Multi-soliton solutions for the supercritical generalized Korteweg-de Vries equations

Vianney Combet (PIMS – UBC)

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The equation

We consider the generalized Korteweg-de Vries equation (gKdV)

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^p) = 0, \\ u(0) = u_0 \in H^1(\mathbb{R}), \end{cases}$$

where $u = u(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}$ and $p \geqslant 2$ is an integer.

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where $u = u(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}$ and $p \geqslant 2$ is an integer.

Theorem (Local existence, Kenig, Ponce and Vega '93)

For all $u_0 \in H^1(\mathbb{R})$, there exist $T^* = T^*(\|u_0\|_{H^1}) > 0$ and $X_T \subset C([0,T], H^1(\mathbb{R}))$ such that (gKdV) has a unique solution $u \in X_T$ with $u(0) = u_0$ for all $T < T^*$. Moreover, $u_0 \mapsto u$ is a continuous map from $H^1(\mathbb{R})$ to $C([0,T], H^1(\mathbb{R}))$, and if $T^* < +\infty$, then $\lim_{t \nearrow T^*} \|u(t)\|_{H^1} = +\infty$.

Conservation laws and Gagliardo-Nirenberg inequality

1. Conservation laws: If $u \in C([0,T], H^1(\mathbb{R}))$ is a solution of (gKdV) with $u(0) = u_0$, then, for all $t \in [0,T]$,

$$m(u(t)) = \int_{\mathbb{R}} u^2(t)\, dx = m(u_0),$$

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}} u_x^2(t)\, dx - \frac{1}{p+1}\int_{\mathbb{R}} u^{p+1}(t)\, dx = E(u_0).$$

2. Gagliardo-Nirenberg inequality: For all $u \in H^1(\mathbb{R})$ and for all $p \geqslant 1$, one has

$$\int |u|^{p+1} \leqslant C(p) \left(\int u^2 \right)^{\frac{p+3}{4}} \left(\int u_x^2 \right)^{\frac{p-1}{4}}.$$

Globalization

- For $p < 5$ (subcritical case), there is global well posedness in $H^1(\mathbb{R})$.
- For $p = 5$ (critical case), blow up occurs for a large class of initial data (Martel and Merle '02).
- For $p > 5$ (supercritical case), the existence of blow up solutions is an open problem.

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Existence of solitons

The (gKdV) equation admits traveling wave solutions (also called solitons):

$$R_{c,x_0}(t,x) = Q_c(x - ct - x_0)$$

where $(c, x_0) \in \mathbb{R}_+^* \times \mathbb{R}$, and Q_c satisfies the ordinary differential equation

$$Q_c'' + Q_c^p = cQ_c,$$

which has a unique solution (up to translations) if we impose $Q_c \in H^1(\mathbb{R})$ and $Q_c > 0$.

Properties of Q_c

1. One has
$$Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}x)$$
where $Q = Q_1$ is given by $Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)}\right)^{\frac{1}{p-1}}$.
2. $Q_c(x) \sim C_p e^{-\sqrt{c}|x|}$ when $x \longrightarrow \pm\infty$.
3. $\|Q_c\|_{L^2}^2 = c^{\frac{5-p}{2(p-1)}} \|Q\|_{L^2}^2$.

Previous stability results

1. Solitons are *orbitally* stable for subcritical gKdV (Benjamin, Bona, Cazenave and Lions, Weinstein, Grillakis *et al.*).
2. Solitons are orbitally unstable for critical gKdV (Martel and Merle) and supercritical gKdV (Bona *et al.*).
3. Solitons are *asymptotically* stable (Pego and Weinstein, Martel and Merle).
4. Sums of decoupled solitons are stable for subcritical gKdV (Martel, Merle and Tsai).

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Definition of N-solitons

Let $N \geqslant 1$. Given $2N$ real parameters

$$0 < c_1 < \cdots < c_N, \quad x_1, \ldots, x_N \in \mathbb{R},$$

we denote $R(t) = \sum_{j=1}^{N} R_{c_j,x_j}(t)$.

Definition

We call a solution φ of (gKdV) an N-soliton if

$$\lim_{t\to+\infty} \|\varphi(t) - R(t)\|_{H^1} = 0.$$

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Variational characterization of Q_c

Proposition (Weinstein '86)

Suppose that $p < 5$. If $u \in H^1(\mathbb{R})$ satisfies $E(u) = E(Q_c)$ and $m(u) = m(Q_c)$, then there exists $a \in \mathbb{R}$ such that $u = Q_c(\cdot + a)$.

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Corollary

Suppose that $p \leqslant 5$. If u is a solution of (gKdV) such that

$$\lim_{t \to +\infty} \|u(t) - R_{c,x_0}(t)\|_{H^1} = 0,$$

then $u = R_{c,x_0}$.

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Preliminary remarks

1. Since we consider only one soliton, we can suppose by scaling invariance that $c = 1$.
2. The proof of the following theorem is an adaptation for (gKdV) of previous works of Duyckaerts and Merle '09, and of the following works of Duyckaerts and Roudenko, on the L^2 supercritical nonlinear Schrödinger equation.

Statement

Theorem (C. '09)

Let $p > 5$.

1. There exists a one-parameter family $(U^A)_{A\in\mathbb{R}}$ of solutions of (gKdV) such that, for all $A \in \mathbb{R}$,

$$\lim_{t\to+\infty} \|U^A(t,\cdot+t) - Q\|_{H^1} = 0,$$

and if $A' \in \mathbb{R}$ satisfies $A' \neq A$, then $U^{A'} \neq U^A$.

Statement

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and if $A' \in \mathbb{R}$ satisfies $A' \neq A$, then $U^{A'} \neq U^A$.

2. Conversely, if u is a solution of (gKdV) such that

$$\lim_{t\to+\infty} \inf_{y\in\mathbb{R}} \|u(t) - Q(\cdot - y)\|_{H^1} = 0,$$

then there exist $A \in \mathbb{R}$, $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $u(t) = U^A(t, \cdot - x_0)$ for $t \geqslant t_0$.

Linearized equation around Q

If $u(t,x) = Q(x-t) + h(t,x-t)$ satisfies (gKdV), then h satisfies

$$\partial_t h + \mathcal{L}h = O(h^2),$$

where $\mathcal{L}h = -(Lh)_x$ and $Lh = -h_{xx} + h - pQ^{p-1}h$.

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Theorem (Pego and Weinstein '92)

Let $\sigma(\mathcal{L})$ be the spectrum of the operator $\mathcal{L}$ defined on $L^2(\mathbb{R})$. Then

$$\sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, e_0\} \textit{ with } e_0 > 0.$$

Furthermore, e_0 and $-e_0$ are simple eigenvalues of $\mathcal{L}$ with eigenfunctions Y^+ and Y^- which have an exponential decay at infinity, and the null space of $\mathcal{L}$ is spanned by Q'.

Properties of the family (U^A)

Proposition

Let $A \in \mathbb{R}$. If $t_0 = t_0(A) \in \mathbb{R}$ is large enough, then there exists a solution $U^A \in C^\infty([t_0, +\infty), H^\infty)$ of (gKdV) such that, for all $s \in \mathbb{R}$, there exists $C > 0$ such that

$$\forall t \geqslant t_0, \quad \| U^A(t, \cdot + t) - Q - Ae^{-e_0 t} Y^+ \|_{H^s} \leqslant Ce^{-2e_0 t}.$$

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Proposition

Up to translations in time and in space, there are only three special solutions: U^1, U^{-1} and Q. More precisely, one has (for t large enough in each case):

(a) *If $A > 0$, then $U^A(t) = U^1(t + t_A, \cdot + t_A)$ for some $t_A \in \mathbb{R}$.*

(b) *If $A = 0$, then $U^0(t) = Q(\cdot - t)$.*

(c) *If $A < 0$, then $U^A(t) = U^{-1}(t + t_A, \cdot + t_A)$ for some $t_A \in \mathbb{R}$.*

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Previous result 1

Let $N \geqslant 2$, $0 < c_1 < \cdots < c_N$, $x_1, \ldots, x_N \in \mathbb{R}$, $R = \sum_{j=1}^N R_{c_j,x_j}$.

Theorem (Martel '05)

Suppose that $p \leqslant 5$. Then there exist $T_0 \in \mathbb{R}$ and a unique solution $\varphi \in C([T_0,+\infty), H^1(\mathbb{R}))$ of (gKdV) such that

$$\lim_{t\to+\infty} \|\varphi(t) - R(t)\|_{H^1} = 0.$$

Moreover, for all $s \geqslant 0$, $\varphi \in C([T_0,+\infty), H^s(\mathbb{R}))$ and there exists $A_s > 0$ such that, for all $t \geqslant T_0$ and for some $\gamma > 0$,

$$\|\varphi(t) - R(t)\|_{H^s(\mathbb{R})} \leqslant A_s e^{-\gamma t}.$$

Remark

This result is based on refinements of a previous work of Martel, Merle and Tsai (2002).

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Previous result 2

Let $N \geqslant 2$, $0 < c_1 < \cdots < c_N$, $x_1, \ldots, x_N \in \mathbb{R}$, $R = \sum_{j=1}^{N} R_{c_j,x_j}$.

Theorem (Côte, Martel and Merle '09)

Suppose that $p > 5$. Then there exist $T_0 \in \mathbb{R}$, $C, \sigma_0 > 0$ and a solution $\varphi \in C([T_0,+\infty), H^1)$ of (gKdV) such that

$$\forall t \in [T_0,+\infty), \quad \|\varphi(t) - R(t)\|_{H^1} \leqslant Ce^{-\sigma_0^{3/2} t}.$$

Remark

This result is also based on refinements of a previous work of Martel, Merle and Tsai (2002), with an additional topological argument and using the precise description of the spectrum of $\mathcal{L}$ by Pego and Weinstein (1992).

Main result

Let $N \geqslant 2$, $0 < c_1 < \cdots < c_N$, $x_1, \ldots, x_N \in \mathbb{R}$, $R = \sum_{j=1}^{N} R_{c_j,x_j}$.

Theorem (C. '10)

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Theorem (C. '10)

Suppose that $p > 5$.

1. There exists an N-parameter family $(\varphi_{A_1,\ldots,A_N})_{(A_1,\ldots,A_N)\in\mathbb{R}^N}$ of solutions of (gKdV) such that, for all $(A_1, \ldots, A_N) \in \mathbb{R}^N$,

$$\lim_{t\to+\infty} \|\varphi_{A_1,\ldots,A_N}(t) - R(t)\|_{H^1} = 0,$$

and if $(A'_1, \ldots, A'_N) \neq (A_1, \ldots, A_N)$, $\varphi_{A'_1,\ldots,A'_N} \neq \varphi_{A_1,\ldots,A_N}$.

Main result

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   and if $(A'_1, \ldots, A'_N) \neq (A_1, \ldots, A_N)$, $\varphi_{A'_1,\ldots,A'_N} \neq \varphi_{A_1,\ldots,A_N}$.
2. Conversely, if u is a solution of (gKdV) such that $\lim_{t\to+\infty} \|u(t) - R(t)\|_{H^1} = 0$, then there exists $(A_1, \ldots, A_N) \in \mathbb{R}^N$ such that $u = \varphi_{A_1,\ldots,A_N}$.

Key proposition for the construction

For all $j \in [\![1, N]\!]$, we denote

$$Y_j^{\pm}(t,x) = c_j^{-1/2} Y^{\pm}(\sqrt{c_j}(x - c_j t - x_j)) \quad \text{and} \quad e_j = c_j^{3/2} e_0.$$

Note that

$$0 < e_1 < e_2 < \cdots < e_N.$$

Proposition (Perturbation along one soliton)

Let $j \in [\![1, N]\!]$, $A_j \in \mathbb{R}$, and φ be any N-soliton of (gKdV). Then there exist $t_0 > 0$ and a solution $u \in C([t_0, +\infty), H^1)$ of (gKdV) such that

$$\forall t \geqslant t_0, \quad \|u(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t},$$

for some $\gamma > 0$.

Construction of $\varphi_{A_1,\dots,A_N}$

Let $(A_1,\dots,A_N)\in\mathbb{R}^N$ and fix φ an N-soliton constructed by Côte, Martel and Merle.

1. Applying the key proposition with φ, there exists φ_{A_1} solution of (gKdV) such that, for all $t\geqslant t_0$,

$$\|\varphi_{A_1}(t)-\varphi(t)-A_1e^{-e_1t}Y_1^+(t)\|_{H^1}\leqslant e^{-(e_1+\gamma)t}.$$

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$$\|\varphi_{A_1}(t) - \varphi(t) - A_1 e^{-e_1 t} Y_1^+(t)\|_{H^1} \leqslant e^{-(e_1+\gamma)t}.$$
2. But φ_{A_1} is also an N-soliton. Thus, there exists φ_{A_1,A_2} such that, for all $t \geqslant t_0'$,
$$\|\varphi_{A_1,A_2}(t) - \varphi_{A_1}(t) - A_2 e^{-e_2 t} Y_2^+(t)\|_{H^1} \leqslant e^{-(e_2+\gamma)t}.$$

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3. Similarly, for all $j\in[\![2,N]\!]$, we construct by induction a solution $\varphi_{A_1,\dots,A_j}$ of (gKdV) such that

$$\|\varphi_{A_1,\dots,A_j}(t)-\varphi_{A_1,\dots,A_{j-1}}(t)-A_je^{-e_jt}Y_j^+(t)\|_{H^1}\leqslant e^{-(e_j+\gamma)t}.$$

Construction of $\varphi_{A_1,\dots,A_N}$

Let $(A_1, \dots, A_N) \in \mathbb{R}^N$ and fix φ an N-soliton constructed by Côte, Martel and Merle.

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3. Similarly, for all $j \in [\![2, N]\!]$, we construct by induction a solution $\varphi_{A_1,\dots,A_j}$ of (gKdV) such that
$$\|\varphi_{A_1,\dots,A_j}(t) - \varphi_{A_1,\dots,A_{j-1}}(t) - A_j e^{-e_j t} Y_j^+(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}.$$
4. Finally, $\varphi_{A_1,\dots,A_N}$ constructed by this way satisfies the theorem.

Construction of $\varphi_{A_1,\dots,A_N}$

Suppose that $\varphi_{A'_1,\dots,A'_N} = \varphi_{A_1,\dots,A_N}$. By construction, we have

$$\varphi_{A_1,\dots,A_N}(t) = \varphi(t) + \sum_{k=1}^{N} A_k e^{-e_k t} Y_k^+(t) + \sum_{k=1}^{N} z_k(t)$$

$$\varphi_{A'_1,\dots,A'_N}(t) = \varphi(t) + \sum_{k=1}^{N} A'_k e^{-e_k t} Y_k^+(t) + \sum_{k=1}^{N} \widetilde{z_k}(t),$$

with $\|z_k(t)\|_{H^1} + \|\widetilde{z_k}(t)\|_{H^1} \leqslant e^{-(e_k+\gamma)t}$. Thus, by difference, we have $e^{-e_1 t}|A_1 - A'_1| \leqslant Ce^{-(e_1+\gamma)t}$, and so $A'_1 = A_1$.

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with $\|z_k(t)\|_{H^1} + \|\widetilde{z_k}(t)\|_{H^1} \leqslant e^{-(e_k+\gamma)t}$. Thus, by difference, we have $e^{-e_1 t}|A_1 - A'_1| \leqslant Ce^{-(e_1+\gamma)t}$, and so $A'_1 = A_1$. Next, we write

$$\varphi_{A_1,\dots,A_N}(t) = \varphi_{A_1}(t) + \sum_{k=2}^{N} A_k e^{-e_k t} Y_k^+(t) + \sum_{k=2}^{N} z_k(t)$$

$$\varphi_{A'_1,\dots,A'_N}(t) = \varphi_{A_1}(t) + \sum_{k=2}^{N} A'_k e^{-e_k t} Y_k^+(t) + \sum_{k=2}^{N} \widetilde{z_k}(t),$$

and we obtain similarly $A'_2 = A_2$, and so on.

Plan

Convergence at small exponential rate γ

Let $p > 5$, $N \geqslant 2$, $0 < c_1 < \cdots < c_N$, $x_1, \ldots, x_N \in \mathbb{R}$. Denote $R_j = R_{c_j,x_j}$, $R = \sum_{j=1}^N R_j$ and φ the multi-soliton used for the construction. Let u be a solution of (gKdV) such that

$$\lim_{t\to+\infty} \|u(t) - R(t)\|_{H^1} = 0.$$

Convergence at small exponential rate γ

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Lemma

There exist $\gamma, t_0, C > 0$ such that, for all $t \geqslant t_0$,

$$\|u(t) - R(t)\|_{H^1} \leqslant Ce^{-\gamma t}.$$

Convergence at small exponential rate γ

Let $p > 5$, $N \geqslant 2$, $0 < c_1 < \cdots < c_N$, $x_1, \ldots, x_N \in \mathbb{R}$. Denote $R_j = R_{c_j,x_j}$, $R = \sum_{j=1}^N R_j$ and φ the multi-soliton used for the construction. Let u be a solution of (gKdV) such that

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Lemma

There exist $\gamma, t_0, C > 0$ such that, for all $t \geqslant t_0$,

$$\|u(t) - R(t)\|_{H^1} \leqslant Ce^{-\gamma t}.$$

Corollary

Let $\varepsilon = u - \varphi$. Then there exist $C, t_0 > 0$ such that, for all $t \geqslant t_0$, $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-\gamma t}$.

Adjoint of $\mathcal{L}_c$

Let $c > 0$. We define

$$\mathcal{L}_c a = -\partial_x(L_c a),\ \ L_c a = -\partial_x^2 a + ca - pQ_c^{p-1} a,$$

and

$$e_c = c^{3/2} e_0 > 0,\ \ Y_c^\pm(x) = c^{-1/2} Y^\pm(\sqrt{c}x).$$

Lemma

Let $Z_c^\pm = L_c Y_c^\pm$. Then the following properties hold:

(i) $L_c(\partial_x Z_c^\pm) = \mp e_c Z_c^\pm$.

(ii) $(Y_c^+, Z_c^+) = (Y_c^-, Z_c^-) = 0$, $(Z_c^+, Q_c') = (Z_c^-, Q_c') = 0$, *and* $(Y_c^+, Z_c^-) = (Y_c^-, Z_c^+) = 1$.

(iii) *There exist $\sigma_c > 0$ and $C > 0$ such that, for all $v_c \in H^1$,*

$$(L_c v_c, v_c) \geqslant \sigma_c \|v_c\|_{H^1}^2 - C(v_c, Z_c^+)^2 - C(v_c, Z_c^-)^2 - C(v_c, Q_c')^2.$$

Convergence at exponential rate e_1

Lemma

Let $\varepsilon = u - \varphi$. Then there exist $C, t_0 > 0$ such that, for all $t \geqslant t_0$, $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-e_1 t}$.

Proof. For $j \in [\![1, N]\!]$, we recall that $e_j = e_{c_j}$,

$$R_j(t,x) = Q_{c_j}(x - c_j t - x_j) \text{ and } Y_j^{\pm}(t,x) = Y_{c_j}^{\pm}(x - c_j t - x_j).$$

We also denote $Z_j^{\pm}(t,x) = Z_{c_j}^{\pm}(x - c_j t - x_j)$ and

$$\alpha_j^{\pm}(t) = \int \varepsilon(t) Z_j^{\pm}(t), \ \boldsymbol{\alpha}(t) = (\alpha_j^{\pm}(t))_{j,\pm}.$$

Finally, we define $\widetilde{\varepsilon}(t) = \varepsilon(t) + \sum_{j=1}^N a_j(t) R_{jx}(t)$ with $a_j(t) = -\|Q'_{c_j}\|_{L^2}^{-2} \cdot \int \varepsilon(t) R_{jx}(t)$, so that

$$\left| \int \widetilde{\varepsilon}(t) R_{jx}(t) \right| \leqslant Ce^{-\gamma t} \|\varepsilon(t)\|_{H^1}.$$

Estimates

We can find the following estimates, for all $j \in [\![1, N]\!]$ and all $t \geqslant t_0$:

$$\left| \frac{d}{dt}\alpha_j^{\pm}(t) \mp e_j \alpha_j(t) \right| \leqslant Ce^{-\gamma t} \|\varepsilon(t)\|_{H^1},$$

$$\|\widetilde{\varepsilon}(t)\|_{H^1}^2 \leqslant Ce^{-2\gamma t} \sup_{t' \geqslant t} \|\varepsilon(t')\|_{H^1}^2 + C\|\boldsymbol{\alpha}(t)\|^2,$$

$$|a_j'(t)| \leqslant Ce^{-\gamma t}\|\varepsilon(t)\|_{H^1} + C\|\widetilde{\varepsilon}(t)\|_{H^1},$$

$$\|\varepsilon(t)\|_{H^1} \leqslant C\|\widetilde{\varepsilon}(t)\|_{H^1} + C\sum_{j=1}^{N} |a_j(t)|.$$

Induction

Recall that we already have $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-\gamma_0 t}$ with $\gamma_0 = \gamma$. Now, we prove that if $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-\gamma_0 t}$ with $\gamma \leqslant \gamma_0 < e_1 - \gamma$, then $\|\varepsilon(t)\|_{H^1} \leqslant C' e^{-(\gamma_0+\gamma)t}$.

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5. Finally, we get $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-(\gamma_0+\gamma)t}$ as expected.

Identification of A_1

Now, we have $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-\gamma_0 t}$ with $e_1 - \gamma < \gamma_0 < e_1$.

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Hence, there exists $A_1 \in \mathbb{R}$ such that

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4. It is easy to conclude that $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-e_1 t}$ by the preliminary estimates. $\square$

Identification of the solution

Defining $\varepsilon_1(t) = u(t) - \varphi_{A_1}(t)$, we have $\|\varepsilon_1(t)\|_{H^1} \leqslant Ce^{-e_1 t}$, and defining z_1 by $z_1(t) = \varphi_{A_1}(t) - \varphi(t) - A_1 e^{-e_1 t} Y_1^+(t)$, we also have

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As $\|z_1(t)\|_{H^1} \leqslant e^{-(e_1+\gamma)t}$, we finally find, for $t \to +\infty$,

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Proposition

For all $j \in [\![1, N]\!]$, there exist $t_0, C > 0$ and $(A_1, \ldots, A_j) \in \mathbb{R}^j$ such that, defining $\varepsilon_j(t) = u(t) - \varphi_{A_1,\ldots,A_j}(t)$, one has

$$\forall t \geqslant t_0, \quad \|\varepsilon_j(t)\|_{H^1} \leqslant Ce^{-e_j t}.$$

Moreover, defining $\alpha_{j,k}^\pm(t) = \int \varepsilon_j(t) Z_k^\pm(t)$ for all $k \in [\![1, N]\!]$, one has $\lim_{t\to+\infty} e^{e_k t}\alpha_{j,k}^-(t) = 0$, for all $k \in [\![1, j]\!]$.

Conclusion

Corollary

There exist $(A_1, \ldots, A_N) \in \mathbb{R}^N$ *and* $C, t_0 > 0$ *such that, defining* $z(t) = u(t) - \varphi_{A_1,\ldots,A_N}(t)$, *one has* $\|z(t)\|_{H^1} \leqslant Ce^{-2e_N t}$ *for* $t \geqslant t_0$.

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For the proof of this final proposition, we consider

$$\theta(t) = \sup_{t' \geqslant t} e^{e_N t'} \|z(t')\|_{H^1},$$

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For the proof of this final proposition, we consider

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and we prove by similar techniques that there exists $C^* > 0$ such that

$$\theta(t) \leqslant C^* e^{-\gamma t_0} \theta(t).$$

Choosing t_0 large enough so that $C^* e^{-\gamma t_0} \leqslant \frac{1}{2}$, we finally find $\theta(t) = 0$.