Curves on surfaces

§1. S sm. pr. sf. / $\mathbb{C}$, $\beta \in H_2(S)$

$$H_\beta := \mathrm{Hilb}_\beta(S) = \{ C \text{ eff. div. on } S,\ [C] = \beta \}$$

$$\mathrm{Pic}^\beta(S) = \{ [L] \in \mathrm{Pic}(S) \mid c(L) = \beta \}$$

$$AJ : H_\beta \to \mathrm{Pic}^\beta,\quad C \mapsto [\mathcal{O}_S(C)]$$

$$\begin{array}{ccc} H_\beta & \to & \mathrm{Pic}^\beta \\ \cup\!\mid & & \Psi \\ \mathbb{P}^{h^0(L)-1} \cong |L| & \to & [L] \end{array}$$

$\overline{C} \subseteq S$: def. $H^0(N_{C/S})$ $\mathcal{O}_C(C)$

ob $H^1(N_{C/S})$

$\mathcal{C} \to H_\beta$ univ., $\pi : H_\beta \times S \to H_\beta$

$R\pi_* \mathcal{O}_{\mathcal{C}}(\mathcal{C})$

Dürr-Kabanov-Okonek:

$\exists$ perf. ob. thy. $F^\bullet = (R\pi_* \mathcal{O}_{\mathcal{C}}(\mathcal{C}))^\vee \longrightarrow \mathbb{L}_{M_\beta}$.

Def. (Behrend-Fantechi)

M scheme, $E^\bullet = \{E^{-1} \longrightarrow E^0\}$ vbs

$\phi : E^\bullet \longrightarrow \underbrace{\mathbb{L}_M}_{\text{cot. cx. } M}$ mph. in $D^b(M)$

s.t. $h^0(\phi)$ iso, $h^{-1}(\phi)$ surj.

This data: perf ob. thy. on M.

$\underline{\Phi}$ gives:

- $\forall p \in M$: def. $h^0(E^{\bullet\vee}|_p)$

  ob. $h^1(E^{\bullet\vee}|_p)$

- $[M]^{vir} \in A_{vd}(M)$

  $vd = \mathrm{rk}\, E^{\bullet} = \dim \mathrm{defs} - \dim \mathrm{ob}.$

$\overline{C} \subseteq S$: $0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$

$\sigma : H^1(N) \to H^2(\mathcal{O}_S)$ semi-reg. map

$\underline{\beta} \in \ker \sigma$

$\overline{A} \subseteq S$ eff. div. s.t.

$\forall\ [L] \in \mathrm{Pic}^{\gamma}$: $h^1(L) = h^2(L) = 0$

$\gamma := \beta + [A]$

(then H_γ sm.)

$$M_\beta \underset{\iota}{\hookrightarrow} \underset{\text{sm.}}{M_\gamma} \;:\; C \mapsto A + C$$

$$D \in M_\gamma : \quad D \in \iota(M_\beta) \Leftrightarrow s_D|_A = 0 \in H^0(\mathcal{O}_A(D))$$

$$\mathcal{D} \to M_\gamma, \quad \pi : M_\gamma \times A \to M_\gamma$$

$$\pi_* \mathcal{O}(\mathcal{D})|_{M_\gamma \times A} \;\; \underset{s}{\rightleftarrows} \;\; M_\gamma \;\; \supset \; s^{-1}(0) = \iota(M_\beta)$$

sheaf!

Assume: $\forall\, [L] \in \mathrm{Pic}^{\beta}$: $h^2(L) = 0$.

$\Rightarrow \pi_* \mathcal{O}_A(\mathcal{D})$ eff. vb on ngh. open $\subset (M_\beta)$

Claim: $\rightsquigarrow$ p.o.t. $F^{\text{red.}} \longrightarrow \mathbb{L}_{M_\beta}$

(ob: $\ker \sigma^{\beta}$

$$\begin{array}{l} E \quad \text{vb.} \\ \downarrow \uparrow s \\ N \overset{\iota}{\supseteq} s^{-1}(0) = M \\ \text{sm.} \end{array}$$

$$E^{\bullet} = \{ E^{*}|_M \longrightarrow \Omega_N|_M \}$$

$$\begin{tikzcd} E^{*}|_M \arrow[r] \arrow[d, "s^{*}"'] & \Omega_N|_M \arrow[d, equal] \\ I_M/I_M^2|_M \arrow[r, "d"] & \Omega_N|_M \end{tikzcd}$$

$$\underbrace{I_M/I_M^2|_M}_{= \mathbb{L}_M}$$

$\phi : E^{\bullet} \longrightarrow \mathbb{L}_M$ p.o.t. $\iota_{*}[M]^{vir} = c_{top}(E)$

$F \rightsquigarrow [M_\beta]^{vir}$ $\quad vd = \frac{\beta(\beta - k)}{2}$

$F^{red.} \underset{(ASI)}{\rightsquigarrow} [M_\beta]^{red}$ $\quad k := c_1(\mathcal{O}(K_S))$

$vd = \text{——} \, n \, \text{——} + h^{0,2}(S)$

DKO

$$\Omega \cdot [M_\beta]^{vir} \longrightarrow P_S^{\pm}(\beta) \in H_*(\mathrm{Pic}^\beta) = \Lambda^* H^2(S,\mathbb{Z})$$

$$[M_{L-\beta}]^{vir} \nearrow$$

Poincaré inv.

Conj. (DKO): $h^{0,2}(S)>0$: $P_S^+(\beta) = P_S^-(\beta) = SW_\beta(S)$

$h^{0,2}(S)=0$: $P_S^{\pm}(\beta) = SW_\beta^{\pm}(\beta)$.

SW = Seiberg-Witten

Thm. (Chang-Kiem). True.

§3. $Z \subseteq C \subseteq S$ $\quad s \circlearrowleft E$
0-dim.

$\mathrm{Hilb}^n(\mathcal{C}/M_\beta) \subseteq S^{[n]} \times M_\beta$

Hilb n pts. on fibres $\mathcal{C}/M_\beta$

Hilb n pts. on S (sm.)

$s^{-1}(0)$ $\quad M^0(\mathcal{O}_Z \subseteq C)$

$\rightsquigarrow$ rel. p.o.t. on $\mathrm{Hilb}^n(\mathcal{C}/_{M_\beta})$ $/_{M_\beta}$

<u>Thm. 1.</u> (K–Thomas)

This rel. p.o.t. $\pm$ $[M_\beta]^{vir}$, $[M_\beta]^{red}$

give $[\mathrm{Hilb}^n(\mathcal{C}/_{M_\beta})]^{vir/red}$.

PT thy: X CY3:

stable pairs: (F, s) : – F pure dim 1

– $s \in H^0(F)$, st. cokers 0-dim.

E.g. $C \subseteq X$ CM cv.

$Z \subseteq C$ Cartier $\mathcal{O}_X \to \mathcal{O}_C(Z)$.

$\chi(F) = \chi \in \mathbb{Z}$, $\operatorname{supp} F = \beta \in H_2$

$P_\chi(X, \beta)$ = modulispace le Potier

$P_\chi(X,\beta) \subseteq$ comp. moduli sp. of cx. in $D^b(X)$

$(F,s) \leftrightarrow \{\overset{0}{\mathcal{O}_X} \to \overset{1}{F}\}$

Thm. (PT). $\exists$ natural p.o.t. on $P_\chi(X,\beta)$ of vd $= 0$.

$$PT_{\chi,\beta}(X) = \int_{[P_\chi(X,\beta)]^{vir}} 1$$

$$GW_{g,\beta}(X) = \int_{[\overline{M}_{g,0}(X,\beta)]^{vir}} 1$$

GW-inv. (discomn.)

$$PT_\beta(X) = \sum_\chi PT_{\chi,\beta}(X)\, q^\chi$$

$$GW_\beta(X) = \sum_g GW_{g,\beta}(X)\, u^{2g-2}.$$

Conj. (GW/PT). $GW_\beta(X) = PT_\beta(X)$ for $-q = e^{iu}$.

$X = \mathrm{Tot}(K_S)$ $\mathbb{C}^*\,\Omega$ noncpt CY3

$P_\chi(X,\beta) \supseteq \underset{\text{cpt.}}{P_\chi(X,\beta)^{\mathbb{C}^*}} = \underbrace{P_\chi(S,\beta)}\sqcup \cdots$

$\chi = 1 - h + n$ $\qquad\cong \mathrm{Hilb}^n(\mathcal{C}/M_\beta)$

$h = \text{arith } \beta$.

$P_\chi(X,\beta)$

$\longrightarrow E^{\circ}_{X,PT}$ PT thy.

$\longrightarrow E^{red.}_{X,PT}$ red. PT thy

(assume (AS2): $\cup\beta : H^1(T_S) \twoheadrightarrow H^2(\mathcal{O}_S)$

(AS2) $\Rightarrow$ (AS1) .)

$\sigma_1, \cdots, \sigma_m \in H^{\cdot}(X, \mathbb{Z})$

$\rightsquigarrow \overline{PT}^{red}_{\chi,\beta}(X, \sigma_1 \cdots \sigma_m)$

def via GP localisation, by int. over $[P_\chi(X,\beta)^{T}]^{vir/red}$

<u>Thm 2.</u> (K–Thomas).

$E^{\cdot}_{PT,X}\Big|^{fix}_{P_\chi(S,\beta)}$, gives $[Hilb^n(\mathcal{C}/H_\beta)]^{vir}$.

Same for red.

§4. Thm. (K)

$\forall S, \beta$ s.t. ① β irred., or

② $-K_S$ nef β suff. ample wrt. h

h = arith. gen β

$X = \mathrm{Tot}(K_S)$

$m := \frac{\beta(\beta - K_S)}{2}$

Then:

$$GW_\beta(X, [pt]^m) \quad u = q = e^{iu}$$

$$PT_\beta(X, [pt]^m)$$

have the same lowest order term

$$\Longleftrightarrow \quad SW_\beta(S) = \int_{[\overline{M}_{h,\beta}(S,\beta)]^{vir}} \prod_{i=1}^{m} ev_i^* \, pt.$$

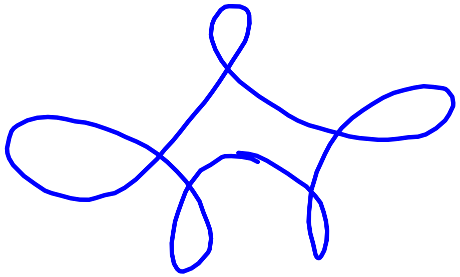
true: Taubes

$\overline{S}, \beta, |L|, c_1(L) = \beta$

$\delta \geq 0$, L suff. ample wrt. δ

$\mathbb{P}^\delta \subseteq |L|$ has finite ~~#~~ of δ-nodal

general crvs.



$a^L_\delta(S)$ = severi degree

$$\overline{M}_{h-\delta,\bullet}(S,\beta) \ni f : C \to S$$

$h = h(\beta)$ arith: $\quad f(C) \in \mathbb{P}^{\delta} \subseteq |L| \subseteq H_\beta$

$$(\ldots)^{BL} = \prod_{i=1}^{b_1} ev_i^* \gamma_i \times \prod_{j=b_1}^{b_1+\chi(L)-1-\delta} ev_i^* pt$$

γ_i basis $H_1(S)$

$$\bullet = b_1 + \chi(L) - 1 - \delta$$

$$\deg(\cdots) = \mathrm{vd}\ [\overline{M}_{h-\delta,\circ}(S,\beta)]^{red}$$

<u>Fact:</u>

$$\int_{[\overline{M}_{h-\delta,\circ}(S,\beta)]^{red}} (\cdots) = a^L_\delta(S).$$

Thm 4. (K-Thomas). u $-q = e^{iu}$

Then: $GW^{red}_{\beta}(X, (\dots))$, $PT^{red}_{\beta}(X, (\dots))$

have same lowest order term $\Longleftrightarrow$

$$a^{S}_{\delta}(L) = \underset{i=0,\dots,\delta}{\text{lin. comb.}}\ e(\mathrm{Hilb}^{i}(\mathcal{C}/\mathbb{P}^{\delta}))$$

true: K-Shende-Thomas