Homogenizations in Perforated Domain

Ki-Ahm Lee

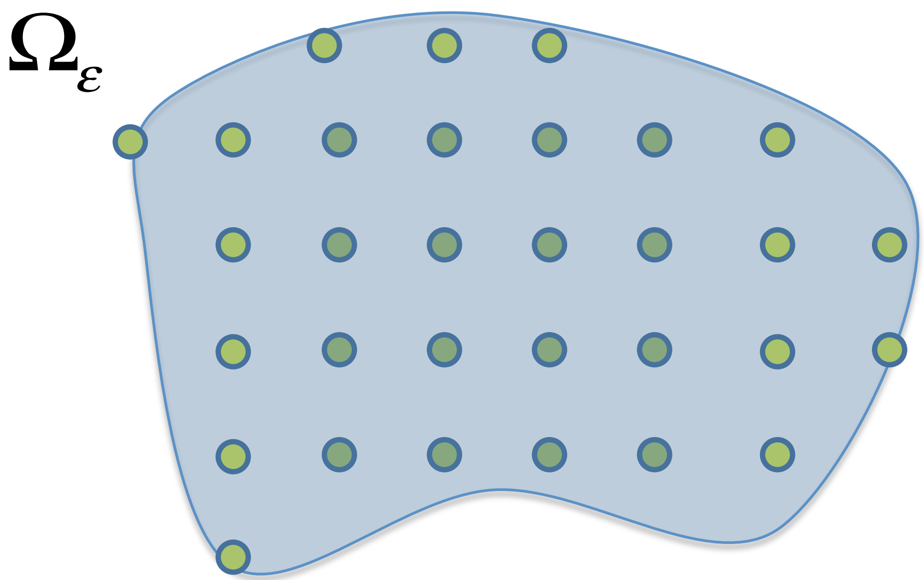
Seoul National University

Outline

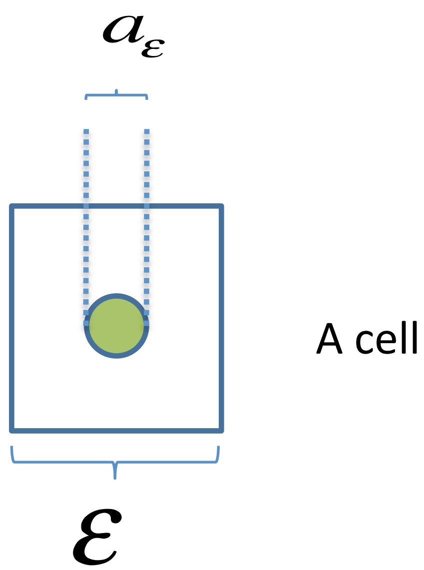
1. Perforated Domain
2. Neumann Problems (joint work with Minha Yoo; interesting discussion with Li-Ming Yeh)

3. Dirichlet Problems
   3.1. Elliptic Case (joint works with L. Caffarelli)
   3.2 Parabolic Case (joint works with Sunghoon Kim)
   3.3. Nonlinear Eigen Value Problem (joint works with Sunghoon Kim)
   3.4 Porous Medium Equation (joint works with Sunghoon Kim)

1. Perforated Domain



$$\Omega_\varepsilon = \Omega \setminus T_{a_\varepsilon}$$



1.1 Linear Equations

(Dirichlet Problems)

Find the solution u_ε such that

$$\Delta \mathrm{u}_\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon$$

$$u_\varepsilon = 0 \text{ in } \quad \partial\Omega_\varepsilon \cap \Omega$$

$$u_\varepsilon = g(x) \text{ on } \quad \partial\Omega$$

(Neumann Problems)

Find the solution u_ε such that

$$\Delta \mathrm{u}_\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon$$

$$\frac{\partial \mathrm{u}_\varepsilon}{\partial \nu} = 0 \text{ in } \quad \partial\Omega_\varepsilon \cap \Omega$$

$$u_\varepsilon = g(x) \text{ on } \quad \partial\Omega$$

1.2 Nonlinear Equations

(Dirichlet Problems)

Find the solution u_ε such that

$$F(D^2 u_\varepsilon) = 0 \quad \text{in} \quad \Omega_\varepsilon$$

$$u_\varepsilon = 0 \text{ in } \quad \partial\Omega_\varepsilon \cap \Omega$$

$$u_\varepsilon = g(x) \text{ on } \quad \partial\Omega$$

(Neumann Problems)

Find the solution u_ε such that

$$F(D^2 u_\varepsilon) = 0 \quad \text{in} \quad \Omega_\varepsilon$$

$$\frac{\partial u_\varepsilon}{\partial \nu} = 0 \text{ in } \quad \partial\Omega_\varepsilon \cap \Omega$$

$$u_\varepsilon = g(x) \text{ on } \quad \partial\Omega$$

1.3 Questions:

(1). What's the uniform estimate satisfied by u_ε to extract convergent subsequence?

(2). If u is a limit of u_ε, what is the equation (or Homogenized Equation, Effective Equation) satisfied by u?

(Rough Idea)

Homogenized Equation satisfied by u is the correctibility condition on u so that u can be corrected to u_ε so that the corrected u_ε satisfies ε-problem locally.

So the condition for the existence of corrector is almost equal to the homogenized equation.

2. Neumann Problem

2.1 Viscosity Method

$$F(D^2 \mathrm{u}_\varepsilon) = 0 \quad \text{in} \quad \Omega_\varepsilon$$

$$B\left(\frac{\partial u_\varepsilon}{\partial \nu}, u_\varepsilon\right) = 0 \text{ in } \quad \partial\Omega_\varepsilon \cap \Omega$$

$$u_\varepsilon = g(x) \text{ on } \quad \partial\Omega$$

for an outward normal direction ν to $\partial\Omega_\varepsilon$.

Conditions:

(1) $F(M)$ is uniformly elliptic i.e. for symmetric matrices M and N

$$\lambda\|N^+\| - \Lambda\|N^-\| \le F(M+N) - F(M) \le \Lambda\|N^+\| - \lambda\|N^-\|$$

where $N = N^+ - N^-$ and $N^+, N^- \ge 0$

(2) $B(q+q', z) \ge B(q,z)$ and $B(q, z+z') \ge B(q,z)$ for $q' \ge 0, z' \ge 0$

(3) $F(\cdot)$, $B(\cdot,\cdot)$ are C^1.

Def. u_ε is a viscosity super-solution if

$$F(D^2 u_\varepsilon) \le 0 \quad \text{in} \quad \Omega_\varepsilon$$

$$B\left(\frac{\partial u_\varepsilon}{\partial \nu}, u_\varepsilon\right) \ge 0 \text{ in } \quad \partial\Omega_\varepsilon \cap \Omega$$

$$u_\varepsilon \ge g(x) \text{ on } \quad \partial\Omega$$

in viscosity sense.

Def. u_ε is a viscosity sub-solution if

$$F(D^2 u_\varepsilon) \ge 0 \quad \text{in} \quad \Omega_\varepsilon$$

$$B\left(\frac{\partial u_\varepsilon}{\partial \nu}, u_\varepsilon\right) \le 0 \text{ in } \quad \partial\Omega_\varepsilon \cap \Omega$$

$$u_\varepsilon \le g(x) \text{ on } \quad \partial\Omega$$

in viscosity sense.

Lemma : (Comparison Principle)

If u_ε^+ and u_ε^- are super - and sub - solutions repectively such that $u_\varepsilon^+ \geq u_\varepsilon^-$ on $\partial\Omega$, then $u_\varepsilon^+ \geq u_\varepsilon^-$ in Ω.

Cor. (Existence)

Main Theorem

Let u_ε be the viscosity solution of

$$F(D^2 u_\varepsilon) = 0 \quad \text{in} \quad \Omega_\varepsilon$$

$$\frac{\partial u_\varepsilon}{\partial \nu} = 0 \text{ in } \quad \partial\Omega_\varepsilon \cap \Omega$$

$$u_\varepsilon = g(x) \text{ on } \quad \partial\Omega \,.$$

where F is homogeneous of degree one.

Then

(1) There is a Lipschitz function $u(x)$ and $\eta(\varepsilon) > 0$ such that

$$|u_\varepsilon(x) - u(x)| < \eta(\varepsilon) \text{ for all } x \in \Omega_\varepsilon$$

and $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

(2) There is a uniformly elliptic operator $\overline{F}(\cdot)$ such that

$u(x)$ is a viscosity solution of

$$\overline{F}(D^2 u) = 0 \quad \text{in} \quad \Omega$$

$$u = g(x) \text{ on } \quad \partial\Omega \,.$$

2.2 The Concept of Convergence

(1) Discrete Gradient Estimate

Lemma. Set $\Delta_e u_\varepsilon = \frac{u_\varepsilon(x+h\varepsilon e)-u_\varepsilon(x)}{h\varepsilon}$ for $h \in Z^n$.

Then $|\Delta_e u_\varepsilon| < C$ uniformly

(Idea of Proof)

Set $Z = \sup_{x\in\Omega_\varepsilon} |\Delta_e u_\varepsilon|^2$.

a. Interior estimate

$$L[Z] = a_{ij}(x) D_{ij} Z \geq 0 \quad \text{in } \Omega_\varepsilon$$

$$\text{for } a_{ij} = \int_0^1 F_{ij}(sD_{ij}u(x+h\varepsilon e) + (1-s)D_{ij}u(x), \frac{x}{\varepsilon})\, ds.$$

$$\frac{\partial Z}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\varepsilon \cap \Omega$$

So there is no maximum in $\Omega_\varepsilon \setminus \partial\Omega$.

b. Boundary

We need a super - solution such that

$h_\varepsilon = 0$ on ∂B_r

$h_\varepsilon = 1$ on ∂B_R

$F(D^2 h_\varepsilon) \le 0$ in $(B_R \setminus B_r)_\varepsilon$

$\dfrac{\partial h_\varepsilon}{\partial \nu} \ge 0$ in $\partial(B_R \setminus B_r)_\varepsilon \cap (B_R \setminus B_r)$

and $|h_\varepsilon(x)| < Cd(x, \partial B_r)$ (Linear growth)

It is not easy to construct such barrier.

(Step1) Choose $R' > R$ and $r' < r$.

Consider the homogenization h'_ε in

$D = (B_{R'} \setminus B_R) \cup (B_R \setminus B_r)_\varepsilon \cup (B_r \setminus B_{r'})$: a perforated in compact subset

Homogenized equation will be uniformly elliptic and gives

the upper bound gradient of the limit h' and then uniform upper bound of h'_ε.

That upper bound is also independent of $r - r'$.

(Step2) Choose a small $r - r'$ so that h'_ε is a super-solution with a small error.

(2) Almost Flatness

Lemma.

$$\operatorname{osc}_{Q_\varepsilon(m)\setminus T_{a_\varepsilon}(m)} u_\varepsilon = O(\varepsilon) \ \text{ uniformly.}$$

(Idea of Proof)

Set $v_\varepsilon(y) = \dfrac{1}{\varepsilon}\left(u_\varepsilon(\varepsilon y) - \min\limits_{B_{4\varepsilon}\setminus T_{a_\varepsilon}} u_\varepsilon \right) \geq 0$ in $B_4(0)$. Then

$$F(D^2 v_\varepsilon) = 0 \quad \text{in} \quad B_4$$

$$\frac{\partial v_\varepsilon}{\partial \nu} = 0 \text{ in } \quad \partial T_a \cap B_4$$

There is $y_0 \in B_4$ and $v(y_0) = 0$

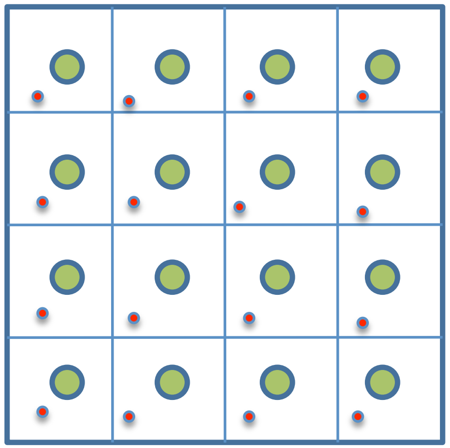
By the Harnack inequality and Discrete gradient estimate,

$$\sup_{B_2} v_\varepsilon \leq C \inf_{B_2} v_\varepsilon \leq C v_\varepsilon(y_0 + m) \leq C\varepsilon$$

for $y_0 + m \in B_2$

Lemma (Golobal ε - Lipschitz Estimate)

$$\left|u_\varepsilon(x) - u_\varepsilon(y)\right| < C(|x-y| + \varepsilon) \text{ for all } x, y \in \Omega_\varepsilon$$



Lemma There is a Lipschitz function $u(x)$ and $\eta(\varepsilon) > 0$ such that

$$\left|u_\varepsilon(x) - u(x)\right| < \eta(\varepsilon) \text{ for all } x \in \Omega_\varepsilon$$

and $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

2.3 Correctors

Lemma (Correctors)

For any given matrix P and a vector ξ, there are ε - periodic functions $w_\varepsilon(\cdot\,;P,\xi)$, a unique constant $\sigma(P,\xi)$, and bounded functions $\sigma_\varepsilon(P,\xi)(x)$ such that

$$\begin{cases} F(P + D^2 w_\varepsilon) = \sigma_\varepsilon & \text{in } R^n \setminus T_{a_\varepsilon} \\ \dfrac{\partial w_\varepsilon}{\partial \nu} = \nu \cdot \xi \text{ - } \sigma_\varepsilon & \text{on } \partial T_{a_\varepsilon} \end{cases}$$

and $\sigma_\varepsilon \to \sigma$ uniformly as $\varepsilon \to 0$.

Let us consider

$$\begin{cases} -w_\varepsilon + F(P + D^2 w_\varepsilon) = 0 & \text{in } R^n \setminus T_{a_\varepsilon} \\ \dfrac{\partial w_\varepsilon}{\partial \nu} + w_\varepsilon = \nu \cdot \xi & \text{on } \partial T_{a_\varepsilon} \end{cases}$$

For large M depending on P and ξ,

$h^{\pm}(x) = \pm M$ are super- and sub-solutions.

Lemma: (Existence of Corrector)

There is a ε-periodic solution w_ε which is uniformly bounded.

Set $\hat{w}_\varepsilon(x) = w_\varepsilon(x) - w_\varepsilon(0)$ and $\hat{w}_\varepsilon(x) = \varepsilon\eta_\varepsilon\left(\dfrac{x}{\varepsilon}\right)$

Then η_ε satisfies $\eta_\varepsilon(0) = 0$ and

$$\begin{cases} -\varepsilon^2\eta_\varepsilon + F(\varepsilon P + D^2 w_\varepsilon) = \varepsilon w_\varepsilon(0) & \text{in } R^n \setminus T_a \\ \dfrac{\partial \eta_\varepsilon}{\partial \nu} + \varepsilon\eta_\varepsilon = \nu \cdot \xi \ \text{-}\ w_\varepsilon(0) & \text{on } \partial T_a \end{cases}$$

By Harnack type estimate and the periodicity,

Lemma :

$\|\eta_\varepsilon\|_\infty < C$ uniformly.

and osc $w_\varepsilon < C\varepsilon$

Set $\sigma_\varepsilon(P,\xi) = w_\varepsilon(x) \to \sigma(P,\xi)$

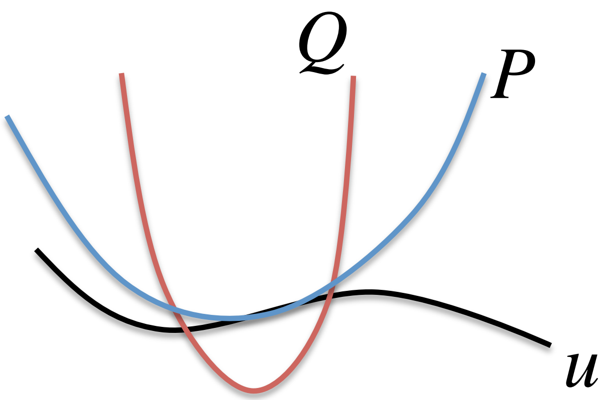
(Uniqueness comes from a simple comparison.)

2.4 The proof of main theorem

Set $\overline{F}(P,\xi)=\sigma(P,\xi)$.

Theorem. When $a_\varepsilon = a_0\varepsilon$, u is a viscosity solution of

$$\overline{F}(D^2u,Du)=0 \quad \text{in } \Omega$$
$$u=0 \quad \text{on } \partial\Omega.$$



Claim: u is a sub-solution.

Otherewise at x_0, $P(x)=\frac{1}{2}P_{ij}x_ix_j+\xi_ix_i+c$

$$\overline{F}(P,\xi)<-2\delta_0<0.$$

and then $\overline{F}(Q,\xi)<-\delta_0<0.$

Let $Q_\varepsilon = Q+w_\varepsilon$.

$$\begin{cases} F(Q_\varepsilon) = F(Q + D^2 w_\varepsilon) = \sigma_\varepsilon < \overline{F}(Q,\xi) + \delta_0/2 < - \delta_0/2 \\ \dfrac{\partial Q_\varepsilon}{\partial \nu} \approx \nu \cdot \xi - \nu \cdot \xi \ - \sigma_\varepsilon \quad > \quad - \overline{F}(Q,\xi) - \delta_0/2 \quad > \delta_0/2 \quad > 0 \qquad \text{on } \partial T_{a_\varepsilon} \end{cases}$$

Threfore Q_ε is a super-solution of ε-problem

in a small neighborhood B_δ of $x_0 = 0$.

And $Q_\varepsilon > u_\varepsilon$ on ∂B_δ from the convergence lemma.

So $Q_\varepsilon > u_\varepsilon$ in B_δ which is a contradiction.

Lemma:

1. $\lambda\|N^+\| - \Lambda\|N^-\| \le \overline{F}(M+N,\xi) - \overline{F}(M,\xi) \le \Lambda\|N^+\| - \lambda\|N^-\|$

2. $\overline{F}(tM, t\xi) = t\overline{F}(M,\xi)$

3. If $F(\cdot)$ is convex, so is $\overline{F}(\cdot,\xi)$.

2.5 Discussions

(1). There are two key estimates:

Discrete gradient estimate and Harnack - type estimate in cell problems

i.e. $\sup_{B_2} v_\varepsilon \le C(\inf_{B_2} v_\varepsilon + \text{"}data\text{"})$

(2). For a bounded function $f(x)$, consider

$$F(D^2 u_\varepsilon) = f(x) \quad \text{in} \quad \Omega_\varepsilon$$

$$\frac{\partial u_\varepsilon}{\partial \nu} = 0 \text{ in } \partial\Omega_\varepsilon \cap \Omega, \ u_\varepsilon = g(x) \text{ on } \partial\Omega .$$

(Discrete Hölder Estimate :)

$$\left|u_\varepsilon(x) - u_\varepsilon(y)\right| < C(|x-y|^\alpha + \varepsilon) \text{ for all } x, y \in \Omega_\varepsilon .$$

(Main idea: Perturbation theory of Schauder type)

(i) Discrete gradient estimate for fixed constant.

(ii) Perturbation lemma

(iii) Improvement approximation

with scaling and Perturbation lemma

(3). $F^1(D^2 u_\varepsilon) = 0 \quad \text{in} \quad \Omega_\varepsilon$

$$F^2(D^2 u_\varepsilon) = 0 \quad \text{in} \quad T_\varepsilon$$

$$\frac{\partial u_\varepsilon}{\partial v^1} + \frac{\partial u_\varepsilon}{\partial v^2} = 0 \quad \text{in} \quad \partial\Omega_\varepsilon \cap \Omega \text{ (Continuity of Flux)}$$

$$u_\varepsilon = g(x) \text{ on } \partial\Omega .$$

(Harnack Estimate :)

(4). (small permeability in perforated domain)

$$F^1(D^2 u_\varepsilon) = 0 \quad \text{in} \quad \Omega_\varepsilon$$

$$F^2(D^2 u_\varepsilon) = 0 \quad \text{in} \quad T_\varepsilon$$

$$\frac{\partial u_\varepsilon}{\partial v^1} + \varepsilon \frac{\partial u_\varepsilon}{\partial v^2} = 0 \quad \text{in} \quad \partial\Omega_\varepsilon \cap \Omega$$

$$u_\varepsilon = g(x) \text{ on } \partial\Omega .$$

(5). Randomly Perforated domain.

We need to consider obstacle problem for Neumann problem
to find subaddative quantity following
L. Caffarelli, P. Souganidis and L. Wang.
We need the stability of this quantity
through discrete nondegeneracy estimate.

And we need a kind of uniform estimate to extract convergent subsequence.

3. Dirichlet Problem

(Dirichlet Problems)

Find the solution u_ε such that

$$F(D^2 u_\varepsilon) = 0 \quad \text{in} \quad \Omega_\varepsilon$$

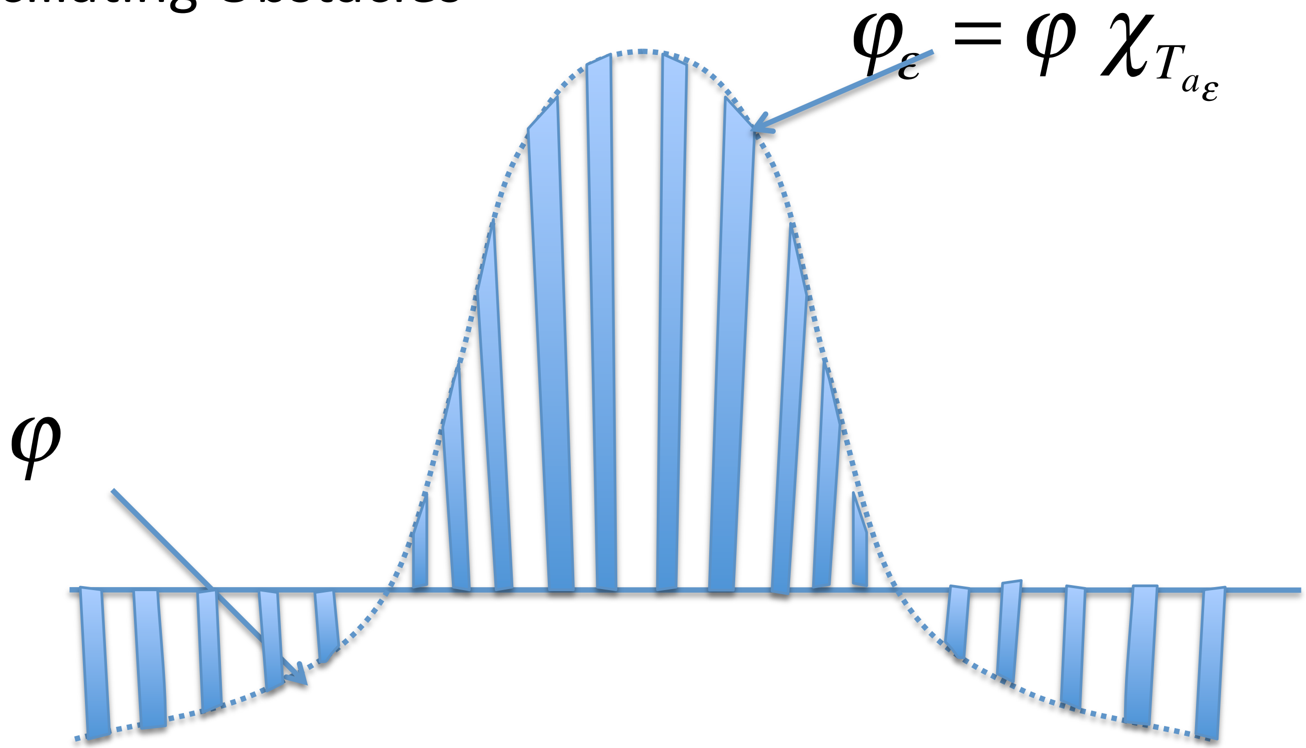
$$u_\varepsilon = 0 \text{ in } \quad \partial\Omega_\varepsilon \cap \Omega$$

$$u_\varepsilon = g(x) \text{ on } \quad \partial\Omega$$

The Dirichlet Problem can be considered as an highly oscillating obstacle problem

Highly Oscillating Obstacle Problems

Oscillating Obstacles



Possible Cases

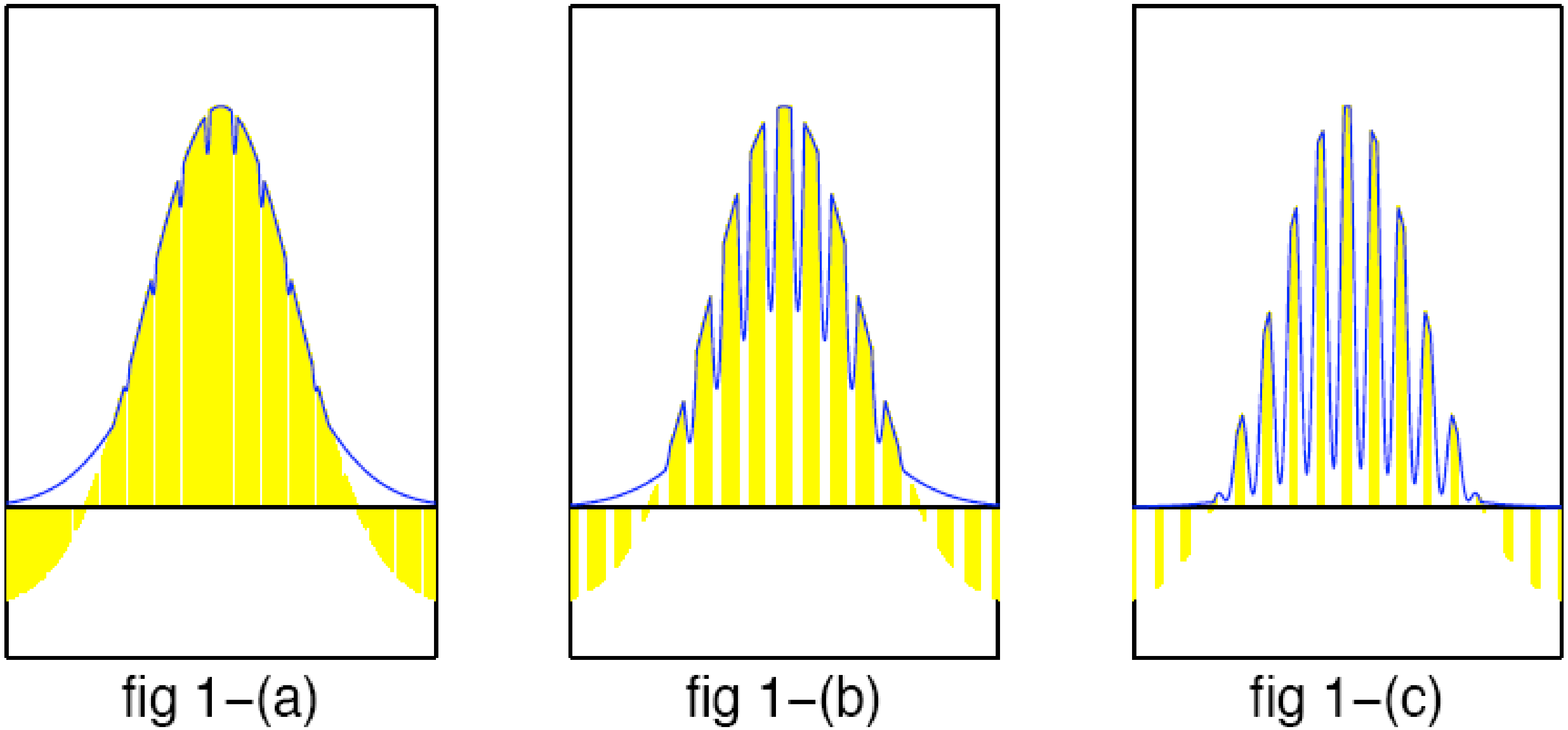


Fig 1 This figures show the oscillation of u_ε when the decary rate of a_ε is subcritical (fig 1-(a)), critical (fig 1-(b)), and supercritical (fig 1-(c))

3.1 Main Theorem For Laplace Equations

3.1.1 Main Theorem For Laplace Equations

(1). (The Concept of Convergence)

There is u and $p>0$ such that $u_\varepsilon \longrightarrow \mathrm{u}$ in L^p.

And for any $\delta > 0$, there is $D_\delta \subset \Omega$ such that

$$u_\varepsilon \longrightarrow \mathrm{u} \text{ uniformly in } D_\delta$$

$$\text{and } \quad |\Omega \setminus \mathrm{D}_\delta| < \delta$$

(2) Let $a_\varepsilon^* = \varepsilon^{\alpha_*}$ for $\alpha_* = \dfrac{n}{n-2}$ for $n=3$ and $\alpha_* = e^{-\frac{1}{\varepsilon^2}}$ for $n=2$.

(a) For $c_0 \alpha_\varepsilon^* \le \alpha_\varepsilon \le C_0 \alpha_\varepsilon^*$, u is a viscosity solution of

$$\Delta \mathrm{u} + \kappa_{\mathrm{B}_{\mathrm{r_o}}} (\varphi - u)_+ = 0 \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

where $\kappa_{\mathrm{B}_{\mathrm{r_o}}}$ is the capacity of B_{r_o} if $r_o = \lim\limits_{\varepsilon \to 0} \frac{\alpha_\varepsilon}{\alpha_\varepsilon^*}$ exists.

(b) If $\alpha_\varepsilon = o(\alpha_\varepsilon^*)$, then u is a viscosity solution of

$$\Delta \mathrm{u} = 0 \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega.$$

(c) If $\alpha_\varepsilon^* = o(\alpha_\varepsilon)$, then u is a viscosity solution of

$$\Delta \mathrm{u} \le 0 \quad \text{in } \Omega$$

$$u \ge \varphi \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega.$$

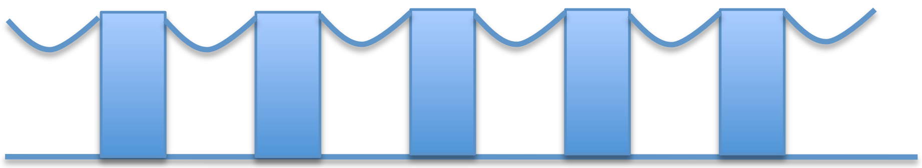
3.1.2 Correctors

$$\begin{cases} \Delta w_\varepsilon = k & \text{in } R^n \\ w_\varepsilon(x) = 1 & \text{in } \cup T_{a_\varepsilon} \end{cases}$$

Lemma. Let $a_\varepsilon = c_0 \varepsilon^\alpha$. There is a unique number $\alpha^* = \dfrac{n}{n-2}$ s.t.

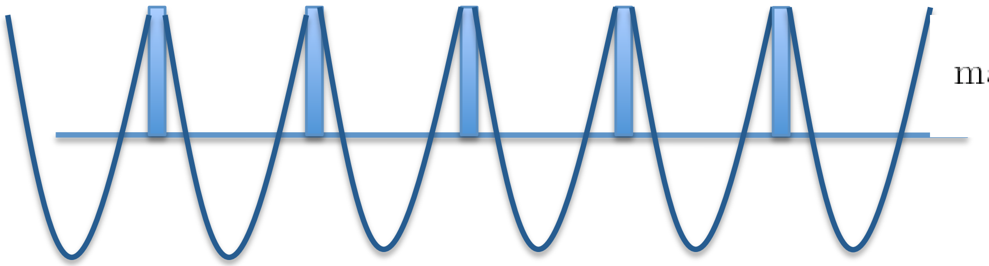
$$\begin{cases} \liminf w_\varepsilon = -\infty & \text{for any } k > 0 \text{ if } \alpha > \alpha_* \\ \liminf w_\varepsilon(x) = 1 & \text{for } \alpha = \alpha_* \text{ and } k = k_\varepsilon \text{ s.t. } k_\varepsilon \to \text{cap}(B_1) \\ \liminf w_\varepsilon = 0 & \text{for any } k > 0 \text{ if } \alpha < \alpha_* \end{cases}$$

$(c)\ a_\varepsilon = o(a_\varepsilon^*)$



$$g_1^-(r) = k_1 \frac{\varepsilon^\beta}{r^{\beta-2}} - k_1 \frac{\varepsilon^\beta}{a_\varepsilon^{\beta-2}} + 1$$

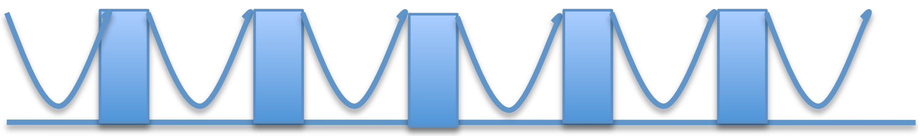
$(b)\ a_\varepsilon^* = o(a_\varepsilon)$



$$\max\left(\min\left(0, \frac{1}{r^{n-2}} - \frac{1}{(a_\varepsilon)^{n-2}}\right), \frac{4^{n-2}}{\varepsilon^{n-2}} - \frac{1}{(a_\varepsilon)^{n-2}}\right)$$

$$h_\varepsilon^-(x) = \max_{m\in\varepsilon\mathbb{Z}^n} \left[g_1^-(|x-m|)\right]$$

$(c)\ a_\varepsilon^* = \varepsilon^{\alpha_*}$



$$g_1(x) = \frac{\varepsilon^n}{r^{n-2}} - \frac{\varepsilon^n}{a_\varepsilon^*}$$

Since $|\nabla g_1| \approx |\nabla |x|^2|$,

$$g^\pm(x) = g_1(x) + c_\pm |x|^2$$

3.1.3 The Concept of Convergence

(1) Discrete Gradient Estimate

Lemma. Set $\Delta_e u_\varepsilon = \frac{u_\varepsilon(x+h\varepsilon e)-u_\varepsilon(x)}{h\varepsilon}$.

Then $|\Delta_e u_\varepsilon| < C$ uniformly

$$-\triangle u_{\varepsilon,\delta}(x) + \beta_\delta(u_{\varepsilon,\delta}(x) - \varphi_\varepsilon(x)) = 0 \quad \text{in } \Omega$$

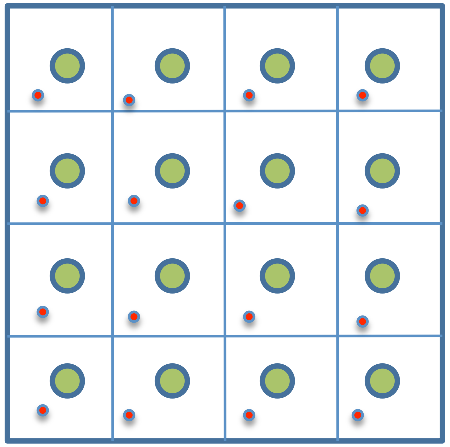
$$u_{\varepsilon,\delta}(x) = 0 \quad \text{on } \partial\Omega$$

$$-\triangle(|\triangle_e u_\varepsilon|^2) + \beta'_\delta(\cdot)(2|\triangle_e u_{\varepsilon,\delta}(x)|^2 - 2\triangle_e u_\varepsilon \triangle_\varepsilon \varphi_\varepsilon) = 0.$$

(2) Almost Flatness

Lemma. Set $a_\varepsilon = (a_\varepsilon^*)^{1/2}$. Then

$$\operatorname{osc}_{B_\varepsilon(m)\setminus B_{\sqrt{a_\varepsilon}}(m)} u_\varepsilon = o(\varepsilon^\gamma) \text{ uniformly.}$$



Nonlinear Equations

$F(D^2u,x)=0$ is uniformly elliptic if

$$\frac{1}{\Lambda}\|N\| \le F(M+N,x)-F(M,x) \le \Lambda\|N\|$$

for $n\times n$ symmetric metrices M,N s.t. $N\ge 0$.

$$(NL_\varepsilon)\quad \begin{cases} F(D^2u_\varepsilon)\le 0 & \text{in } \Omega \\ u_\varepsilon(x)=0 & \text{on } \partial\Omega \\ u_\varepsilon(x)\ge \varphi_\varepsilon(x) & \text{in } \Omega \end{cases}$$

$Ex)$

$$F(D^2u)=k_1\lambda_+(D^2u)+k_2\lambda_-(D^2u)=0$$

Let $u=r^\alpha$.

Then $F(D^2r^\alpha)=\frac{\alpha}{r^{\alpha+2}}(k_1(\alpha+1)-k_2)=0.$

Homogeneous Solutions

Proposition 1.5.
Let $F(D^2u)$ be homogeneous of degree 1. Then there is a homogeneous solution $V(x)$ for $F(D^2V) = 0$ in $\mathbb{R}^n \backslash \{0\}$ which is one of the following three types.

(1) **(Type I)** *$V(x)$ has negative degree $-\lambda < 0$ and satisfies $\lim_{|x|\to\infty} V = 0$ and $\lim_{|x|\to 0} V = \infty$. It is also unique up to constant ratio and has the form*

$$V(x) = |x|^{-\lambda}\Phi(\theta)$$

for $\theta = \frac{x}{|x|}$.

(2) **(Type II)** *$V(x)$ has zero degree and satisfies $\lim_{|x|\to\infty} V = \infty$ and $\lim_{|x|\to 0} V = -\infty$. It is also unique up to constant ratio and has the form*

$$V(x) = \log(|x|) + \Phi(\theta)$$

for $\theta = \frac{x}{|x|}$.

(3) **(Type III)** *$V(x)$ has positive degree $\lambda > 0$ and satisfies $\lim_{|x|\to\infty} V = \infty$ and $\lim_{|x|\to 0} V = 0$. It is also unique up to constant ratio and has the form*

$$V(x) = |x|^{\lambda}\Phi(\theta)$$

for $\theta = \frac{x}{|x|}$.

Theorem II

(1). (The Concept of Convergence)

There is u and $p > 0$ such that $u_\varepsilon \longrightarrow$ u in L^p.

And for any $\delta > 0$, there is $D_\delta \subset \Omega$ such that

$$u_\varepsilon \longrightarrow \text{u uniformly in } D_\delta$$

$$\text{and } |\Omega \setminus \mathrm{D}_\delta| < \delta$$

(2) In (Type III), for any a_ε, the limit u is a least viscosity super-solution of

$$F(D^2\mathrm{u}) \le 0 \quad \text{in } \Omega$$
$$u \ge \varphi \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega.$$

(3) *Set* $\alpha_\varepsilon^* = \varepsilon^{\alpha_*}$ *for* $\alpha_* = \frac{\lambda+2}{\lambda}$ *in (Type I) and* $\alpha_\varepsilon^* = e^{-\frac{1}{\varepsilon}}$ *in (Type II). Then*

(a) *For* $c_o\alpha_\varepsilon^* \leq \alpha_\varepsilon \leq C_o\alpha_\varepsilon^*$, *there is a uniform elliptic operator* $\overline{F}(D^2u, (\varphi - u)_+)$ *such that the limit* u *is a viscosity solution of*

$$\begin{aligned} \overline{F}(D^2u, (\varphi - u)_+) &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

And $\overline{F}(0, c) = k_{B_1}c$.

(b) *If* $\alpha_\varepsilon = o(\alpha_\varepsilon^*)$ *then* u *is a viscosity solution of*

$$\begin{aligned} F(D^2u) &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

(c) *If* $\alpha_\varepsilon^* = o(\alpha_\varepsilon)$ *then* u *is a least viscosity super solution of*

$$\begin{aligned} F(D^2u) &\leq 0 \quad \text{in } \Omega \\ u &\geq \varphi \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

3.2. Parabolic Problems

Find the smallest super - solution u_ε such that

$$\Delta \mathrm{u}_\varepsilon - \partial_t u_\varepsilon \le 0 \quad \text{in} \quad \Omega \times (0,\mathrm{T}]$$

$$u_\varepsilon \ge \varphi_\varepsilon(x) \text{ in } \Omega \times (0,\mathrm{T}]$$

$$u_\varepsilon = 0 \text{ on } \partial\Omega \times (0,\mathrm{T}]$$

$$u_\varepsilon = g \text{ on } \Omega \times \{0\}$$

Theorem. Let $a_\varepsilon^* = \varepsilon^{\alpha_*}$ for $\alpha_* = \dfrac{n}{n-2}$ for $n = 3$ and $\alpha_* = e^{-\frac{1}{\varepsilon^2}}$ for $n = 2$.

For $c_0\alpha_\varepsilon^* \le \alpha_\varepsilon \le C_0\alpha_\varepsilon^*$, u is a viscosity solution of

$$\Delta \mathrm{u} + \kappa_{\mathrm{B}_{\mathrm{r_o}}}(\varphi - u)_+ - u_t = 0 \quad \text{in } Q_T$$

$$u = 0 \quad \text{on } \partial_l Q_T$$

$$u = g(x) \text{ in } \Omega \times \{t = 0\}$$

where $\kappa_{\mathrm{B}_{\mathrm{r_o}}}$ is the capacity of B_{r_o} if $r_o = \lim_{\varepsilon\to 0} \frac{\alpha_\varepsilon}{\alpha_\varepsilon^*}$ exists.

Idea : $Q_\varepsilon(x,t) = Q(x,t) + (\varphi(x_0) - Q(x_0,t_0))w_\varepsilon(x)$

3.3. Nonlinear Eigen Value Problems

Let us consider nonlinear eigen value problems:

$$\begin{cases} \Delta\varphi_\varepsilon = -\lambda\,\varphi_\varepsilon{}^p & \text{in} \quad \Omega_\varepsilon \\ \varphi_\varepsilon > 0 & \text{in} \quad \Omega_\varepsilon \\ \varphi_\varepsilon = 0 & \text{on} \quad \partial T_{a_\varepsilon} \\ \varphi_\varepsilon = 0 & \text{on} \quad \partial\Omega \end{cases}$$

for $0 < p < 1$.

Let $\lambda = 1$.

3.3.1. Homogenized Equation

Theorem: (1). (The Concept of Convergence)

There is u and $q>0$ such that $u_\varepsilon \longrightarrow \mathrm{u}$ in L^q.

And for any $\delta>0$, there is $D_\delta \subset \Omega$ such that

$u_\varepsilon \longrightarrow \mathrm{u}$ uniformly in D_δ and $|\Omega \setminus \mathrm{D}_\delta| < \delta$

(2) Let $a_\varepsilon^* = \varepsilon^{\alpha_*}$ for $\alpha_* = \dfrac{n}{n-2}$ for $n=3$ and $\alpha_* = e^{-\frac{1}{\varepsilon^2}}$ for $n=2$.

(a) For $c_0 \alpha_\varepsilon^* \le \alpha_\varepsilon \le C_0 \alpha_\varepsilon^*$, u is a viscosity solution of

$$\Delta \mathrm{u} - \kappa_{\mathrm{B}_{\mathrm{r_o}}} u_+ = -\lambda u^p \quad \text{in } \Omega$$

$$u > 0 \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

where $\kappa_{\mathrm{B}_{\mathrm{r_o}}}$ is the capacity of B_{r_o} if $r_o = \lim\limits_{\varepsilon \to 0} \frac{\alpha_\varepsilon}{\alpha_\varepsilon^*}$ exists.

3.3.2. Main Steps

a. Discrete Gradient Estimate for the Green Function for Laplace operator.

idea) • $G_{\Omega_\varepsilon}(x,y) = G_\Omega(x,y) + h_\varepsilon(x;y)$

where $\begin{cases} \Delta h_\varepsilon(x;y) = 0 & \text{in } \Omega_\varepsilon \\ h_\varepsilon(x;y) = -G_\Omega(x,y) & \text{on } \partial\Omega_\varepsilon \end{cases}$

• $h_\varepsilon(x;y)$ has discrete gradient estimate with the order of $|\nabla G_\Omega|$.

$\Rightarrow$ Discrete Gradient estimate of u_ε from Green's representation.

b. Almost Flatness

$$v_\varepsilon(x) = u_\varepsilon(\sqrt{a_\varepsilon^*}\, x) \Rightarrow \Delta v_\varepsilon = -a_\varepsilon^* v_\varepsilon{}^p$$

c. Correctibility Condition (or Homogenizd Equation)

$$-\Delta w_\varepsilon + \tfrac{1}{b}(b - bw_\varepsilon)^p = k_\varepsilon$$

Set $v_\varepsilon(x) = w_\varepsilon(a_\varepsilon^* x)$.

$$\begin{cases} -\Delta v_\varepsilon + \frac{(a_\varepsilon^*)^2}{b}(b - bv_\varepsilon)^p = k_\varepsilon (a_\varepsilon^*)^2 \quad \text{in} \quad Q_{\frac{\varepsilon}{a_\varepsilon^*}} \setminus D \\ v_\varepsilon = 1 \quad \text{on } \partial D \\ v_\varepsilon = |\nabla v_\varepsilon| = 0 \quad \text{on } \partial Q_{\frac{\varepsilon}{a_\varepsilon^*}} \end{cases}$$

$$-\int_{Q_{\frac{\varepsilon}{a_\varepsilon^*}} \setminus D} \Delta v_\varepsilon \, dx = (a_\varepsilon^*)^2 \int_{Q_{\frac{\varepsilon}{a_\varepsilon^*}} \setminus D} k_\varepsilon - b^{p-1}(1 - v_\varepsilon)^p \, dx$$

When $\varepsilon \to 0$, $-\kappa(D) = k - b^{p-1} \Rightarrow k = -\kappa(D) + b^{p-1}$

$P(x)$ is correctible $\approx$ $P_\varepsilon(x) = P(x) - P(x_0)w_\varepsilon$ is a solution of ε - problem

$$0 = \Delta P_\varepsilon + (P_\varepsilon(x))^p \approx \Delta P - P(x_0)\Delta w_\varepsilon + P(x_0)^p(1 - w_\varepsilon)^p$$

$$\approx \Delta P + P(x_0)k$$

$$\approx \Delta P - \kappa(D)P(x_0) + P(x_0)^p$$

d. Discrete Hopf Principle

idea) Creat a barrier from the oscillating obstacle problem

$\Rightarrow$ Discrete nondegeneracy of u_ε

3.4. Porous Medium Equations in a fixed Perforated domain

$$\Delta u_\varepsilon^m - \partial_t u_\varepsilon = 0 \quad \text{in} \quad \Omega_{a_\varepsilon} \times (0,T]$$

$$u_\varepsilon = 0 \text{ on } \quad \partial\Omega_{a_\varepsilon} \times (0,T]$$

$$u_\varepsilon = g_\varepsilon \text{ on } \quad \Omega_{a_\varepsilon} \times \{0\}$$

Set $v_\varepsilon = u_\varepsilon^m$ which is a flux.

$\nabla v_\varepsilon \neq 0$ on a fixed boundary $\partial\Omega$.

For $v > \delta > 0$, we can use the results on Laplacian.

$$v_\varepsilon^{1-\frac{1}{m}} \Delta v_\varepsilon - \partial_t v_\varepsilon = 0 \quad \text{in} \quad \Omega_{a_\varepsilon} \times (0,T]$$

$$v_\varepsilon = 0 \text{ on } \quad \partial\Omega_{a_\varepsilon} \times (0,T]$$

$$v_\varepsilon = g_\varepsilon^m \text{ on } \quad \Omega_{a_\varepsilon} \times \{0\}$$

Theorem. Let $a_\varepsilon^* = \varepsilon^{\alpha_*}$ for $\alpha_* = \dfrac{n}{n-2}$ for $n = 3$ and $\alpha_* = e^{-\frac{1}{\varepsilon^2}}$ for $n = 2$.

For $c_0\alpha_\varepsilon^* \le \alpha_\varepsilon \le C_0\alpha_\varepsilon^*$, u is a viscosity solution of

$$v^{1-\frac{1}{m}}\,(\,\Delta v - \kappa_{\mathrm{B}_{\mathrm{r_o}}} v_+)\; - v_t = 0 \quad \text{in } \; Q_T$$

$$v = 0 \quad \text{on } \partial_l Q_T$$

$$v = g(x)^m \;\; \text{in } \; \Omega\times\{t=0\}$$

where $\kappa_{\mathrm{B}_{\mathrm{r_o}}}$ is the capacity of B_{r_o} if $r_o = \lim\limits_{\varepsilon\to 0}\frac{\alpha_\varepsilon}{\alpha_\varepsilon^*}$ exists.

3.4.1. Main Steps

a. Discrete Nondegeneracy

idea) $U_\varepsilon(x,t) = \dfrac{\varphi_\varepsilon(x)}{(1+t)^{1/(m(m-1))}}$

$c_1\, U_\varepsilon(x,t) \le u_\varepsilon \le c_2\, U_\varepsilon(x,t)$ for some $0 < c_1 < c_2 < \infty$

$\Rightarrow$ Discrete nondegeneracy of u_ε

$\Rightarrow$ $u > 0$ in Ω

b.Discrete Gradient Estimate for $u_{\varepsilon,t}(x,0) \le 0$

by applying maximum principle on

$Z = |D_e^{\varepsilon h} u|^2$

c. Almost Flatness

idea) $U_\varepsilon(x,t) = \dfrac{\varphi_\varepsilon(x)}{(1+t)^{1/(m(m-1))}}$

$c_1\, U_\varepsilon(x,t) \le u_\varepsilon \le c_2\, U_\varepsilon(x,t)$ for some $0 < c_1 < \mathrm{c}_2 < \infty$

Set $T_{\tilde{a}_\varepsilon} = \{u_\varepsilon(x,t) < \delta_0\}$.

On $\Omega_{\tilde{a}_\varepsilon} = \Omega \setminus T_{\tilde{a}_\varepsilon}$, the equation is uniformly parabolic.

Similar method can be applicable.

Thank you!