \in \mathbb{Z}[x^{\pm 1}] \setminus \{0\}$

(a long history, complete generality due to Yuzvinsky)

This framework extends directly to polynomials in several (commuting) variables:

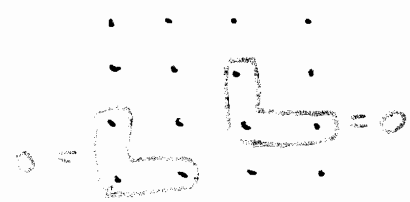
$$f = f(x,y) = \sum_{i,j} f_{ij}\, x^i y^j \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}] \rightsquigarrow \mathbb{Z}^2\text{-action } \alpha_f \text{ on } X_f$$

$$\mathbb{T}^{\mathbb{Z}^2} \supset X_f := \left\{ t \in \mathbb{T}^{\mathbb{Z}^2} : \sum_{i,j} f_{ij}\, t_{i+m,\, j+n} = 0 \quad \forall m,n \right\}$$

$$\alpha_f : \mathbb{Z}^2\text{-action gen by } \leftarrow, \downarrow \; \Big|_{X_f}$$

Ex: $f(x,y) = 1 + x + y$,

$$X_f = \left\{ t \in \mathbb{T}^{\mathbb{Z}^2} : t_{m,n} + t_{m+1,n} + t_{m,n+1} = 0 \quad \forall m,n \right\}$$



Here X_f is dual to $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}] / \langle f(x,y) \rangle$

α_f is dual to $\mathbb{Z}^2$-action gen. by $\times x$, $\times y$ on

Entropy is measured by volume decrease:

$$\mu\Big(\bigcap_{j \in \{0,\dots,n-1\}^2} \alpha_f^{-j}(B_\varepsilon) \Big) \asymp e^{-h(\alpha_f)\cdot n^2}$$

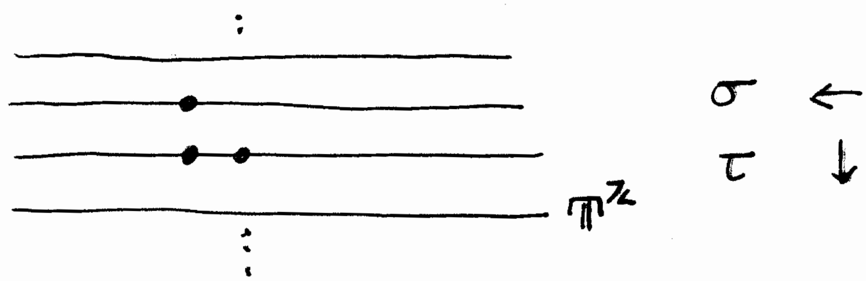
⑥

$$m(f) := \int_0^1 \int_0^1 \log |f(e^{2\pi i s}, e^{2\pi i t})| \, ds \, dt$$

<u>Thm</u>: (L-Schmidt-Ward) $\quad h(\alpha_f) = m(f)$

Rough idea for $f(x,y) = 1 + x + y$:

① Slice up $X_f \subset \mathbb{T}^{\mathbb{Z}^2}$ into horizontal layers



② σ on $\mathbb{T}^{\mathbb{Z}}$ has a continuum of "eigenspaces" V_θ indexed by $e^{2\pi i \theta} \in \mathbb{S}$, $V_\theta = \mathbb{C} \cdot v_\theta$,

$$v_\theta = (\ldots, e^{-2\pi i \theta}, 1, e^{2\pi i \theta}, e^{4\pi i \theta}, \ldots)$$

③ On V_θ, $\tau = -I - \sigma$ acts by multiplication by $-1 - e^{2\pi i \theta}$

④ By analogy with toral auts, only those θ with $|-1-e^{2\pi i \theta}| > 1$ contribute to volume decrease, so should have

$$h(\alpha_f) = \int_0^1 \log^+ |-1-e^{2\pi i \theta}| \, d\theta = \int_0^1 \int_0^1 |1+e^{2\pi i \theta}+e^{2\pi i \phi}| \, d\theta \, d\phi$$

Jensen

$$= m(1+x+y)$$

So far looked at $\mathbb{Z}[x^{\pm 1}] = \mathbb{Z}[\mathbb{Z}]$ and $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}] = \mathbb{Z}[\mathbb{Z}^2]$. But the framework extends directly to $\mathbb{Z}[\Gamma]$:

Γ: discrete countable group, written multiplicatively

$\{F_n\}$: Følner sequence of "averaging sets" analogous to $\{0, \dots, n-1\}^d \subset \mathbb{Z}^d$, so Γ is amenable

$\mathbb{Z}[\Gamma]$: integral group ring of Γ

For $f = \sum_\gamma f_\gamma \cdot \gamma \in \mathbb{Z}[\Gamma]$ define a Γ-action α_f on X_f by:

$$\mathbb{T}^\Gamma \supset X_f := \{ t = (t_\gamma) \in \mathbb{T}^\Gamma : \sum_\gamma f_\gamma t_{\gamma\theta} = 0 \quad \forall \theta \in \Gamma \}$$

α_f = restriction to X_f of left Γ-action on $\mathbb{T}^\Gamma$

$h(\alpha_f)$ = entropy of Γ-action α_f:

$$\mu\Big(\bigcap_{\gamma \in F_n} (\alpha_f^\gamma)^{-1}(B_\varepsilon) \Big) \asymp e^{-h(\alpha_f) \cdot |F_n|} \quad (n \to \infty)$$

Now turn the LSW thm around, and define Mahler measure as entropy:

$$m_\Gamma(f) := h(\alpha_f)$$

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When $\Gamma = \mathbb{Z}^d$, the integral formula for $m(f)$ easily shows $m(fg) = m(f) + m(g)$, but for general Γ not clear.

<u>Thm</u> Suppose $g \in \mathbb{Z}[\Gamma]$ is not a right zero-divisor in $\mathbb{Z}[\Gamma]$. Then $m(fg) = m(f) + m(g)$ for every $f \in \mathbb{Z}[\Gamma]$.

<u>Rk</u>: If Γ is torsion-free, it is a notorious open problem whether $\mathbb{Z}[\Gamma]$ can <u>ever</u> have zero divisors!

The proof starts by showing α_{fg} is an extension of α_f by α_g, uses the Abromov – Rohlin thm for amenable actions due to Ward-Zhang, a simple volume estimate, and an adaptation of the proof of the variational principle for amenable actions due to Ollagnier.

<u>Basic Problem</u>: How to compute, exactly or numerically, $m(f)$?

Focus on the simplest noncomm. example:

Discrete Heisenberg group

$$\Gamma = \left\{ \begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{Z} \right\}$$

$$x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad z = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{so}$$

$$x^a y^b z^c = \begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}$$

$$yx = xyz, \quad zx = xz, \quad zy = yz$$

$Z(\Gamma) = \text{center}(\Gamma) = \{ z^n : n \in \mathbb{Z} \}$

$\Gamma / Z(\Gamma) \cong \mathbb{Z}^2$, so Γ is a $\mathbb{Z}$-ext. of $\mathbb{Z}^2$.

A typical $f \in \mathbb{Z}[\Gamma]$ has the form

$$f = f(x,y,z) = \sum_{i,j,k} f_{ijk}\, x^i y^j z^k,$$

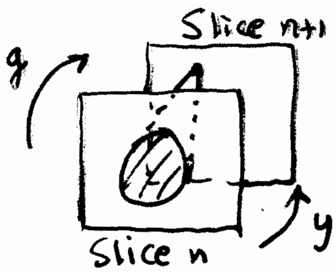
like a Laurent polynomial, except now the variables obey the one noncommutative relation $yx = xyz$.

This leads to some very interesting phenomena.

Special case: linear in y:

$$f(x,y,z) = y - g(x,z)$$

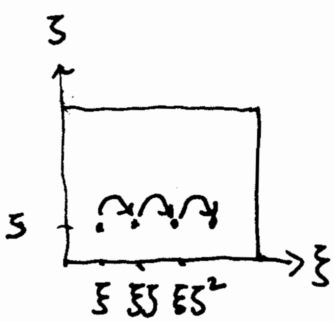
There is, as before, a spectral analysis, using "xz-slices" of X_f. For each $(\xi, \zeta) \in \mathbb{S}^2$ get an



"xz-eigenspace" $V_{\xi,\zeta}$, and going from n^{th} slice to $(n+1)^{st}$ slice is same as multiplying by $g(\xi\zeta^n, \zeta)$. So

$V_{\xi,\zeta}$ contributes to volume decrease iff

$$\limsup_{n\to\infty} \frac{1}{n} \log | g(\xi,\zeta)\, g(\xi\zeta,\zeta)\, g(\xi\zeta^2,\zeta) \cdots g(\xi\zeta^{n-1},\zeta)| > 0$$



$$\Big\downarrow \ (\zeta \text{ irrational})$$

$$\int_{\mathbb{S}} \log |g(\xi,\zeta)|\, d\xi \qquad (= m(g(\cdot,\zeta)))$$

So

$$m(y - g(x,z)) = \int_{\mathbb{S}} \left[\int_{\mathbb{S}} \log|g(\xi,\zeta)|\, d\xi \right]^+ d\zeta$$

<u>Ex</u>: $m_\Gamma(1 + x + y) = 0$ (!)

$m_\Gamma(y - (z^3 - z - 1)(x-1)) \approx 0.427288...$

(11)

Thm: $m_\Gamma(y - g(x,z)) = 0$ iff $g(x,z)$ is the product of a monomial $x^a z^c$ and a finite number of generalized cyclotomics of the form $c(x^r z^s)$ with $r \neq 0$, $c(u)$ = cyclotomic in u.

Quadratic in y examples:

$$f(x,y,z) = y^2 - g_1(x,z)y - g_0(x,z).$$

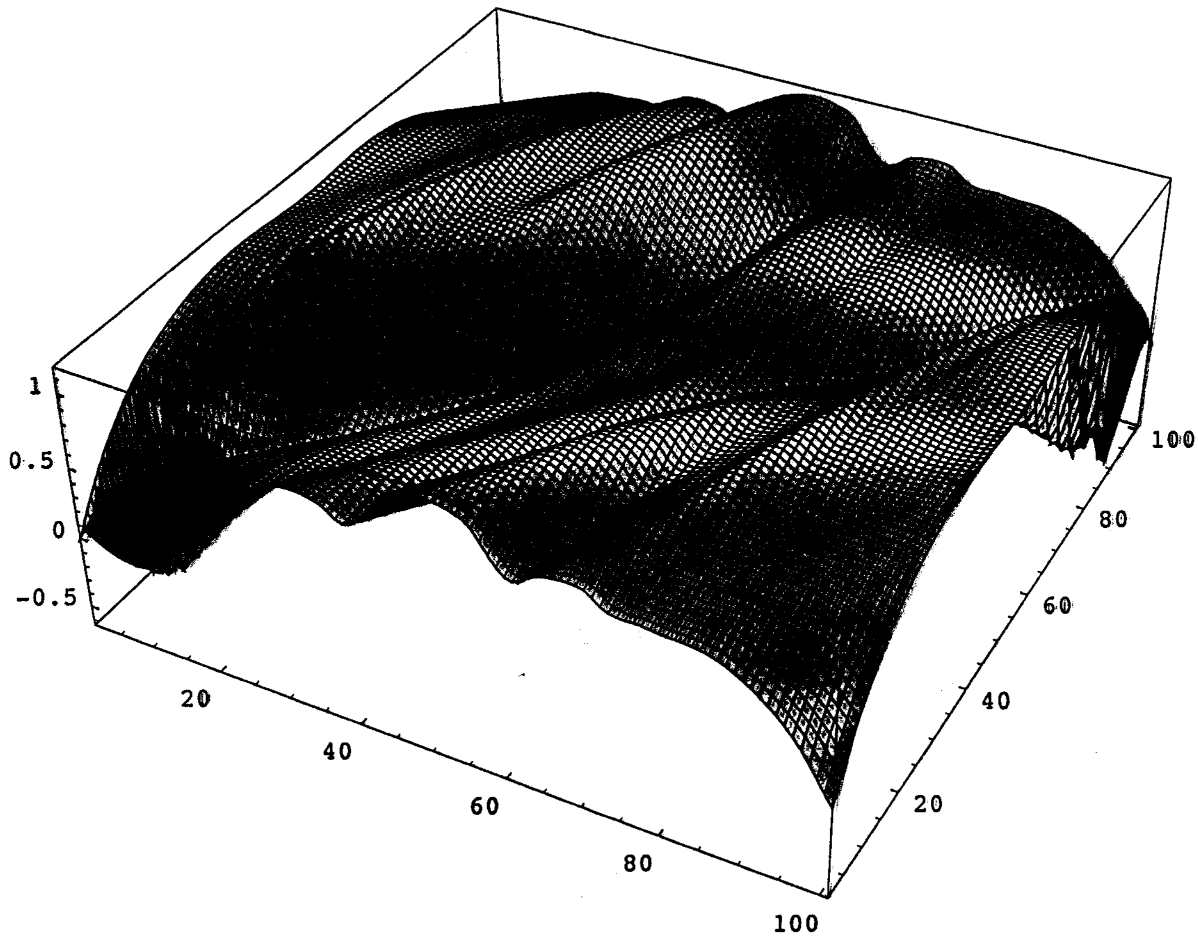
Now use adjacent pairs of xz-slices, and so pairs of $V_{\xi,\zeta}$ eigenspaces. Here the dynamics is controlled by products of noncommuting matrices

$$P_n(\xi,\zeta) = \prod_{j=n-1}^{0} \begin{bmatrix} 0 & 1 \\ g_0(\xi\zeta^j,\zeta) & g_1(\xi\zeta^j,\zeta) \end{bmatrix}$$

The Multiplicative Ergodic Theorem says that for almost every ζ the growth of $P_n(\xi,\zeta)$ is the same for almost every ξ, and measured by the Lyapunov exponents $\Lambda_1(\zeta)$, $\Lambda_2(\zeta)$

so:

$$h(\alpha_f) = m_\Gamma(f) = \int_{\mathbb{S}} [\Lambda_1^+(\zeta) + \Lambda_2^+(\zeta)]\, d\zeta$$



Lyapunov Surface of $y^2-(2x-1)y+1$

Using mathematica, can run the product $P_n(\xi,\zeta)$ backwards to find the dominant direction at (ξ,ζ) and the Lyapunov multiplier $\lambda(\xi,\zeta)$ going from (ξ,ζ) to $(\xi\zeta,\zeta)$.

Plotting $\log|\lambda(\xi,\zeta)|$ gives the <u>Lyapunov surface</u> for f.

<u>Ex</u>: $f(x,y,z) = y^2 - (x+1)y - (x+z)$

Using $P_{300}(\xi,\zeta)$ and 700 decimal places,

$$m_{\mathbb{P}}(f) \approx 0.3219$$

<u>Ex</u>: $f(x,y,z) = y^2 - xy - 1$. Linear in x, get $m_{\mathbb{P}}(f) = 0$. Quadratic in y, so obtain that for almost every $(\xi,\zeta) \in \mathbb{T}^2$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \left\| \prod_{j=n-1}^{0} \begin{bmatrix} 0 & 1 \\ 1 & \xi\zeta^j \end{bmatrix} \right\| = 0$$

So computing entropy in two ways can be interesting!

Many many open questions:

① If $f = \sum_\gamma f_\gamma \cdot \gamma \in \mathbb{Z}[\Gamma]$, let $f^* = \sum_\gamma f_\gamma \cdot \gamma^{-1}$.
Is $m(f) = m(f^*)$? OK if Γ is abelian, or if α_f is expansive (Schmidt-Deninger)

② Characterize f's for which α_f expansive, even for Heisenberg

③ Lehmer! Is there Γ, $f \in \mathbb{Z}[\Gamma]$ with
$0 < m_\Gamma(f) < 0.17$?

④ Characterize f with $m(f) = 0$.

⑤ When if α_f mixing? Bernoulli?
($\Gamma = \mathbb{Z}^d$ Bernoullicity settled by Rudolph and Schmidt)

⑥ Get rigorous numerical approximations for $m(f)$

⑦ So far we've just used principal left ideals $\mathbb{Z}[\Gamma] \cdot f$. What happens with non-principal ideals?