A simple kinetic market economy

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Outline

1. Distribution of wealth, Gibrat's Law
2. Binary interaction model
3. Financial Market model
4. Numerical Results

Modelling of income or wealth distributions

Pareto [1897] observes power law tail i.e. $f(w)$ being the PDF of individuals with wealth/income $w \geq 0$, we have

$$\int_w^\infty f(w_*)\, dw_* \sim w^{-\mu},$$

is the number of people having income greater or equal than w.

R. Gibrat (1904-1980) proposes a lognormal law
$f(w) = \exp(-(\log w)^2)/(\sqrt{\pi} w)$ for the middle income range.

Gibrat's law or law of proportional effects

The simplest microscopic market dynamic that originates a lognormal behavior was proposed by R.Gibrat in *Les inégalités économiques, Paris, (PhD, 1931) through the multiplicative random process, or "law of proportional effects"*

$$w \to w' = b(t)w$$

where $b(t)$ *is a nonnegative random variable. This yields* $\log(w)$ *normally distributed and thus* w *lognormal.*

The law of proportionnal effects applies in several domains

(Non exhaustive list)

– Distribution of size of firms, of cities *
– Distribution of income, of wealth, of consumption †
– Distribution of river flows ‡

*Gibrat 1931 ; Sutton Jal of Economic Literature 1997

†Gibrat 1931 ; Battistin et al. 2007

‡Gibrat 1932, S. El Adlouni et al 2008

Part 2 : Binary Microscopic dynamics

Microscopic trade dynamics between agents with money (w, w_*)

$$
\begin{aligned}
w' &= (1-\gamma)w + \gamma w_* + \eta w \\
w'_* &= (1-\gamma)w_* + \gamma w + \eta_* w_*
\end{aligned}
$$

where (w, w_*) denotes the money of two arbitrary agents before the trade and (w', w'_*) the money after the trade.

The transaction coefficient $\gamma \in (0,1)$ is constant and η and η_* are random variables with the same distribution with variance σ^2 and zero mean.

The first term is related to the **marginal saving propensity** , the second corresponds to the **monetary transaction**, and the last are the effects of **speculative trading**

Kinetic equations

The PDF $f(w,t)$ obeys *

$$\frac{\partial f}{\partial t} = \int_{I\!R^2}\int_0^\infty (\beta_{('w,'w_*)\to(w,w_*)}\frac{1}{J}f('w)f('w_*) - \beta_{(w,w_*)\to(w',w'_*)}f f_*)dw_*\, d\eta\, d\eta_*$$

where $('w,'w_*)$ are the pre-trading money that generates the couple (w,w_*). In the equation J is **jacobian** of the transformation of (w,w_*) into (w',w'_*) and β is related to the probability of the interactions.

*S.C., L.Pareschi, G.Toscani, Jal of Stat. Phys. (2005).

Continuous trading scaling

For large number of small exchange of money (asymptotics related to quasi-elastic limit for granular gases *) we obtain a linear **Fokker-Planck equation** †

$$\frac{\partial g}{\partial \tau} = \frac{\lambda}{2} \frac{\partial^2}{\partial w^2} \left(w^2 g \right) + \frac{\partial}{\partial w} \left((w - m) g \right).$$

*G. Toscani, M2AN (2000)
J.A. Carrillo, S.C., G. Toscani, DCDS-A (2009)

†J.P.Bouchaud, M.Mézard, Physica A, (2000).

Equilibrium states

$$g_\infty(w) = \frac{(\mu-1)^\mu}{\Gamma(\mu)} \frac{\exp\left(-\frac{\mu-1}{w}\right)}{w^{1+\mu}}, \qquad \mu = 1 + \frac{2}{\lambda} > 1.$$

Thus, the obtained stationary distribution exhibits a **Pareto power law** tail for large w.

$$g_\infty(w) \approx C w^{-(1+\mu)}, \quad w \to \infty.$$

Reversible dynamics

For any time reversible interaction, it is proved * that the corresponding stationary state of the money distribution function $f(w,t)$ is characterized by a Boltzmann-Gibbs (exponential) law

$$f(w) = \frac{\exp(-w/\bar{w})}{\bar{w}}$$

Conclusion :

Kinetic theory in economy is a powerful tool and can help to derive mathematical models of binary interactions which yields to for income (wealth) distributions with Pareto or Boltzmann-Gibbs law.

One main limitation is that the probability of trading is , by construction, independant of the agent.

*L.Pareschi, G. Toscani Jal of Stat. Phys. (2006).

Part 3 : a simple model of financial market

– We consider now a simple kinetic model that describes the behavior of a financial market related to the Levy-Levy-Solomon (LLS) microscopic model. We derive and analyze the model in the case of a single stock [††]

– We consider a set of financial agents $i = 1, \ldots, N$. We denote by w_i the money (wealth) of agent i and by n_i the number of stocks of the agent. Additionally we use the notations S for the price of the stock and n for the total number of stocks.

$$\gamma_i w_i = n_i S$$

– The essence of the dynamic is the choice of the agent's portfolio. More precisely at each time step each agent selects which fraction of money to invest in bonds and which fraction γ in stocks. We indicate with r the (constant) interest rate of bonds.

[††] H.Levy, M.Levy, S.Solomon, Academic Press, New York, (2000)

The microscopic dynamic

If an agent has invested $\gamma_i w_i$ money in stocks and $(1-\gamma_i)w_i$ money in bonds at the next time step he will achieve

$$\begin{aligned} w_i' &= (1-\gamma_i)w_i(1+r) + \gamma_i w_i(1+x'), \\ &= w_i + w_i(1-\gamma_i)r + n_i(S' - S + D'), \end{aligned}$$

where x' is the rate of return of the stock given by

$$x' = \frac{S' - S + D'}{S}.$$

S' is the new price of the stock and D' is a stochastic variable taking into account the dividends paid by the company at the end of the time period.

The demand curve

– In practice for any hypothetical price S^h each investor find the optimal proportion γ_i^h which maximize his/her expected utility. Thus we have a demand curve $\gamma_i^h(S^h)$ as a function of the price. Typically this demand curve is a non increasing function of S^h.

– Note that, if we assume that all investors share the same informations and have the same risk aversion then they will have the same proportion of investment in stock regardless of their money, thus

$$\gamma_i^h(S^h) = \gamma^h(S^h).$$

– To make the model more realistic typically a source of stochastic noise, which characterizes all factors causing the investor to deviate from his optimal portfolio, is introduced in the optimal proportion of investments $\gamma^h(S^h)$.

Market clearance

Next, each agent formulates a demand curve

$$n_i^h = n_i^h(S^h) = \frac{\gamma^h(S^h)w_i^h(S^h)}{S^h}$$

characterizing the desired number of stocks as a function of S^h and w_i^h. This number of share demands is a monotonically non increasing function of the hypothetical price S^h.

As the total number of stocks

$$n = \sum_{i=1}^{N} n_i$$

is preserved the new price S' is given by the market clearance condition

$$\sum_{i=1}^{N} n_i^h(S') = n.$$

This will fix the value w' and the model can be advanced in time.

The kinetic model

Let us define $f = f(w,t)$, $w \in \mathbb{R}_+$, $t > 0$ the distribution of money w. We assume that the percentage of wealth invested $\gamma(\xi) = \mu(S(t)) + \xi$, where ξ is a random variable in $[-z, z]$, $z = \min\{-\mu(S(t)), 1 - \mu(S(t))\}$ distributed accordingly to $\Phi(\mu(S(t), \xi)$ with zero mean and variance ζ^2. Here $\mu(S(t))$ is assumed to be a monotonically non increasing function of the price $S(t) \geq 0$ such that $0 < \mu(0) < 1$.

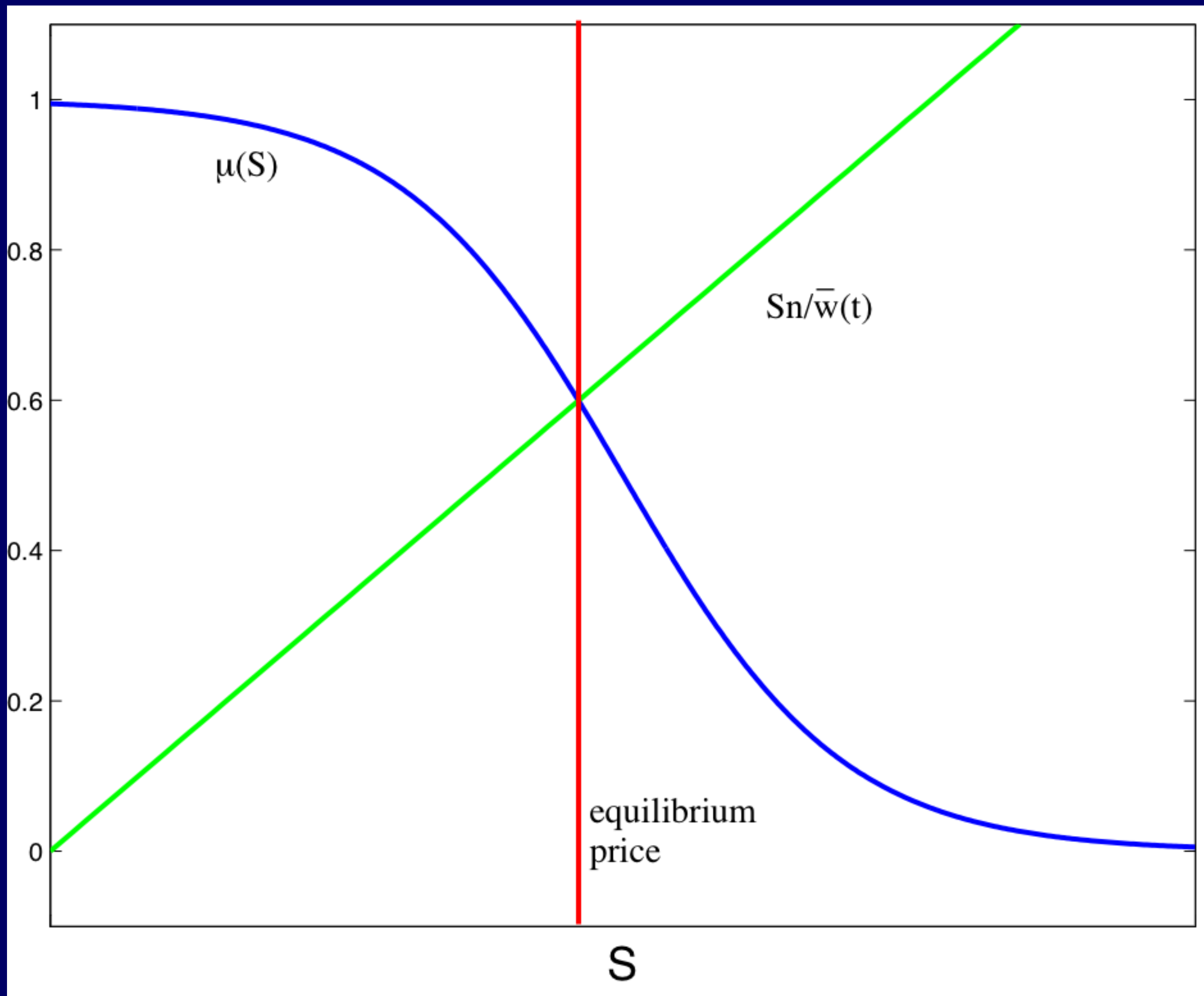
Note that given $f(w,t)$ the actual stock price $S(t)$ is determined as the unique solution of the demand-supply relation

$$S(t) = \frac{1}{n} E[\gamma w] = \frac{1}{n} \int_0^\infty \int_{-z}^{z} \Phi(\mu(S(t)), \xi) \gamma(\xi) f(w,t) w \, d\xi, dw,$$

where for simplicity the total number of agents has been normalized $\rho = \int_0^\infty f(w,t) dw = 1$. More precisely since γ and w are independent we have

$$S(t) = \frac{1}{n} E[\gamma] E[w] = \frac{1}{n} \mu(S(t)) \bar{w}(t), \quad \bar{w}(t) = \int_0^\infty f(w,t) w dw.$$

The equilibrium price



The new wealth

At the next round, the new wealth of the investor will depend on the future price S' and the percentage of money invested γ accordingly with

$$w'(S'(t), \gamma, \eta) = ((1-\gamma)(1+r) + \gamma(1 + x(S'(t), \eta)))w,$$

where the expected rate of return of stocks is given by

$$x(S'(t), \eta) = \frac{S'(t) - S(t) + D + \eta}{S(t)}.$$

where η is a symmetric random variable in distributed accordingly to $\Theta(\eta)$ with zero mean and variance σ^2 which takes into account the effects of the dividends paid by the company. The above equation requires to estimate the future price S' which is unknown.

The new portfolio

The dynamic is then determined by the new fraction of money invested $\gamma'(\xi') = \mu(S'(t)) + \xi'$ where ξ' is a random variable in $[-z', z']$, where $z' = \min\{\mu(S'(t)), 1 - \mu(S'(t))\}$ distributed accordingly to $\Phi(\mu(S'(t)), \xi')$.

We have the demand-supply relation

$$S'(t) = \frac{1}{n} E[\gamma' w'],$$

which permits to write the following equation for the future price

$$S'(t) = \frac{1}{n} E[\gamma'] E[w'] = \frac{1}{n} \mu(S'(t)) E[w'].$$

The future price

Now

$$\begin{aligned} E[w'] &= E[w](1+r) + E[\gamma w](E[x(S',\eta)] - r), \\ &= \bar{w}(1+r) + \mu(S)\bar{w}\left(\frac{S'+D-S}{S} - r\right). \end{aligned}$$

This gives the identity

$$S' = \frac{1}{n}\mu(S')\bar{w}\left[(1+r) + \mu(S)\left(\frac{S'+D-S}{S} - r\right)\right].$$

Using the equation for the price, we can eliminate the dependence on the mean wealth and write

$$\begin{aligned} S' &= \frac{\mu(S')}{\mu(S)}\left[(1-\mu(S))S(1+r) + \mu(S)(S'+D)\right] \\ S' &= \frac{(1-\mu(S))\mu(S')}{(1-\mu(S'))\mu(S)}(1+r)S + \frac{\mu(S')}{1-\mu(S')}D. \end{aligned}$$

Price behavior

The equation for the future price deserves some remarks

– If $\mu(\cdot) = C$, with $C \in (0,1)$ constant then

$$S' = (1+r)S + \frac{C}{1-C}D$$

which corresponds to a dynamic of grow of the prices at rate r.

– In the general case if we set

$$g(S) = \frac{1-\mu(S)}{\mu(S)}S$$

the future price is given by the implicit equation

$$g(S') = g(S)(1+r) + D$$

for a given S. The function $g(S)$ is monotonically increasing with respect to S. This gives the existence of a unique fixed point. Moreover if $r = 0$ and $D = 0$ the unique fixed point is $S' = S$ and the price is unchanged in time.

A linear kinetic equation

evolution of the PDF of wealth is given by the linear kinetic equation

$$\frac{\partial f(w,t)}{\partial t} = \int_{I\!R}\int_{-z}^{z}\left(\beta('w \to w)\frac{1}{j(\xi,\eta,t)}f('w,t) - \beta(w \to w')f(w,t)\right)d\xi\, d\eta.$$

The value $'w$ is obtained simply by inverting the dynamics to get

$$'w = \frac{w}{j(\xi,\eta,t)}, \quad j(\xi,\eta,t) = 1 + r + \gamma(\xi)(x(S'(t),\eta) - r),$$

where the value $S'(t)$ is given as the unique fixed point of the future price equation. The presence of the term j in the integral is needed in order to preserve the total number of agents. β takes the form

$$\beta(w \to w') = \Psi(w' \geq 0)\Phi(\mu(S(t)),\xi)\Theta(\eta),$$

Properties

The equation in weak form takes the simpler form

$$\frac{d}{dt}\int_0^\infty f(w,t)\phi(w)dw = \int_0^\infty \int_{I\!R}\int_{-z}^{z} \Phi(\xi)\Theta(\eta)f(w,t)(\phi(w')-\phi(w))d\xi\, d\eta\, dw.$$

From this follows the conservation of the total number of investors taking $\phi(w)=1$. The choice $\phi(w)=w$ gives the time evolution of the average wealth which characterizes the price behavior

$$\frac{d}{dt}\bar{w}(t) = \left((1-\mu(S(t)))r + \mu(S(t))\frac{S'(t)+D-S(t)}{S(t)}\right)\bar{w}(t).$$

If we now set

$$m(t) = \min\left\{r, \frac{1}{t}\int_0^t \frac{S'(s)+D-S(s)}{S(s)}ds\right\}, \quad M(t) = \max\left\{r, \frac{1}{t}\int_0^t \frac{S'(s)+D-S(s)}{S(s)}ds\right\}$$

we have the bound

$$\bar{w}(0)\exp\left(m(t)\,t\right) \le \bar{w}(t) \le \bar{w}(0)\exp\left(M(t)\,t\right).$$

Analogous bounds for moments of higher order can be obtained.

Evolution of the price

$$\frac{d}{dt}S = \frac{\mu(S)}{\mu(S) - \dot{\mu}(S)S}\Big((1 - \mu(S))r + \mu(S)\bar{x}(S')\Big)\, S,$$

For a constant $\mu(\cdot) = C$, $C \in (0, 1)$ we have the explicit expression for the growth of the wealth (and consequently of the price)

$$\bar{w}(t) = \bar{w}(0)\exp(rt) + \frac{nD}{1 - C}(\exp(rt) - 1).$$

Fokker-Planck asymptotics

As for binary trading models, it is difficult to study in details the asymptotic behavior. Particular asymptotics result in simplified models of Fokker-Planck type, for which it is easier to analyze their asymptotic behavior

We start from the weak form of the kinetic equation and consider a second order Taylor expansion of ϕ around w

$$\phi(w') - \phi(w) = w(r + \gamma(x(S'(t), \eta) - r))\phi'(w) + \frac{1}{2} w^2 (r + \gamma(x(S'(t), \eta) - r))^2 \phi''(\tilde{w}),$$

where $\tilde{w} = \theta w' + (1-\theta)w$, for some $0 \leq \theta \leq 1$.

Asymptotic limit

Consider $r \to 0$. In order for such limit to have a sense and preserve the characteristics of the model, we must rescale the time $\tau = rt$, assume that

$$\lim_{r\to 0} \frac{\sigma^2}{r} = \nu, \quad \lim_{r\to 0} \frac{D}{r} = \delta, (Perpetuity)$$

Note that the above limits imply that $\lim_{r\to 0} S' = S$.

Fokker-Planck equation

Standard computations yields to Fokker-Planck equation

$$\frac{\partial}{\partial \tau} f = \frac{\partial}{\partial w}\left[-A(S(\tau),\delta) w f + \frac{1}{2} B(S(\tau),\nu) \frac{\partial}{\partial w} w^2 f\right],$$

with

$$A(S,\delta) = 1 + \mu(S)\left((\kappa(S)-1) + \frac{\mu(S)(\kappa(S)-1)+1}{1-\mu(S)}\frac{\delta}{S}\right)$$

$$B(S,\nu) = (\mu(S)^2 + \zeta^2)\nu/S^2,$$

and

$$\kappa(S) = \frac{\mu(S)(1-\mu(S))}{\mu(S)(1-\mu(S)) - \dot{\mu}(S)} \leq 1$$

We remark that even for the Fokker-Planck model the mean wealth is increasing with time.

Lognormal behavior

In order to search for self-similar solutions we consider the scaling

$$f(w,\tau) = \frac{1}{w} g(\chi,\tau), \quad \chi = \log(w).$$

Then $g(\chi,\tau)$ satisfy the linear convection-diffusion equation

$$\frac{\partial}{\partial \tau} g(\chi,\tau) = \left(\frac{B}{2} - A\right) \frac{\partial}{\partial \chi} g(\chi,\tau) + \frac{B}{2} \frac{\partial^2}{\partial \chi^2} g(\chi,\tau),$$

which admits the self-similar solution

$$g(\chi,\tau) = \frac{1}{((2\pi)(2\tau+1))^{1/2}} \exp\left(-\frac{(\chi + (B/2 - A)\tau)^2}{B(2\tau+1)}\right).$$

Reverting to the original variables we obtain the lognormal asymptotic behavior of the model

$$f(w,\tau) = \frac{1}{w((2\pi)(2\tau+1))^{1/2}} \exp\left(-\frac{(\log(w) + (B/2 - A)\tau)^2}{B(2\tau+1)}\right).$$

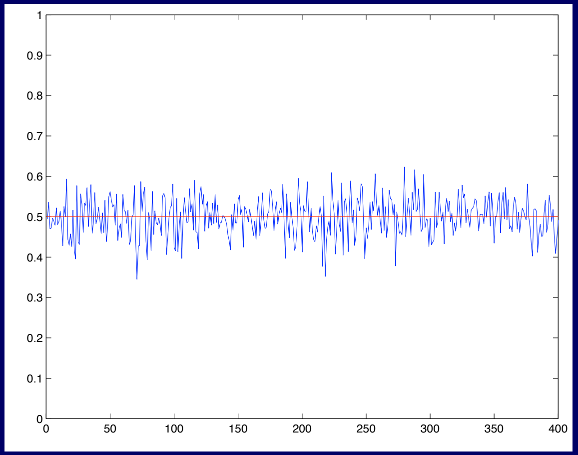
Part 4 - Numerical results

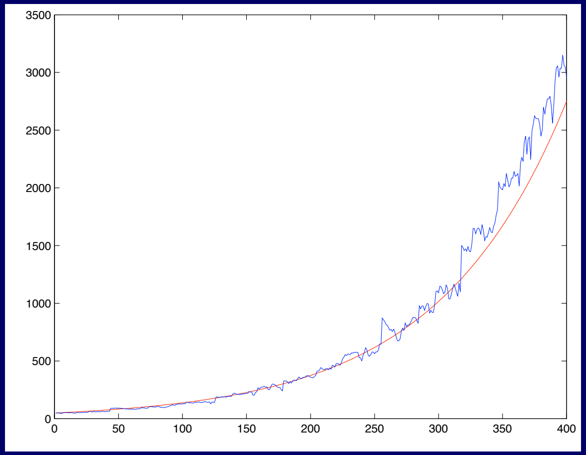
Performed using Monte Carlo simulations

- In all the numerical tests we use $N = 1000$ agents, $n = 10000$ shares, a riskless interest rate $r = 0.01$ and an average dividend growth rate $D = 0.015$.
- Initially each investor has a total wealth of 1000 composed of 10 shares, at a value of 50 per share, and 500 in bonds.
- The random variables ξ and η have been supposed distributed accordingly to a truncated normal distribution so that negative wealth values are avoided
- We compare the results obtained with one single run of the simulation with a direct solution of the price equation.

Test 1

We assume that the agents simply follow a constant investments rule $\mu(S(t)) = C$, with $0 < C < 1$ constant. As a consequence of our choice of parameters we have $C = 0.5$.





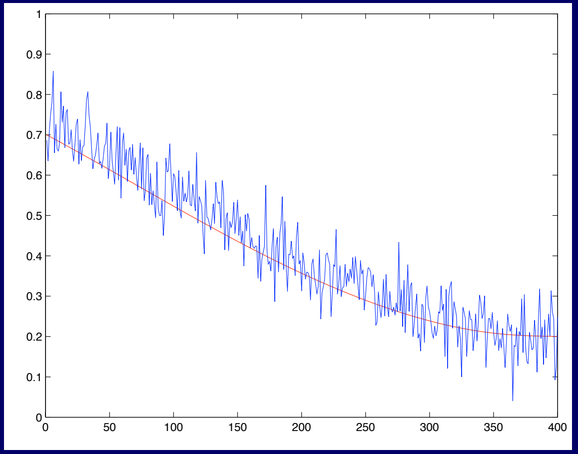
Investments (left) and price (right) in time

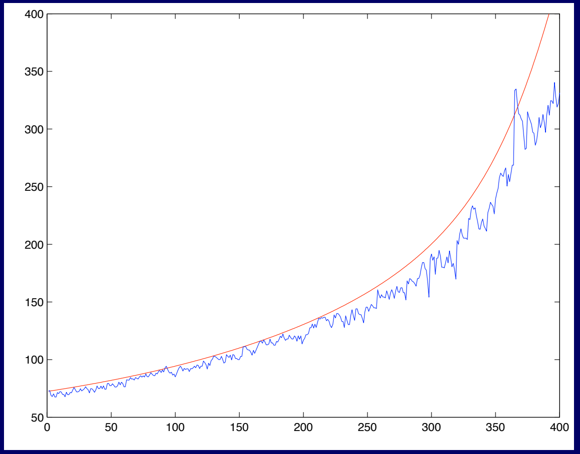
Test 2

$\mu(S(t))$ monotone decreasing function of the price $S(t)$. More precisely we take

$$\mu(S(t)) = C_1 + (1 - C_1)e^{-C_2 S(t)}$$

with $C_1 = 0.2$ and $C_2 = 0.01$.

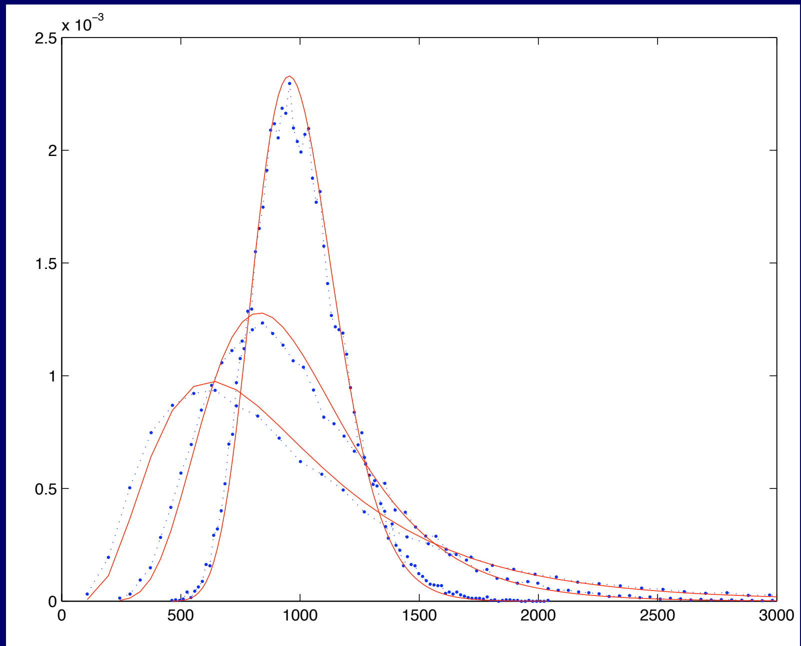




Investments (left) and price (right) in time

Test 3

Compare Boltzmann model and Fokker-Planck model. To this end, we consider the self-similar scaling and compute the solution for the values $r = 0.001$, $D = 0.0015$ with ξ and $\eta/S(0)$ distributed with standard deviation 0.05. Constant value of $\mu = 0.5$ at different times $t = 50, 200, 500$



Test 3. Distribution function at $t = 50, 200, 500$. The continuous line is the lognormal Fokker-Planck solution.

Test 3- Gini coefficient

We compute the Lorentz curve $L(F(w,t))$ defined as

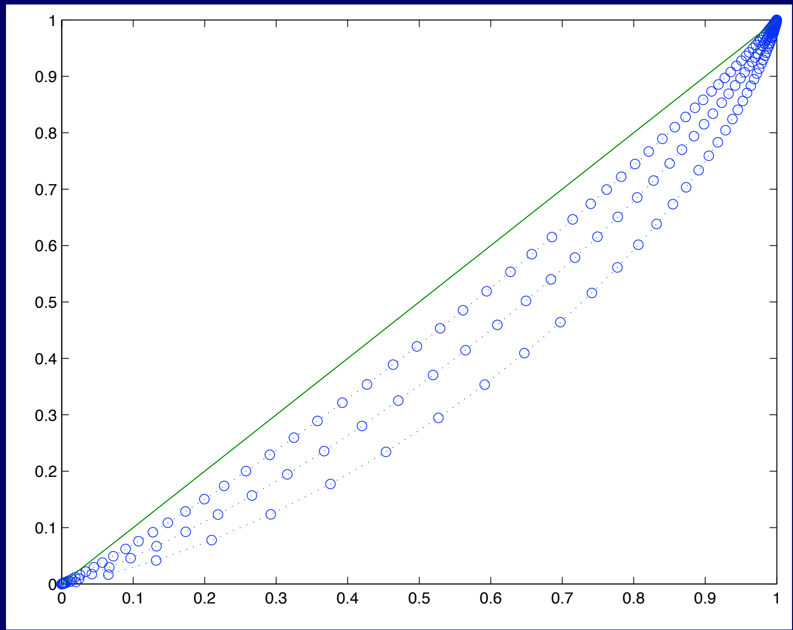
$$L(F(w,t)) = \frac{\int_0^w f(v,t)v\,dv}{\int_0^\infty f(v,t)v\,dv}, \qquad F(w,t) = \int_0^w f(v,t)\,dv,$$

Gini coefficient $G \in [0,1]$ is defined by

$$G = 1 - 2\int_0^1 L(F(w,t))\,dw.$$

measures inequality in the wealth distribution. A value of 0 corresponds to the line of perfect equality.

Test 3- Gini coefficient



Test 3. Lorentz curves. Gini coefficients are $G = 0.1$, $G = 0.2$ and $G = 0.3$ respectively.

Inequalities grow in time due to the speculative dynamics.

Conclusions

- We have derived a simple linear kinetic model which describes a financial market under the assumption that the strategy of investments is given as a function of the price.
- The model is able to describe the exponential grow of the price of the stock and of the wealth above the rate produced by simple investments in bonds.
- The long time behavior of the model has been studied in the Fokker-Planck approximation. It leads to lognormal wealth distribution
- In order to produce the effect of market booms, cycles and crashes, the distribution of investments should be a function of time $\mu(S(t), t)$.
- S.C. , L. Pareschi, C. Piatecki, J. Stat. Phys. (2009)