Workshop on Topics in Kinetic Theory
(PIMS, Victoria, June 29 – July 3, 2009)

Approach to steady motion of a plate moving in a collision less gas under a constant external force

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Topics

- Approach to steady motion of a plate in a free-molecular (collisionless or Knudsen) gas

  A, Tsuji, & Cavallaro, *Phys, Rev. E* (09)

- Approach to equilibrium of a free-molecular gas

  Tsuji & A (work in progress)

Numerical study

Free-molecular gas

Highly rarefied gas

Effect of collisions: Neglected

$$\mathrm{Kn} = l/L = \infty$$

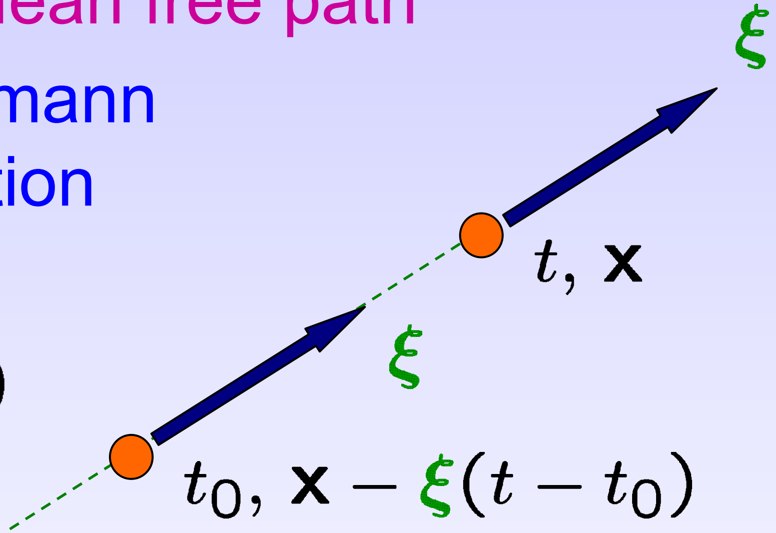
Mean free path

Molecular velocity

$$\frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial f}{\partial \mathbf{x}} = 0$$

Boltzmann equation

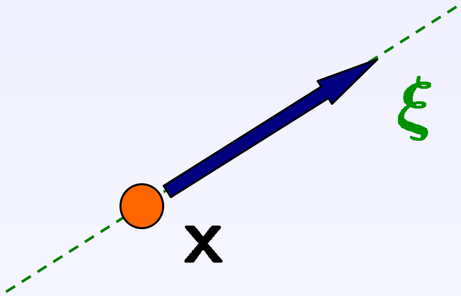
$$f(t, \mathbf{x}, \boldsymbol{\xi}) = f(t_0, \mathbf{x} - \boldsymbol{\xi}(t - t_0), \boldsymbol{\xi})$$



Time-independent case

$$f(\mathbf{x}, \boldsymbol{\xi}) = f(\mathbf{x} - \boldsymbol{\xi} s, \boldsymbol{\xi})$$

parameter



Initial-value problem

(Infinite domain)

Initial condition: $f(t_0, \mathbf{x}, \boldsymbol{\xi}) = g(\mathbf{x}, \boldsymbol{\xi})$

Solution: $f(t, \mathbf{x}, \boldsymbol{\xi}) = g(\mathbf{x} - \boldsymbol{\xi}(t - t_0), \boldsymbol{\xi})$

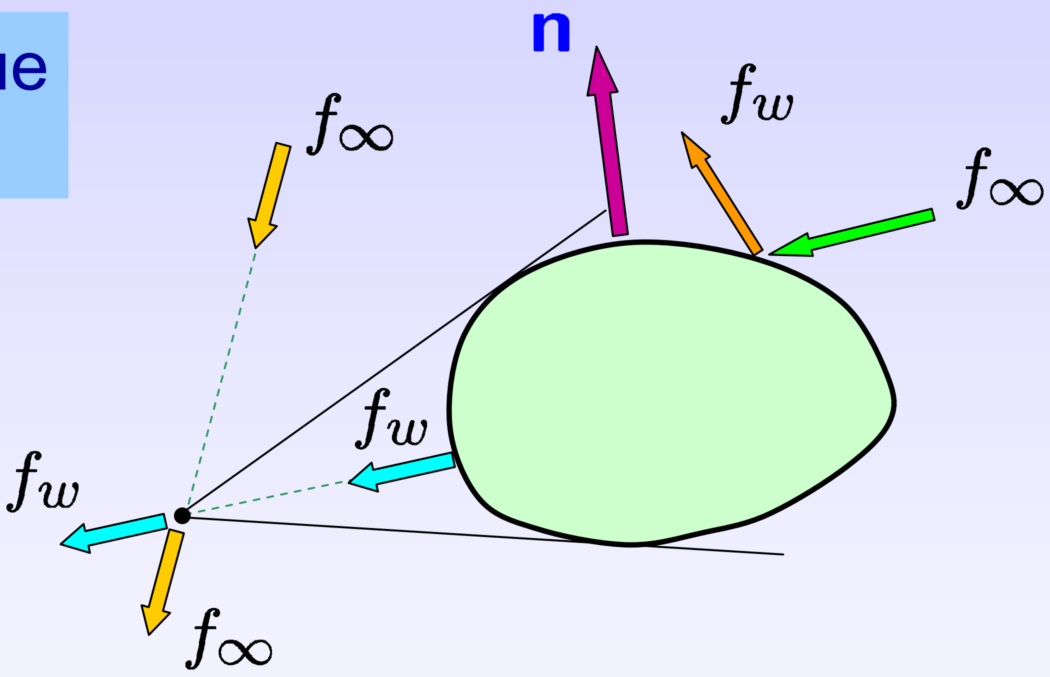
(Steady) boundary-value problem

Single convex body

f_∞ : given

f_w : from BC

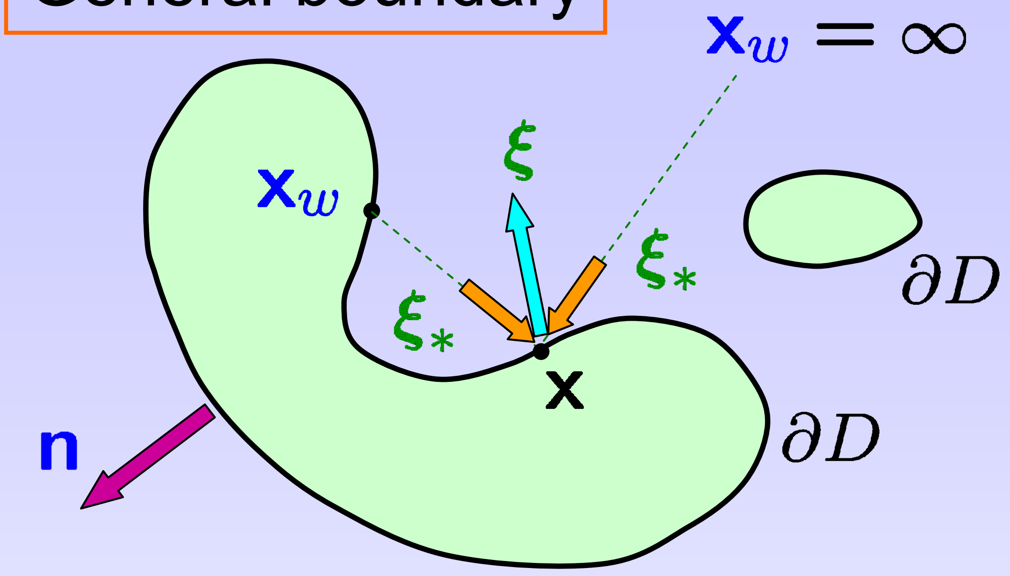
Solved!



$\mathbf{v}_w \cdot \mathbf{n} = 0$

BC : $f(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}_* \cdot \mathbf{n} < 0} K(\boldsymbol{\xi}, \boldsymbol{\xi}_*) f(\boldsymbol{\xi}_*) d\boldsymbol{\xi}_*, \quad (\boldsymbol{\xi} \cdot \mathbf{n} > 0)$

General boundary



$\mathbf{x} \in \partial D$

$\mathbf{x}_w = \mathbf{x}_w(\mathbf{x},\ \boldsymbol{\xi}_*/|\boldsymbol{\xi}_*|)$

BC

↓

$(\mathbf{x} \in \partial D,\ \boldsymbol{\xi}\cdot\mathbf{n} > 0)$

$$f(\mathbf{x},\ \boldsymbol{\xi}) = \int_{\boldsymbol{\xi}_*\cdot\mathbf{n}<0} K(\mathbf{x},\ \boldsymbol{\xi},\ \boldsymbol{\xi}_*) f(\mathbf{x}_w(\mathbf{x},\ \boldsymbol{\xi}_*/|\boldsymbol{\xi}_*|),\ \boldsymbol{\xi}_*) d\boldsymbol{\xi}_*$$

Integral equation for $f(\mathbf{x},\ \boldsymbol{\xi})$ $(\boldsymbol{\xi}\cdot\mathbf{n} > 0)$

Diffuse reflection: Integral equation for $\sigma_w(\mathbf{x})$ $(\mathbf{x} \in \partial D)$

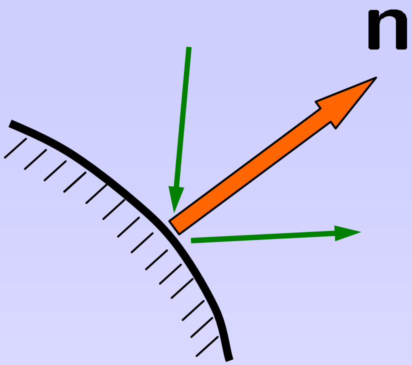
Maxwell type: Exact solution ! Sone, *J. Mec. Theor. Appl.* (84,85)

General situation, effect of boundary temperature

Y. Sone, Molecular Gas Dynamics: Theory, Techniques, and Applications
(Birkhäuser, 2007)

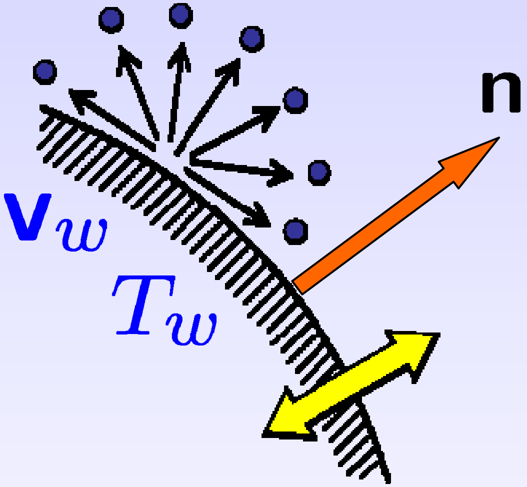
Conventional boundary condition

$$(\boldsymbol{\xi} - \mathbf{v}_w)\cdot\mathbf{n} > 0 \qquad [(t,\, \mathbf{x}) : \text{omitted}\,]$$



- Specular reflection

$$f(\boldsymbol{\xi}) = f(\boldsymbol{\xi} - 2\,[(\boldsymbol{\xi} - \mathbf{v}_w)\cdot\mathbf{n}\,]\,\mathbf{n})$$



- Diffuse reflection

$$f(\boldsymbol{\xi}) = \frac{\sigma_w}{(2\pi R T_w)^{3/2}} \times \exp\left(-\frac{|\boldsymbol{\xi} - \mathbf{v}_w|^2}{2RT_w}\right)$$

$$\sigma_w = -\left(\frac{2\pi}{RT_w}\right)^{1/2} \int_{(\boldsymbol{\xi}-\mathbf{v}_w)\cdot\mathbf{n}<0} (\boldsymbol{\xi} - \mathbf{v}_w)\cdot\mathbf{n} f(\boldsymbol{\xi})d\boldsymbol{\xi}$$

No net mass flux across the boundary

- Maxwell type

$$\alpha \times (\text{diffuse}) + (1 - \alpha) \times (\text{specular})$$

Accommodation coefficient

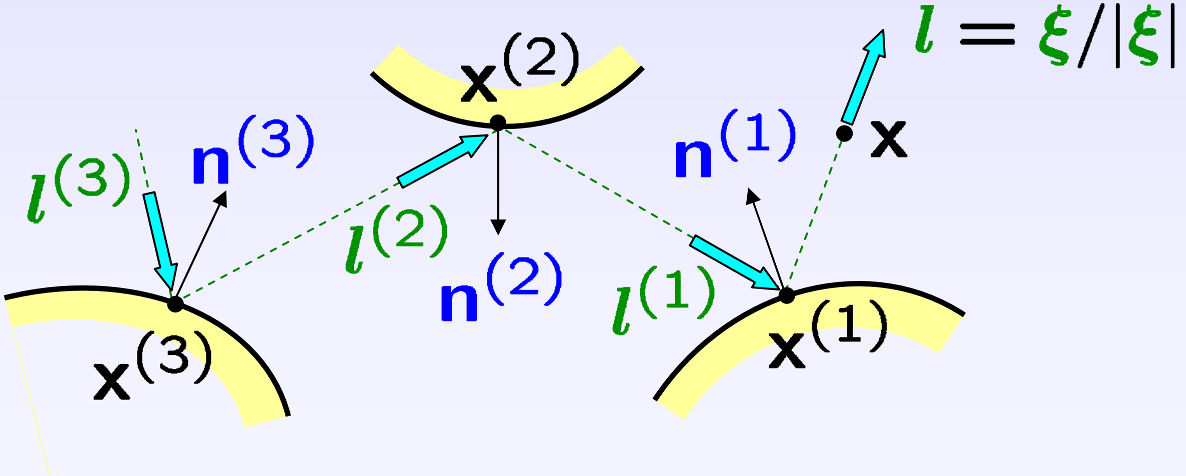
- Cercignani-Lampis model

Cercignani, Lampis (72)

Statics: Effect of boundary temperature

Sone, J Mec. Theor. Appl. (84, 85)

- Closed or open domain, boundary at rest arbitrary shape and arrangement
- Maxwell-type (diffuse-specular) condition
- Arbitrary distribution of boundary temperature, accommodation coefficient



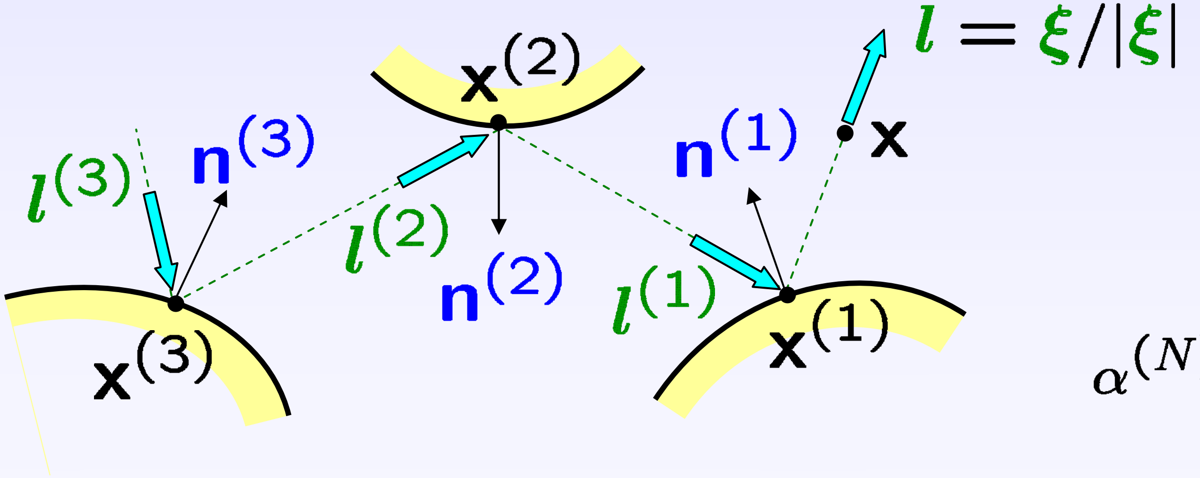
Exact Solution !

Path of a specularly reflected molecule

Exact solution

$$\begin{aligned} f(\mathbf{x}, |\boldsymbol{\xi}|\boldsymbol{l}) &= C\alpha^{(1)}\mathcal{M}^{(1)} + C(1-\alpha^{(1)})\alpha^{(2)}\mathcal{M}^{(2)} \\ &\quad + C(1-\alpha^{(1)})(1-\alpha^{(2)})\alpha^{(3)}\mathcal{M}^{(3)} + \cdots \\ &= C\sum_{m=1}^{\infty}\prod_{h=1}^{m-1}(1-\alpha^{(h)})\alpha^{(m)}\mathcal{M}^{(m)} \end{aligned}$$

$$\begin{cases} \alpha^{(m)} = \alpha(\mathbf{x}^{(m)}), \qquad C = \text{const} \\ \mathcal{M}^{(m)} = \dfrac{1}{[2RT_w(\mathbf{x}^{(m)})]^2}\exp\left(-\dfrac{|\boldsymbol{\xi}|^2}{2RT_w(\mathbf{x}^{(m)})}\right) \end{cases}$$

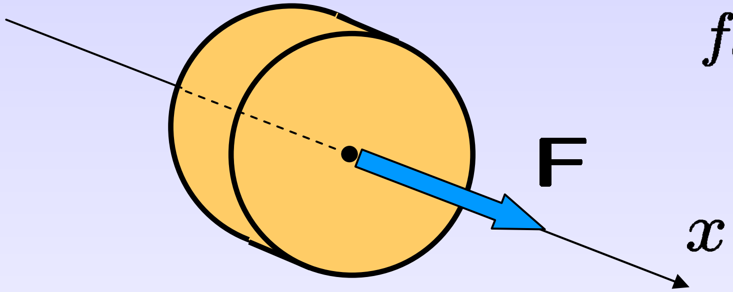


$\alpha^{(N)} = 1$ if $\mathbf{x}^{(N)} = \infty$

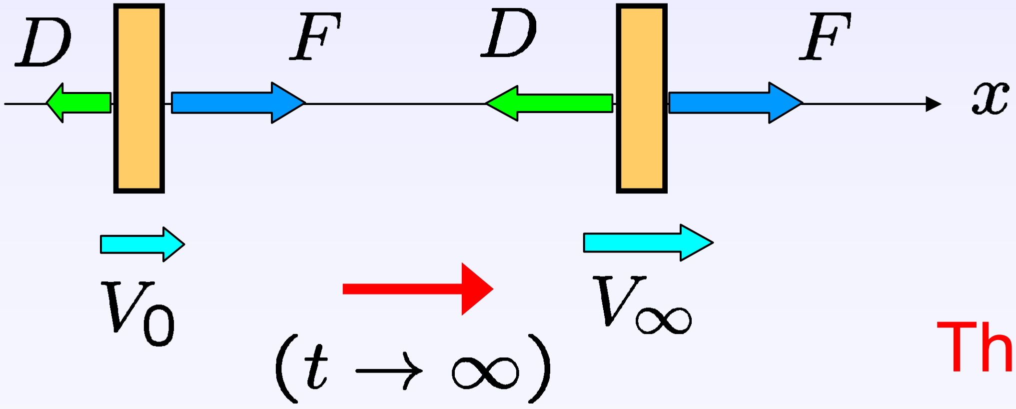
Approach to steady motion of a plate

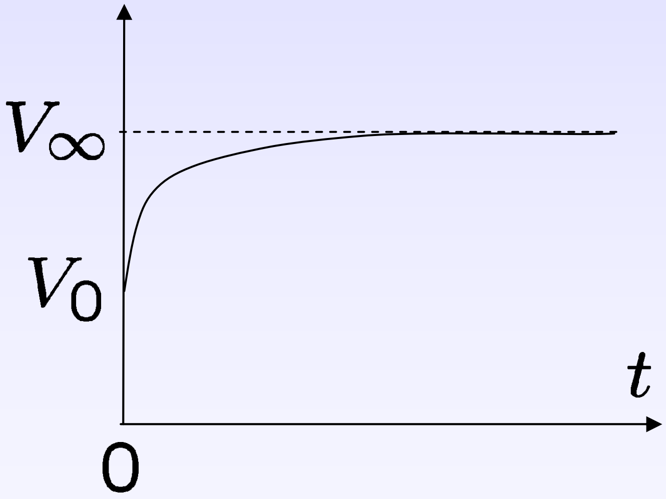
Initial and boundary-value problem

A disk accelerated by a constant external force **F**
in a free-molecular gas (no force on gas molecules)



$$f_\infty = \frac{\rho_\infty}{(2\pi R T_\infty)^{3/2}} \exp\left(-\frac{|\boldsymbol{\xi}|^2}{2RT_\infty}\right)$$





The manner of approach ??

Equation of motion of the disk : $M\frac{dV(t)}{dt} = F - D(t)$

If $D(t) = k\,V(t)$, then

$$V(t) = (V_0 - V_\infty)\,\mathrm{exp}\left(-\frac{k}{M}t\right) + V_\infty, \qquad V_\infty = F/k$$

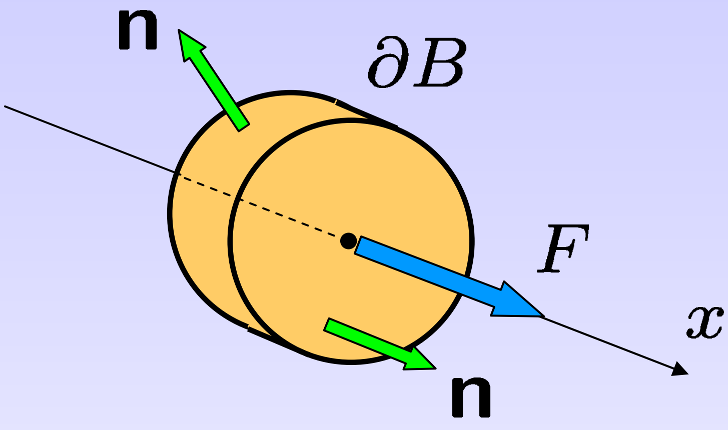
Exponential approach (usual case)

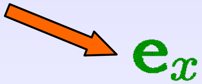
Free-molecular gas ???

Free-molecular gas

$f_\infty :$ ρ_∞ T_∞ (p_∞)

$\mathbf{v}_\infty = 0$





Gas:

EQ: $\partial_t f + \boldsymbol{\xi} \cdot \nabla_x f = 0$

IC: $f = f_\infty \quad (t = 0)$

$$\left[\, f_\infty = \frac{\rho_\infty}{(2\pi R T_\infty)^{3/2}} \exp\left(-\frac{|\boldsymbol{\xi}|^2}{2RT_\infty}\right) \,\right]$$

BC: Specular or Diffuse reflection on body surface

Body:

EQ: $\dot{X}(t) = V(t), \quad M\dot{V}(t) = F - D(t)$

IC: $X(0) = 0, \quad V(0) = V_0$

$$D(t) = \int_{\partial B} \int_{\xi \in \mathbf{R}^3} \xi_x \left[\, \boldsymbol{\xi} - V(t)\, \mathbf{e}_x \,\right] \cdot \mathbf{n}\, f \, d\boldsymbol{\xi}\, dS$$

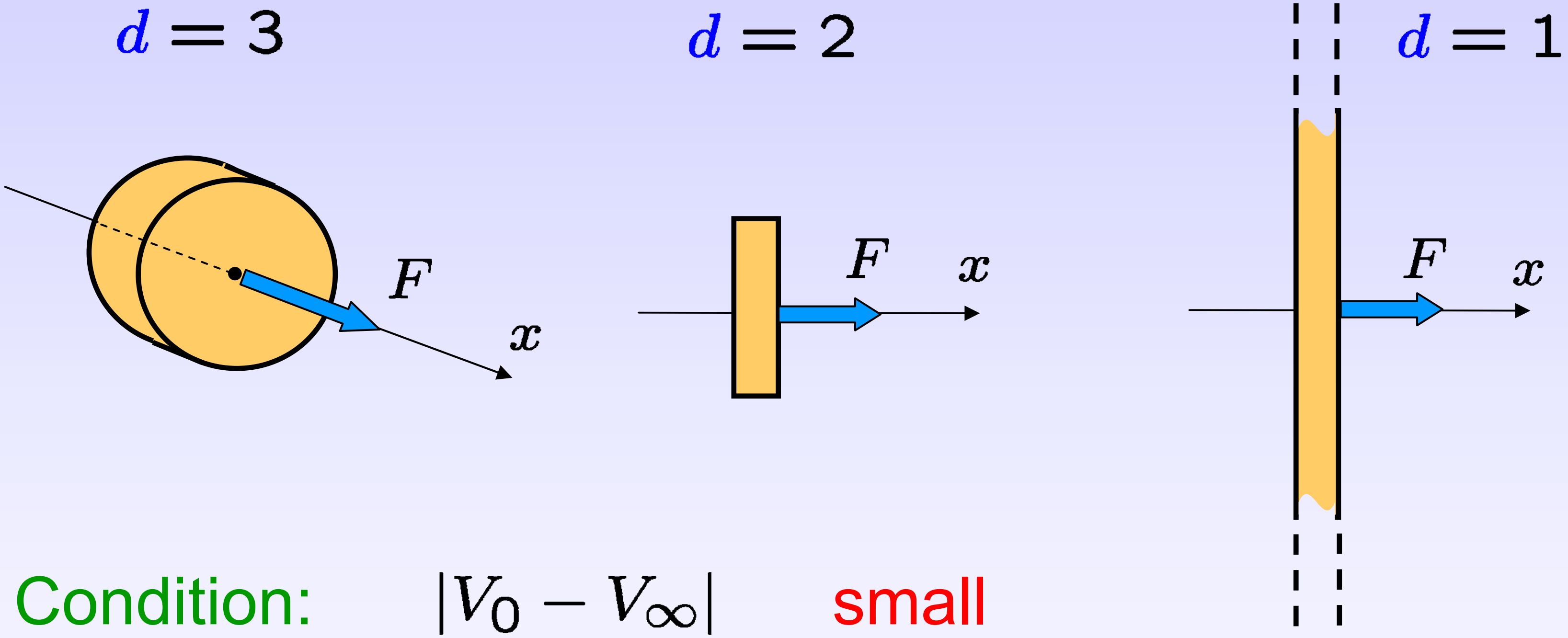
Approach to V_∞

Mathematical study

BC: specular reflection

$$|V(t) - V_\infty| \approx \frac{C}{t^{d+2}}$$

d : dimension



Condition: $|V_0 - V_\infty|$ small

Caprino, Marchioro, & Pulvirenti, *Commun. Math. Phys.* (06)
Caprino, Cavallaro, & Marchioro, *M³AS* (07)
Cavallaro, *Rend. Mat.* (07)

Approach to V_∞

BC: specular reflection $|V(t) - V_\infty| \approx \frac{C}{t^{d+2}}$

d : dimension

Caprino, Marchioro, & Pulvirenti, *Commun. Math. Phys.* (06)
Caprino, Cavallaro, & Marchioro, *M³AS* (07)
Cavallaro, *Rend. Mat.* (07)

BC: diffuse reflection $|V(t) - V_\infty| \approx \frac{C}{t^{d+1}}$

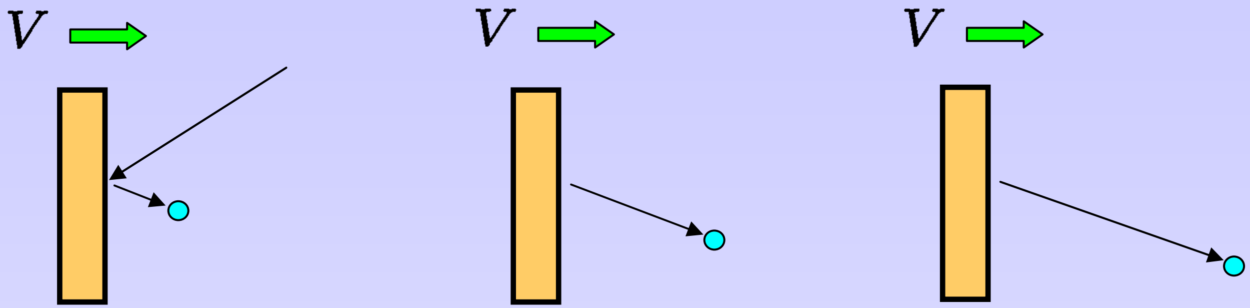
A, Cavallaro, Marchioro, & Pulvirenti, *M²NA* (08)

Condition: $|V_0 - V_\infty|$ small

Cause of non-exponential (power-law) decay

Effect of recollision

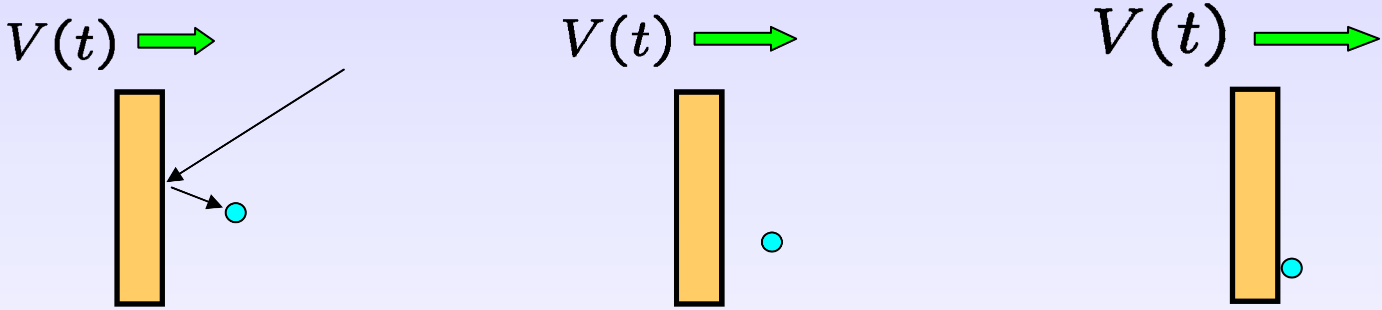
Steady motion ($V =$ const)



incident molecules

$= f_\infty$

Acceleration [$V(t)$: increasing]



incident molecules

$\neq f_\infty$

recollision

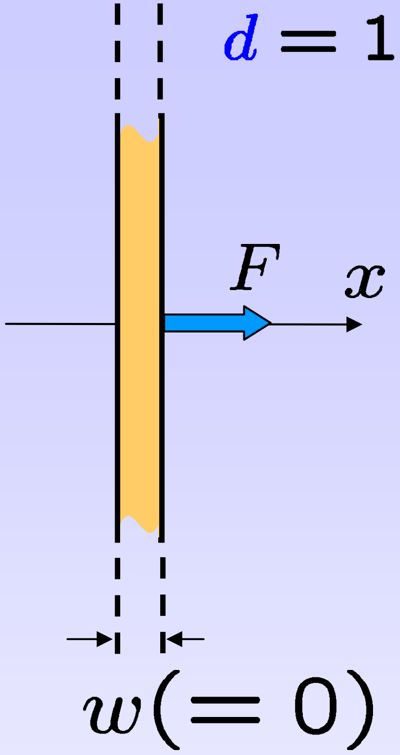
- If we neglect recollision, we obtain exponential approach.
- Diffuse reflection : more chances of recollision (more slow molecules)

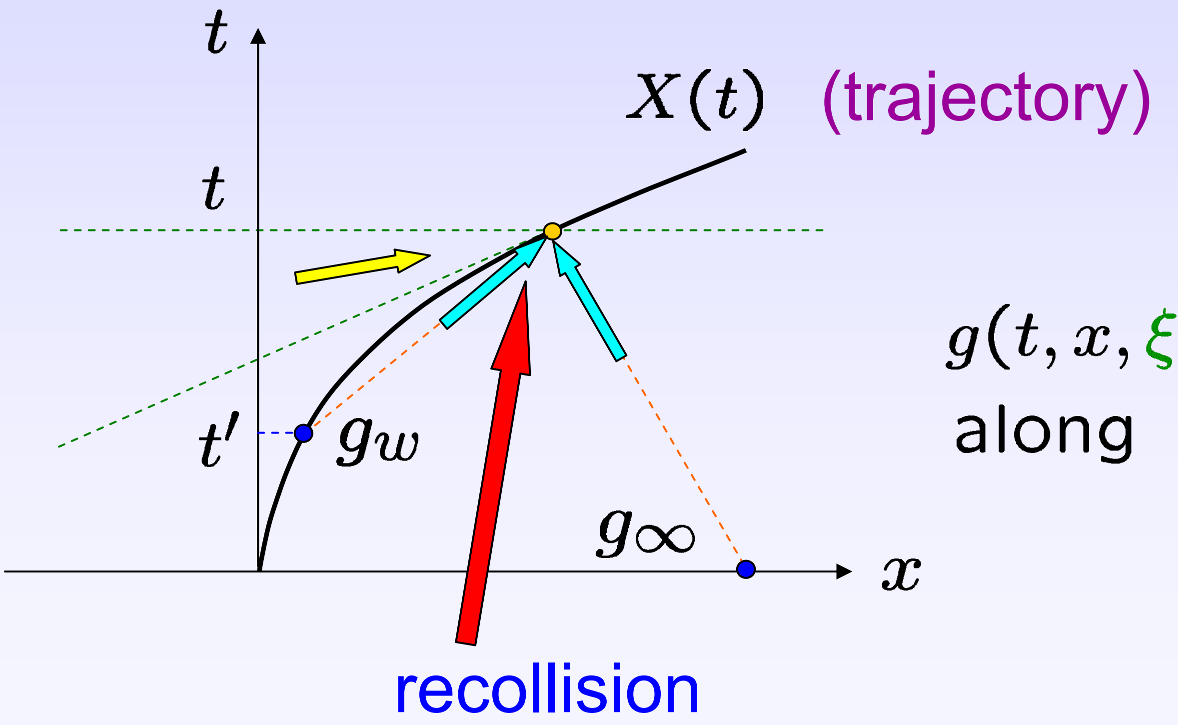
Illustration

1D ($d = 1$)

$$\frac{\partial g}{\partial t} + \xi_x \frac{\partial g}{\partial x} = 0$$

$$g(t, x, \xi_x) = \int\int f(t, x, \boldsymbol{\xi})\, d\xi_y d\xi_z$$





$g(t, x, \xi_x) = \text{const}$

along $x - \xi_x t = \text{const}$

(Rough) sketch of proof

Specular, 3D $(d = 3)$

Caprino, Marchioro, & Pulvirenti (06)

EQ: $\dot{X}(t) = V(t), \quad M\dot{V}(t) = F - D(t)$

IC: $X(0) = 0, \quad V(0) = V_0$

$$D(t) = \int_{\partial B} \int_{\xi \in \mathbf{R}^3} \xi_x \, [\, \boldsymbol{\xi} - V(t)\, \mathbf{e}_x \,] \cdot \mathbf{n} \, f \, d\boldsymbol{\xi} \, dS$$

Assumption: $V_\infty > V_0 > 0, \quad V_\infty - V_0$: small

$$D(t) = D_0(V(t)) + r^+(t) + r^-(t)$$

$D_0(V(t))$: Drag without recollision

$r^+(t)$: Correction (recollision on right face)

$r^-(t)$: Correction (recollision on left face)

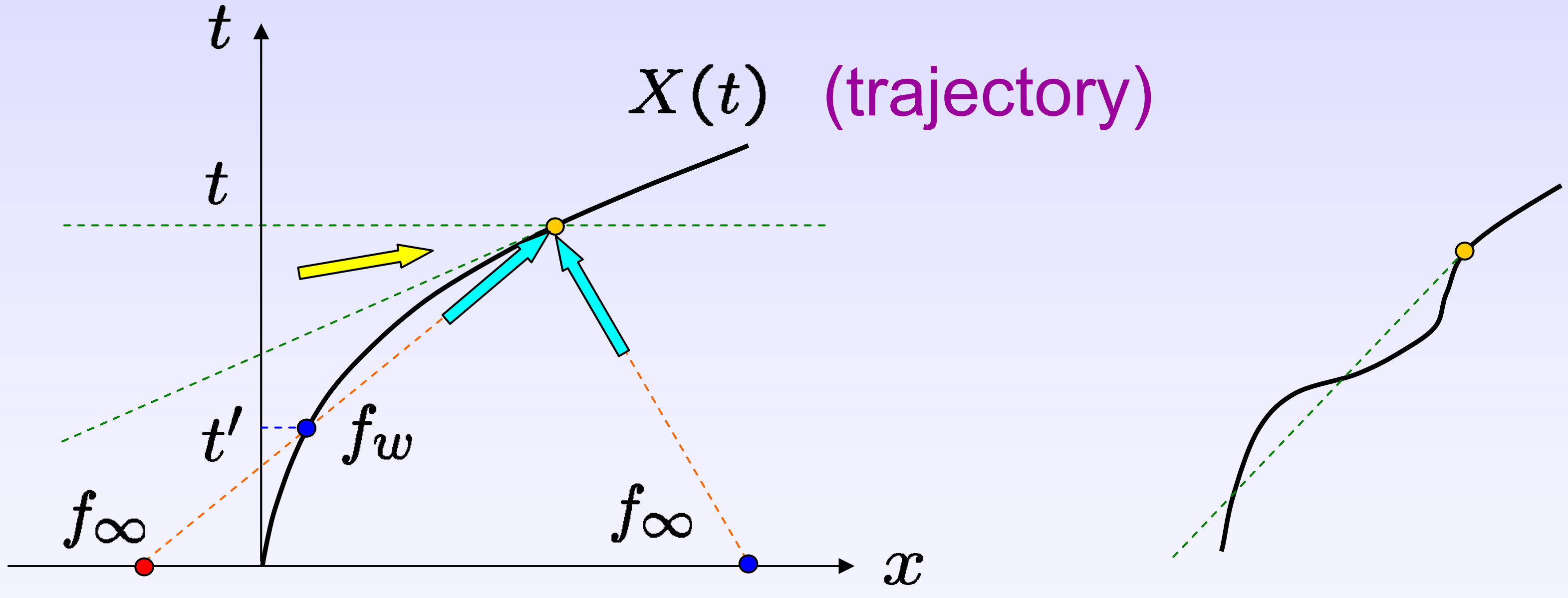
EQ: $\dot{X}(t) = V(t), \quad M\dot{V}(t) = F - D_0(V(t)) - r^+(t) - r^-(t)$

$$D(t) = D_0(V(t)) + r^+(t) + r^-(t)$$

$D_0(V(t))$: Drag without recollision

$r^+(t)$: Correction (recollision on right face)

$r^-(t)$: Correction (recollision on left face)



$$\dot{X}(t) = V(t), \qquad M\dot{V}(t) = F - D_0(V(t)) - r^+(t) - r^-(t)$$

- $V(t) \to W(t)$: given function

$$X_W(t) = \int_0^t W(\tau)d\tau, \quad f, \quad \underbrace{r_W^+(t), \quad r_W^-(t)} : \text{ known}$$

trajectory sol. of B eq. corrections

- Modified problem [for $V_W(t)$]

$$M\dot{V}_W(t) = F - D_0(V_W(t)) - r_W^+(t) - r_W^-(t)$$

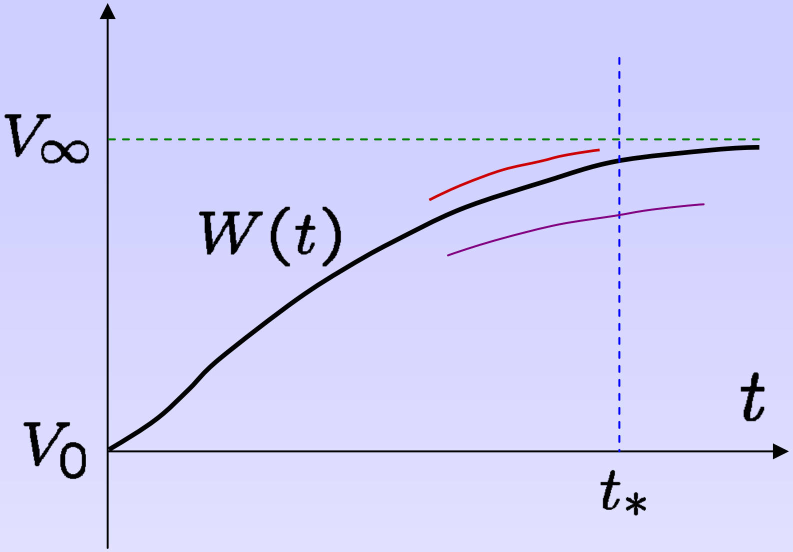
$$\Longrightarrow \quad V_W(t)$$

Map from $W(t)$ to $V_W(t)$: $V_W(t) = \mathcal{K}\,[W(t)]$

I. $\gamma = V_\infty - V_0$: small $(V_\infty > V_0 > 0)$

$W(t)$: Differentiable

Bouded derivative



(i) Increasing in $[0, t_*]$

$$t_* = \frac{1}{2C_-} \ln \frac{C_+}{\gamma} \text{ : large}$$

(ii) $$V_\infty - W(t) \geq e^{-C_- t}\gamma$$

$$V_\infty - W(t) \leq e^{-C_+ t}\gamma + \frac{A_+}{(1+t)^5}\gamma^3$$

⇒ $V_W(t)$ enjoys the same properties !!!

II. $V_{n+1}(t) = \mathcal{K}\left[V_n(t)\right]$ ⇒ $\{V_n(t)\}$ (equibounded equicontinuous)

$V_1(t) = W(t)$

$\{V_{n_m}(t)\} \to V(t)$: Limiting function

$(V(t),\ f)$ solves the original problem

III. $(V(t),\ f)$: any solution of the original problem

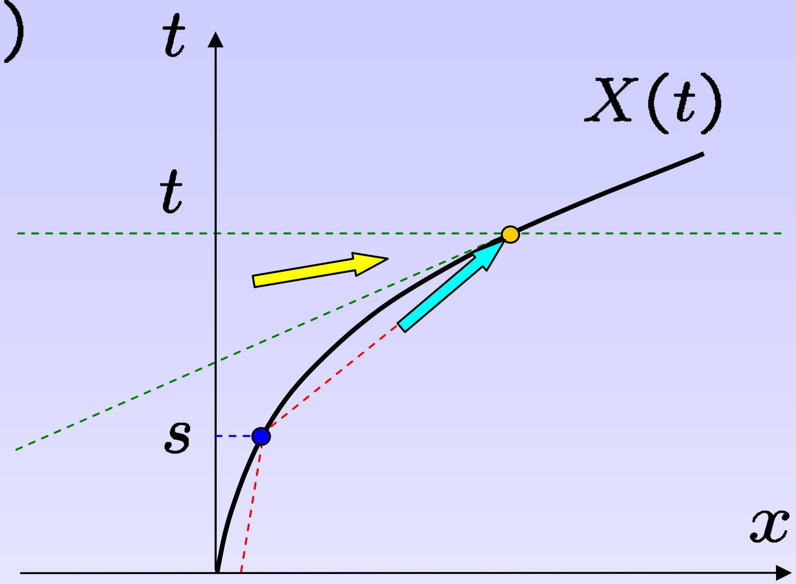
$V(t)$: Same properties as $W(t)$

$$V_\infty - V(t) \leq e^{-C_+ t}\gamma + \frac{A_+}{(1+t)^5}\gamma^3$$

$$V_\infty - V(t) \geq e^{-C_- t}\gamma$$

Improvement

$$\left(r^+(t) \geq C\frac{\gamma^4}{t^5}\right)$$



$$V_\infty - V(t) \geq e^{-C_- t}\gamma + \frac{A_-}{t^5}\gamma^4\chi(\{t > \bar{t}\}) \qquad (\bar{t} = 2t_*)$$

$$\boxed{|V(t) - V_\infty| \approx \frac{C}{t^{d+2}}}$$

$V_0 > V_\infty \geq 0,$ $V_0 - V_\infty$: small

Cavallaro, Caprino, & Marchioro (07)

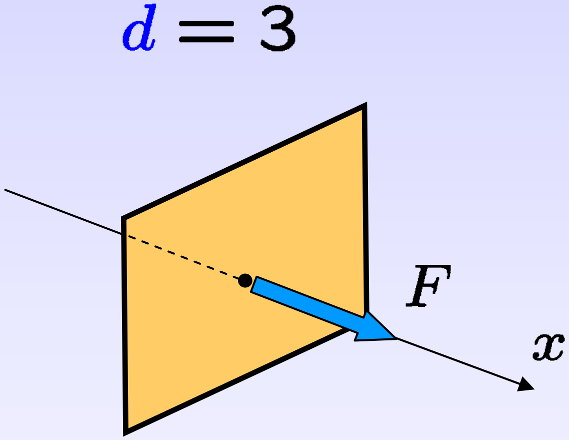
Numerical study

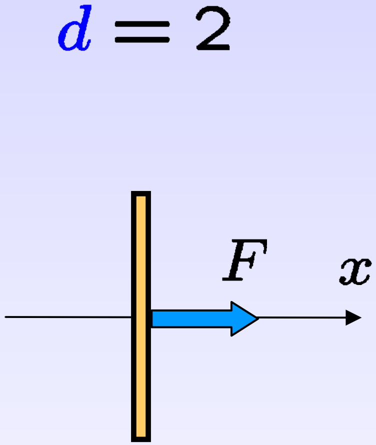
A, Tsuji, & Cavallaro, *Phys. Rev. E* (09)

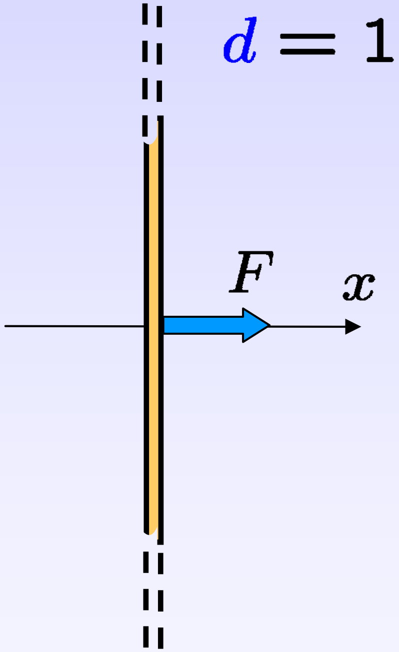
Arbitrary $|V_0 - V_\infty|$

Diffuse reflection (more difficult mathematically)

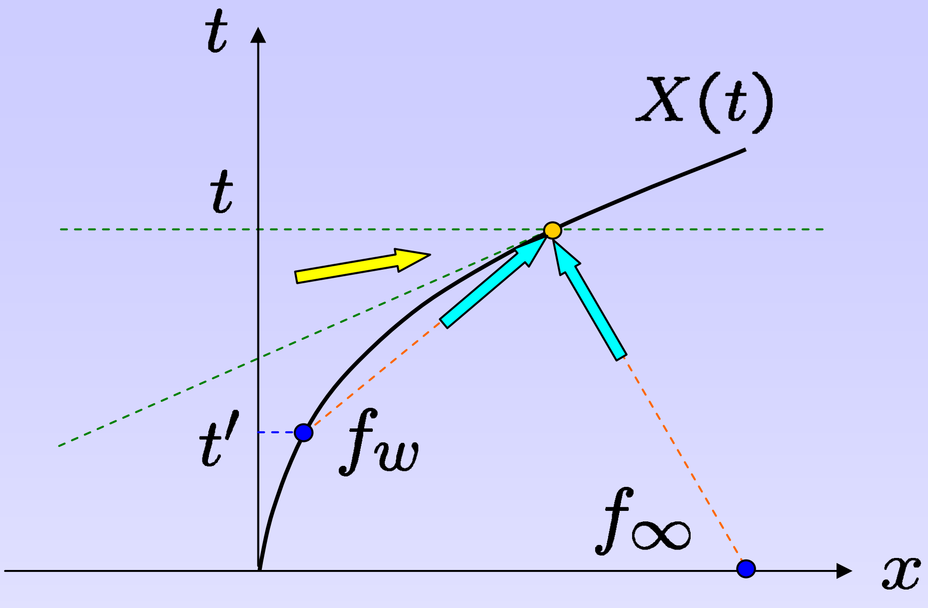
Plate without thickness $(T_w = T_\infty)$







Finite-difference method



Diffuse reflection

$\sigma_w(t'),\ V(t')$

(Macro variables)

$\Downarrow$

$f_w(t')$

$$f_w(\boldsymbol{\xi}) = \frac{\sigma_w}{(2\pi R T_w)^{3/2}} \exp\left(-\frac{|\boldsymbol{\xi} - \mathbf{v}_w|^2}{2RT_w} \right)$$

$$\left(\sigma_w = -\left(\frac{2\pi}{RT_w} \right)^{1/2} \int_{(\boldsymbol{\xi}-\mathbf{v}_w)\cdot\mathbf{n}<0} (\boldsymbol{\xi} - \mathbf{v}_w)\cdot\mathbf{n} f(\boldsymbol{\xi}) d\boldsymbol{\xi} \right)$$

Numerical results

Dimensionless variables

$V(t),\ X(t)$

$V_0,\ V_\infty$ → $u_w(t),\ x_w(t)$

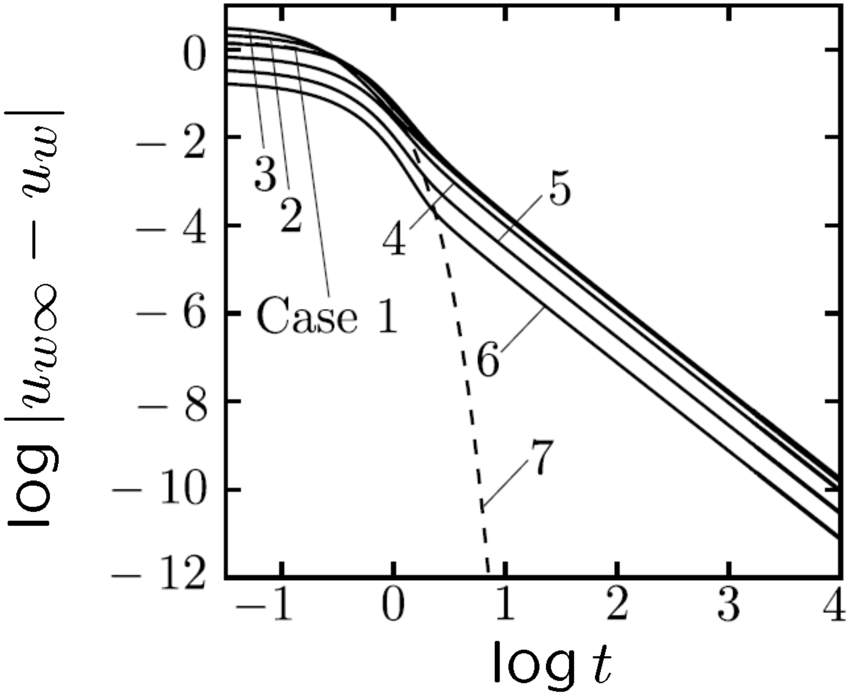
$u_{w0},\ u_{w\infty}$

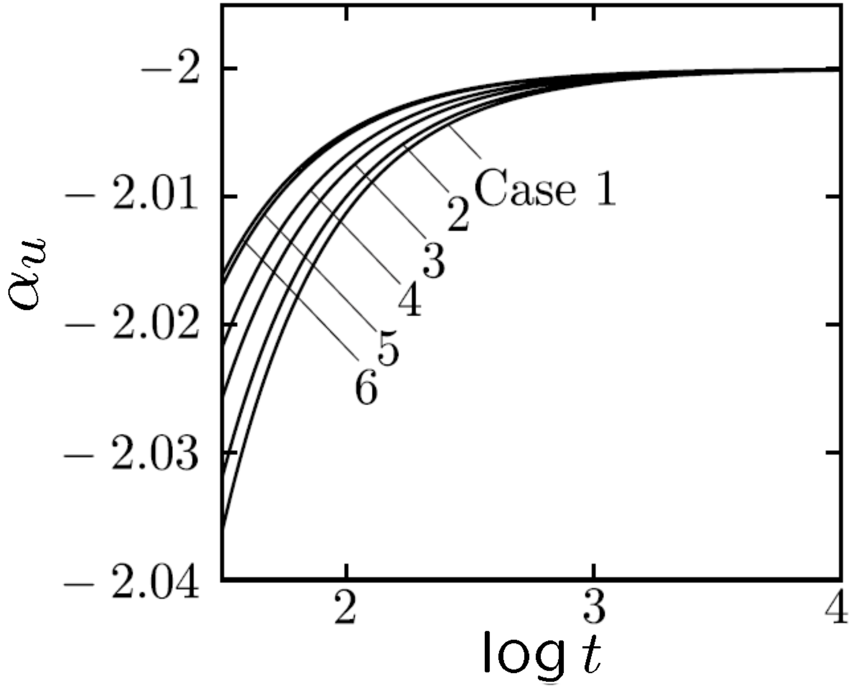
F ↗

M : Specified suitably

1D ($d = 1$)

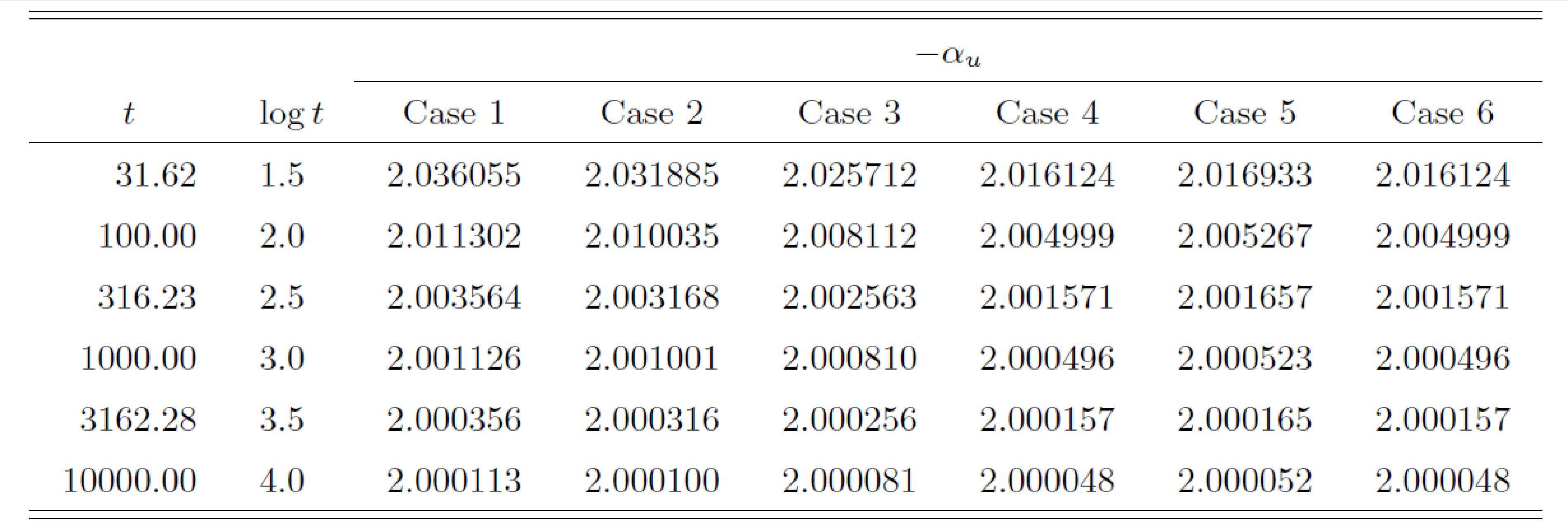
$\log(\cdot) = \log_{10}(\cdot)$





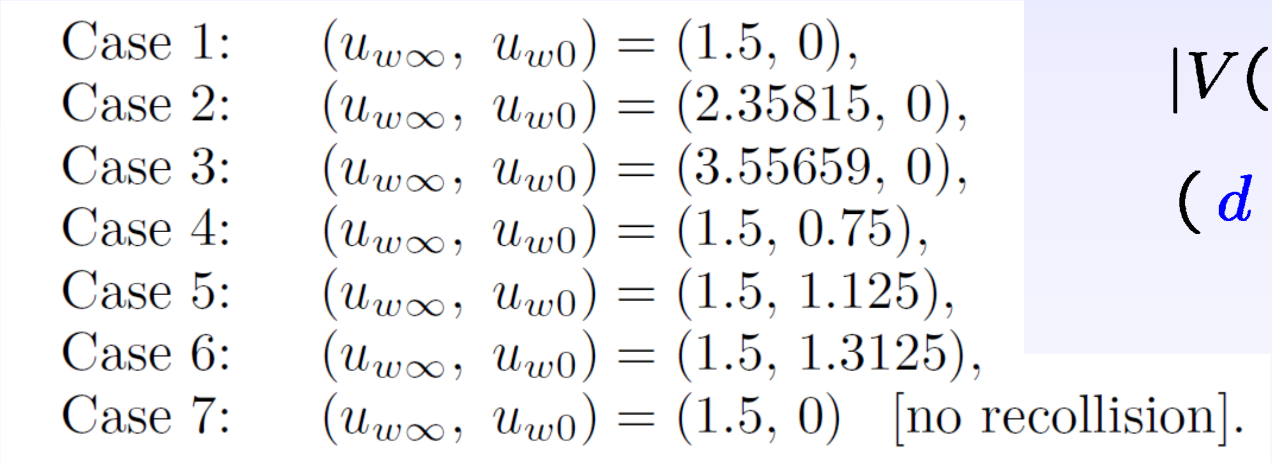
→ $|u_w(t) - u_{w\infty}| \approx \dfrac{C}{t^2}$

$$\alpha_u = \frac{d \log |u_{w\infty} - u_w|}{d \log t}$$



		$-\alpha_u$					
t	$\log t$	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
31.62	1.5	2.036055	2.031885	2.025712	2.016124	2.016933	2.016124
100.00	2.0	2.011302	2.010035	2.008112	2.004999	2.005267	2.004999
316.23	2.5	2.003564	2.003168	2.002563	2.001571	2.001657	2.001571
1000.00	3.0	2.001126	2.001001	2.000810	2.000496	2.000523	2.000496
3162.28	3.5	2.000356	2.000316	2.000256	2.000157	2.000165	2.000157
10000.00	4.0	2.000113	2.000100	2.000081	2.000048	2.000052	2.000048

$$V_\infty > V_0 \geq 0 \quad (u_{w\infty} > u_{w0} \geq 0)$$



Case 1: $(u_{w\infty},\ u_{w0}) = (1.5,\ 0)$,

Case 2: $(u_{w\infty},\ u_{w0}) = (2.35815,\ 0)$,

Case 3: $(u_{w\infty},\ u_{w0}) = (3.55659,\ 0)$,

Case 4: $(u_{w\infty},\ u_{w0}) = (1.5,\ 0.75)$,

Case 5: $(u_{w\infty},\ u_{w0}) = (1.5,\ 1.125)$,

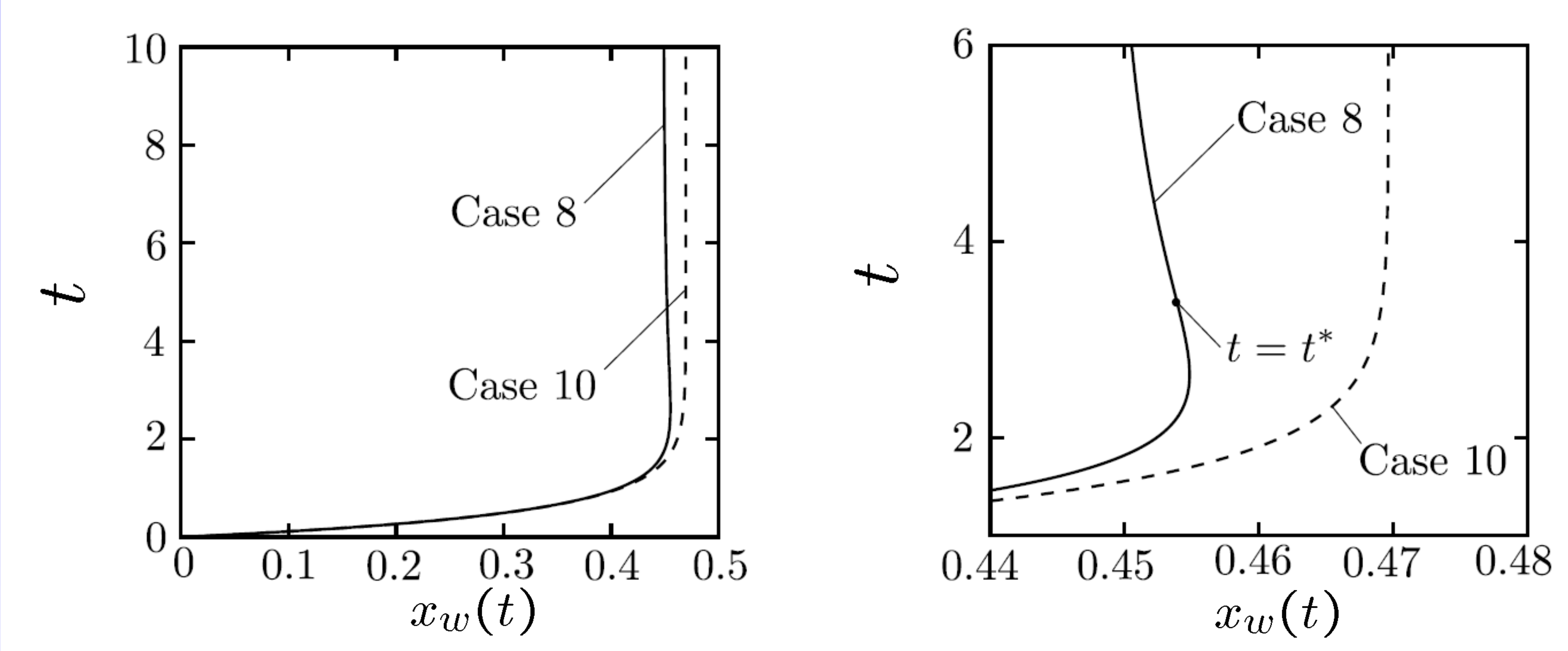
Case 6: $(u_{w\infty},\ u_{w0}) = (1.5,\ 1.3125)$,

Case 7: $(u_{w\infty},\ u_{w0}) = (1.5,\ 0)$ [no recollision].

$$|V(t) - V_\infty| \approx \frac{C}{t^{d+1}}$$

$$(\, d = 1 \,)$$

$V_0 > V_\infty \geq 0 \quad (u_{w0} > u_{w\infty} \geq 0)$ Overshoot !



Case 8: $(u_{w\infty},\ u_{w0}) = (0,\ 1)$,

Case 9: $(u_{w\infty},\ u_{w0}) = (1.5,\ 6)$,

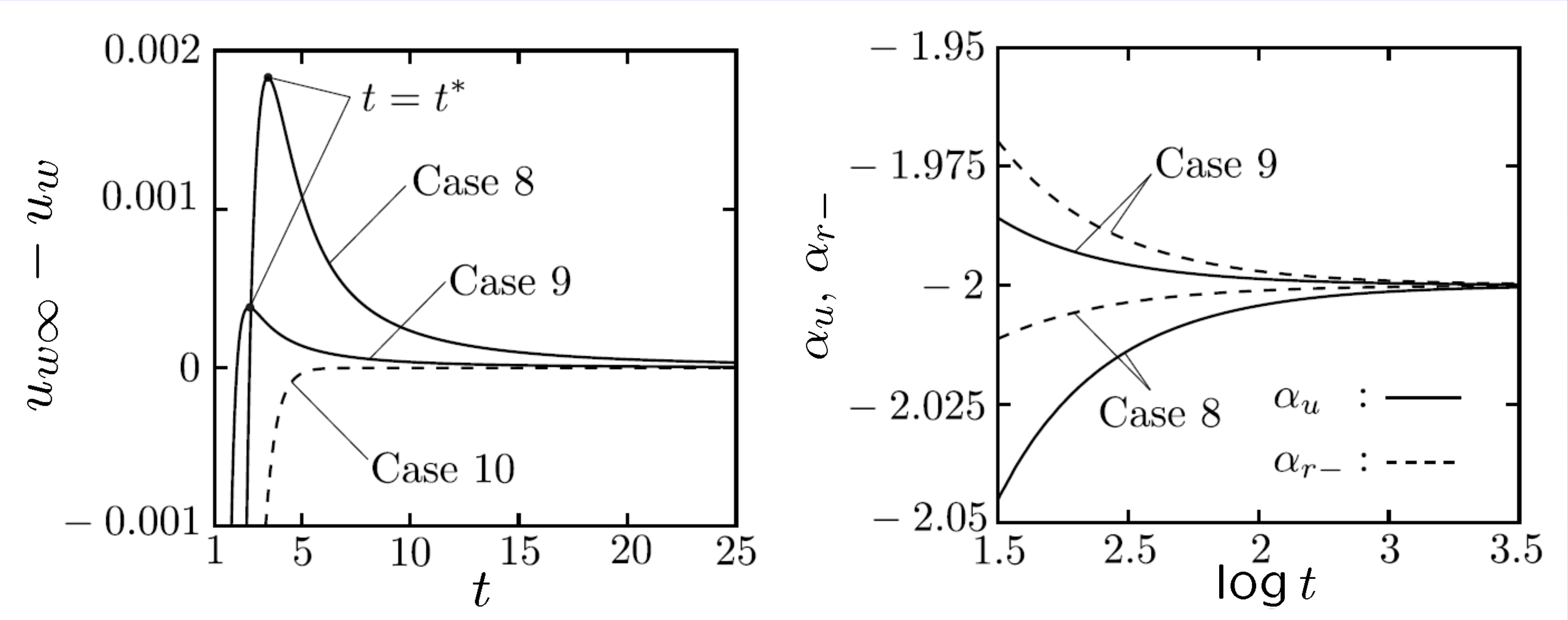
Case 10: $(u_{w\infty},\ u_{w0}) = (0,\ 1)$ [no recollision].

Overshoot is proven in

Cavallaro, Caprino, & Marchioro (07)

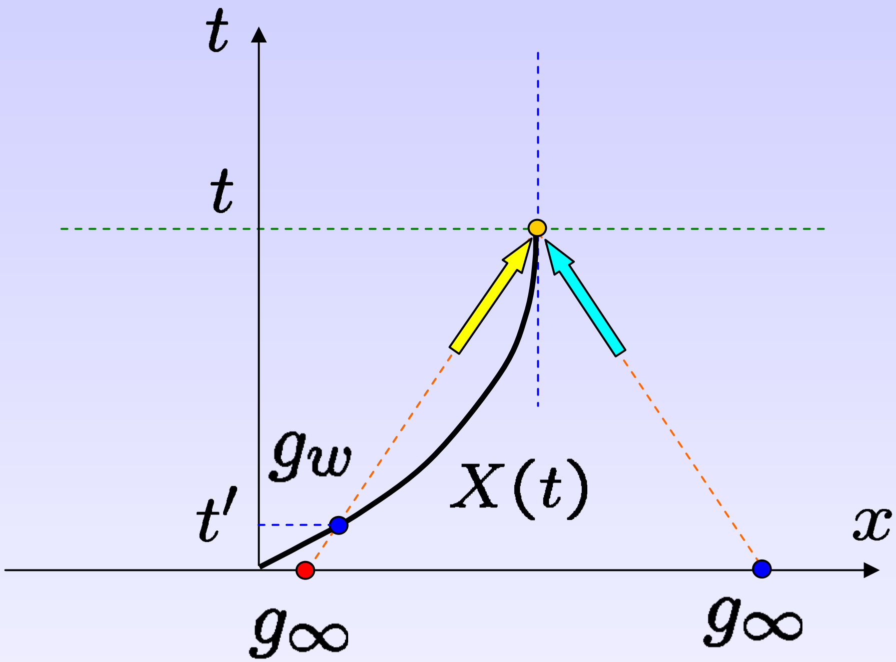
Specular

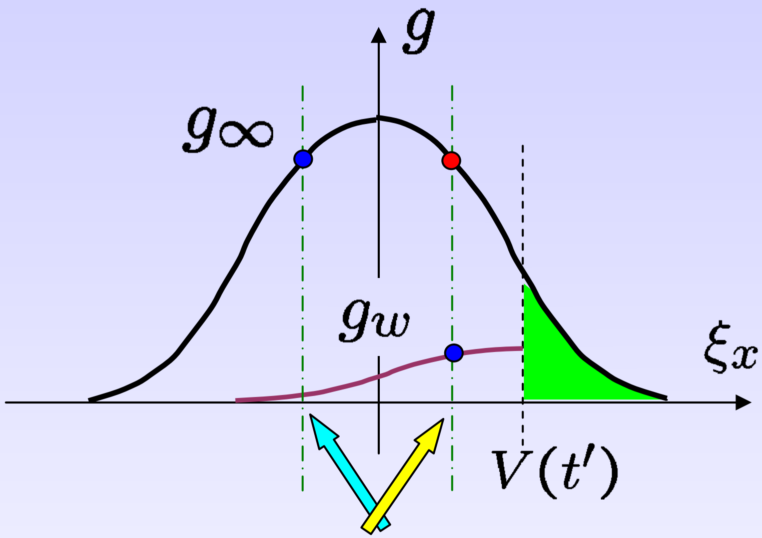
$V_0 - V_\infty$: small



$$\alpha_u = \frac{d\log|u_{w\infty} - u_w|}{d\log t}$$

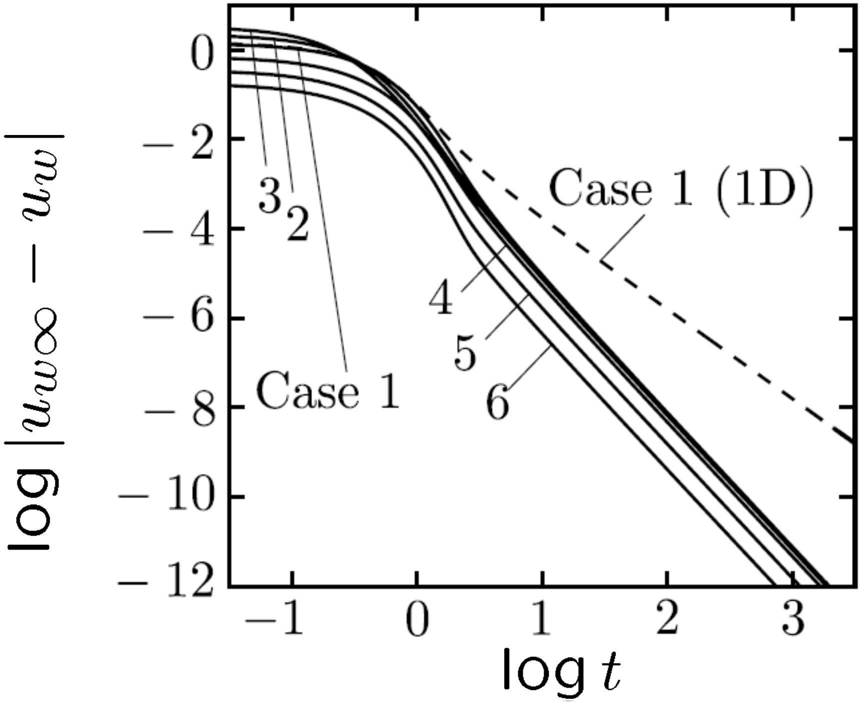
Overshoot (physical explanation)

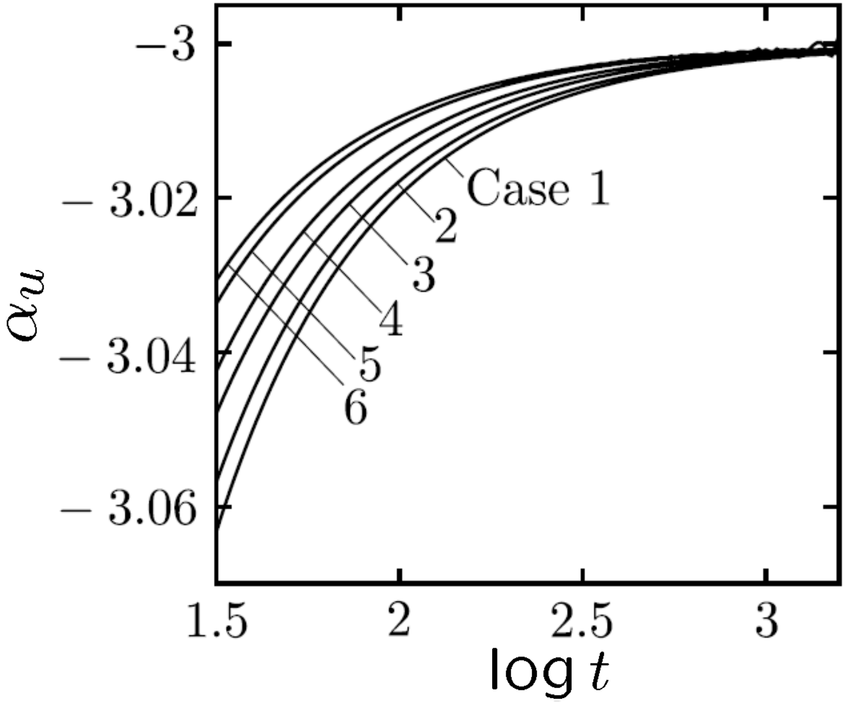




$$(T_w = T_\infty)$$

2D ($d = 2$)





$$\alpha_u = \frac{d \log |u_{w\infty} - u_w|}{d \log t}$$

$$\Longrightarrow \quad |u_w(t) - u_{w\infty}| \approx \frac{C}{t^3}$$

		$-\alpha_u$					
t	$\log t$	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
31.62	1.5	3.063329	3.056900	3.048011	3.042502	3.033773	3.030717
100.00	2.0	3.019708	3.017834	3.015121	3.013234	3.010493	3.009530
316.23	2.5	3.006203	3.005624	3.004769	3.004169	3.003297	3.003054
1000.00	3.0	3.001961	3.001778	3.001516	3.001286	3.000979	3.000809

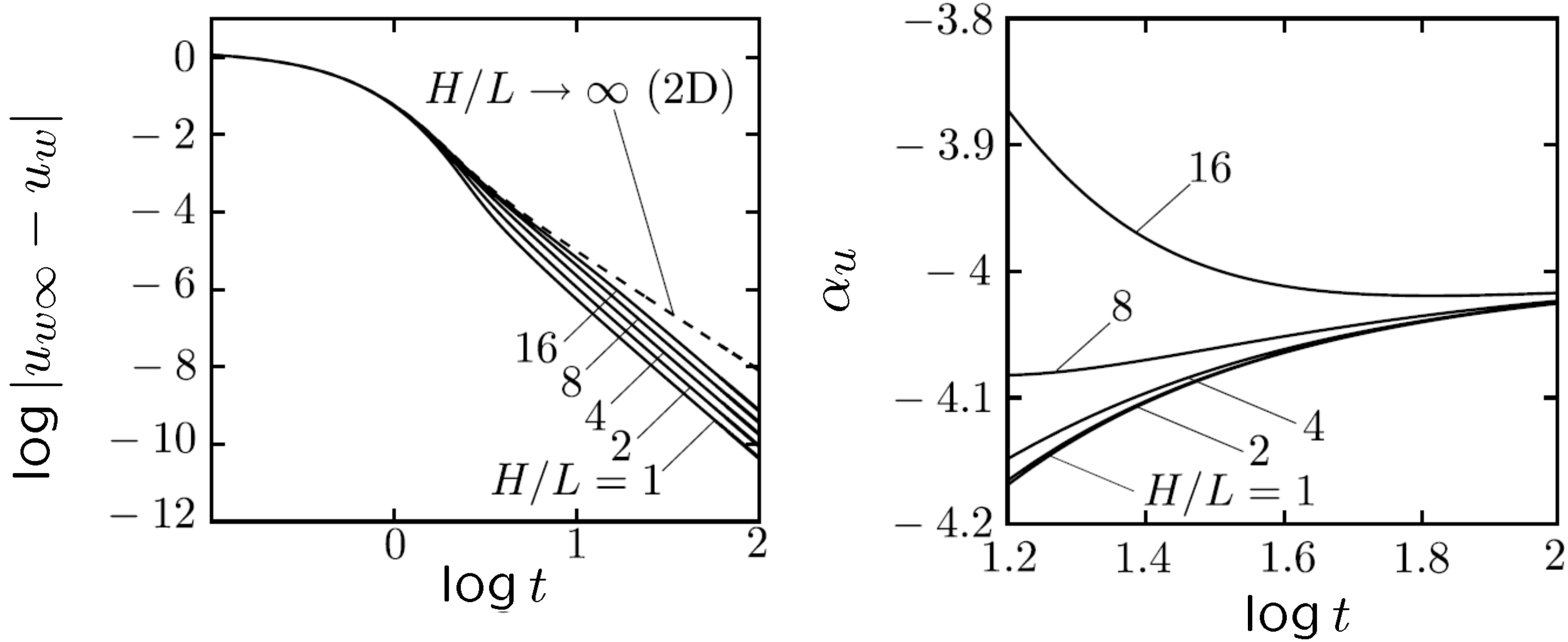
$$V_\infty > V_0 \geq 0 \quad (u_{w\infty} > u_{w0} \geq 0)$$

Case 1: $(u_{w\infty},\ u_{w0}) = (1.5,\ 0)$,
Case 2: $(u_{w\infty},\ u_{w0}) = (2.35815,\ 0)$,
Case 3: $(u_{w\infty},\ u_{w0}) = (3.55659,\ 0)$,
Case 4: $(u_{w\infty},\ u_{w0}) = (1.5,\ 0.75)$,
Case 5: $(u_{w\infty},\ u_{w0}) = (1.5,\ 1.125)$,
Case 6: $(u_{w\infty},\ u_{w0}) = (1.5,\ 1.3125)$.

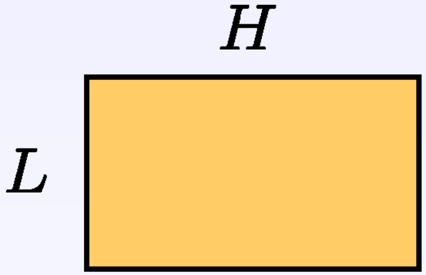
$$|V(t) - V_\infty| \approx \frac{C}{t^{d+1}}$$

$(\ d = 2\ )$

3D（$d = 3$）



Case 1: $(u_{w\infty},\ u_{w0}) = (1.5,\ 0)$

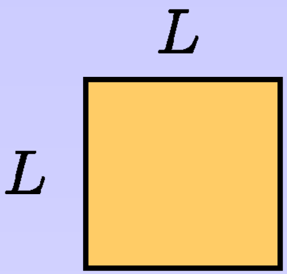


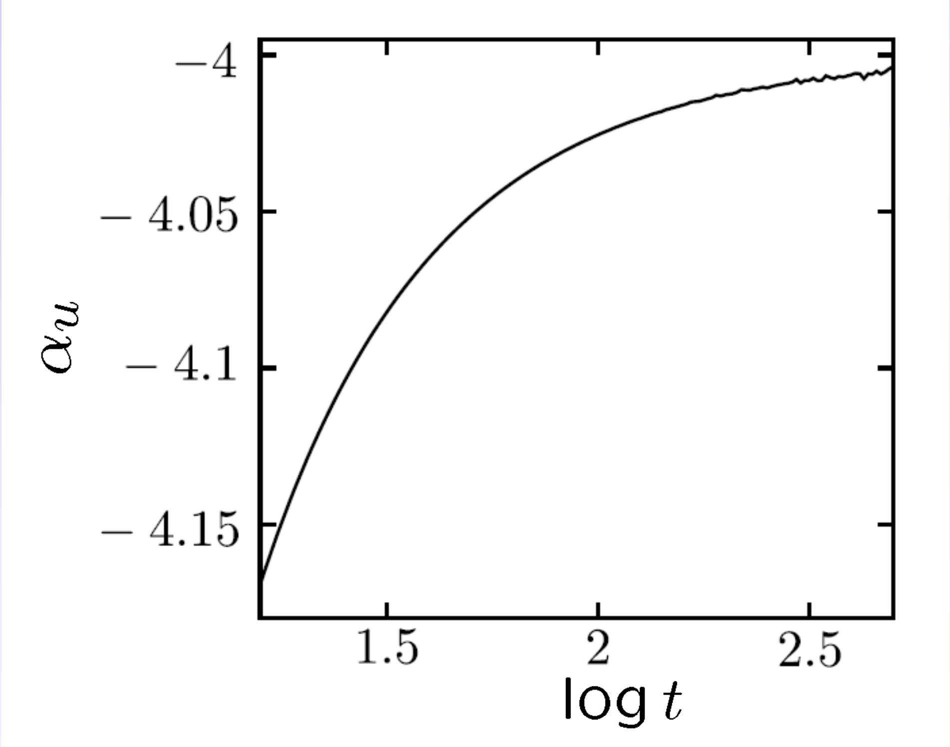
$$\alpha_u = \frac{d \log|u_{w\infty} - u_w|}{d \log t}$$

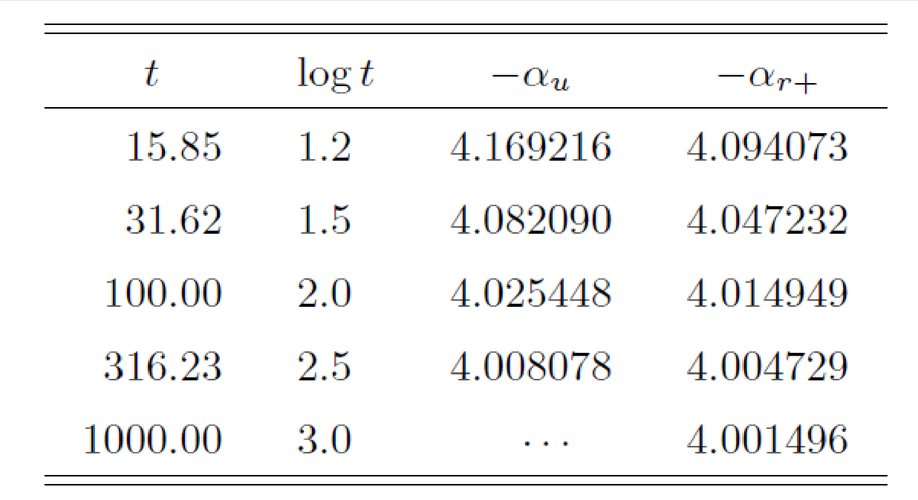
$$\Rightarrow \quad |u_w(t) - u_{w\infty}| \approx \frac{C}{t^4}$$

Case 1: $(u_{w\infty},\ u_{w0}) = (1.5,\ 0)$

Square plate







t	$\log t$	$-\alpha_u$	$-\alpha_{r+}$
15.85	1.2	4.169216	4.094073
31.62	1.5	4.082090	4.047232
100.00	2.0	4.025448	4.014949
316.23	2.5	4.008078	4.004729
1000.00	3.0	$\cdots$	4.001496

$$|V(t) - V_\infty| \approx \frac{C}{t^{d+1}}$$

$$(\ d = 3\ )$$

$$\alpha_u = \frac{d \log |u_{w\infty} - u_w|}{d \log t}$$

Approach to V_∞

BC: specular reflection $$|V(t) - V_\infty| \approx \frac{C}{t^{d+2}}$$

d : dimension

Caprino, Marchioro, & Pulvirenti, *Commun. Math. Phys.* (06)
Caprino, Cavallaro, & Marchioro, *M*^{3}*AS* (07)
Cavallaro, *Rend. Mat.* (07)

BC: diffuse reflection $$|V(t) - V_\infty| \approx \frac{C}{t^{d+1}}$$

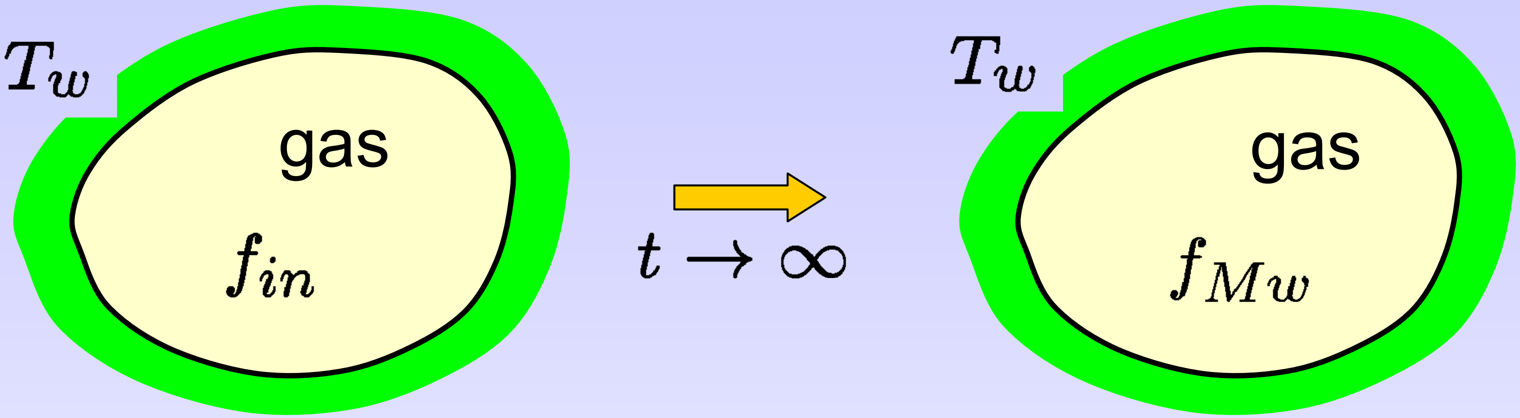
A, Cavallaro, Marchioro, & Pulvirenti, *M*^{2}*NA* (08)

Condition: ~~$|V_0 - V_\infty|$ small~~

Numerical evidence

Approach to equilibrium of a free-molecular gas

Trend to equilibrium



$$f_{Mw} = \frac{\rho_0}{(2\pi R T_w)^{3/2}} \exp\left(-\frac{|\boldsymbol{\xi}|^2}{2RT_w}\right)$$

$$\left(\int_D \int_{\mathbf{R}^3} f_{Mw} d\boldsymbol{\xi} d\mathbf{x} = \int_D \int_{\mathbf{R}^3} f_{in} d\boldsymbol{\xi} d\mathbf{x} \right)$$

Boltzmann equation (with collisions)

Grad, Cercignani, Illner, Arkeryd, Bobylev, Toscani, …
Villani, Mouhot, Desvillettes, Wennberg, Carlen, …
Guo

- Specularly (or backwardly) reflecting boundary
- Periodic box

Desvillettes & Villani, *Invent. Math.* (04)

$$f \to f_{M\infty} : \quad O(t^{-\kappa}) \quad (\forall \kappa > 0)$$

$$f_{M\infty} = \frac{\rho_0}{(2\pi R T_\infty)^{3/2}} \exp\left(- \frac{|\boldsymbol{\xi}|^2}{2RT_\infty} \right)$$

$$\left(\int_D \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ |\boldsymbol{\xi}|^2 \end{pmatrix} f_{M\infty} d\boldsymbol{\xi} d\mathbf{x} = \int_D \int_{\mathbf{R}^3} \begin{pmatrix} 1 \\ |\boldsymbol{\xi}|^2 \end{pmatrix} f_{in} d\boldsymbol{\xi} d\mathbf{x} \right)$$

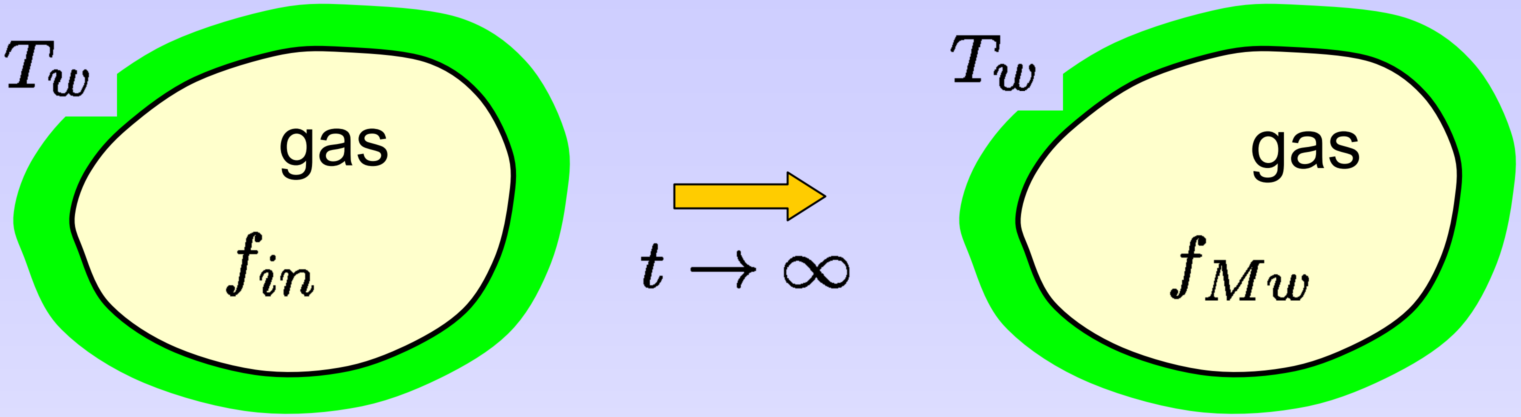
$$H(f|f_{M\infty}) = \int_D \int_{\mathbf{R}^3} f \ln\left(\frac{f}{f_{M\infty}} \right) d\boldsymbol{\xi} d\mathbf{x} < C_\kappa t^{-\kappa}$$

Diffuse reflection

Villani (07?), Guo (09?)

$$f \to f_{Mw} : \quad O(t^{-\kappa}) \quad (\forall \kappa > 0)$$

Free-molecular gas (Diffuse reflection)



$$f_{Mw} = \frac{\rho_0}{(2\pi R T_w)^{3/2}} \exp\left(-\frac{|\boldsymbol{\xi}|^2}{2RT_w} \right)$$

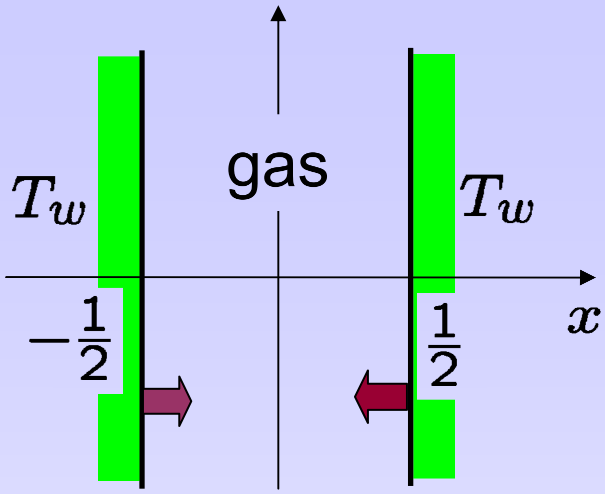
Arkeryd & Nouri, *Mh. Math.* (97)

Slow approach is expected.

$f \to f_{Mw}$: $O(t^{-d})$ (d : dimension of box) guess

1D box ($d = 1$)

Slab



EQ: $\dfrac{\partial f}{\partial t} + \xi_x \dfrac{\partial f}{\partial x} = 0$

IC: $f(0, x, \boldsymbol{\xi}) = f_{in}(x, \boldsymbol{\xi})$

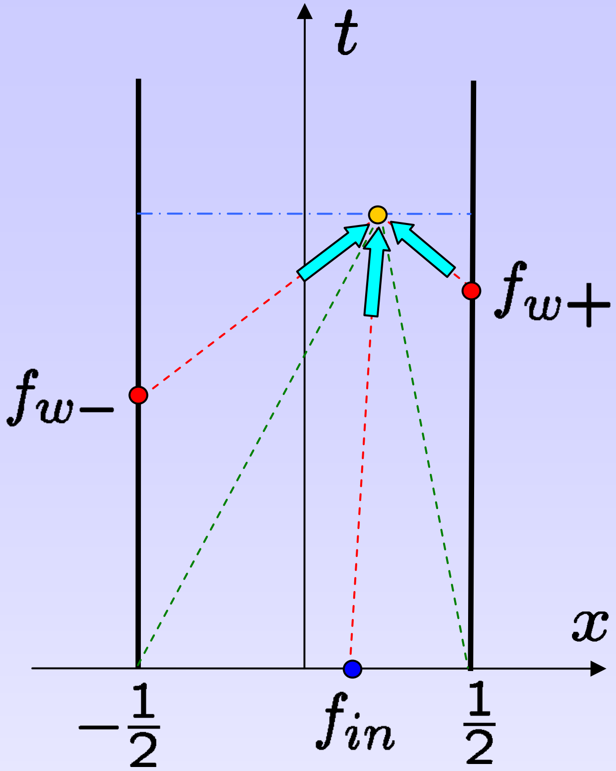
BC: $f = f_{w\pm}, \quad \left(x = \pm\dfrac{1}{2}, \quad \mp\xi_x > 0\right)$

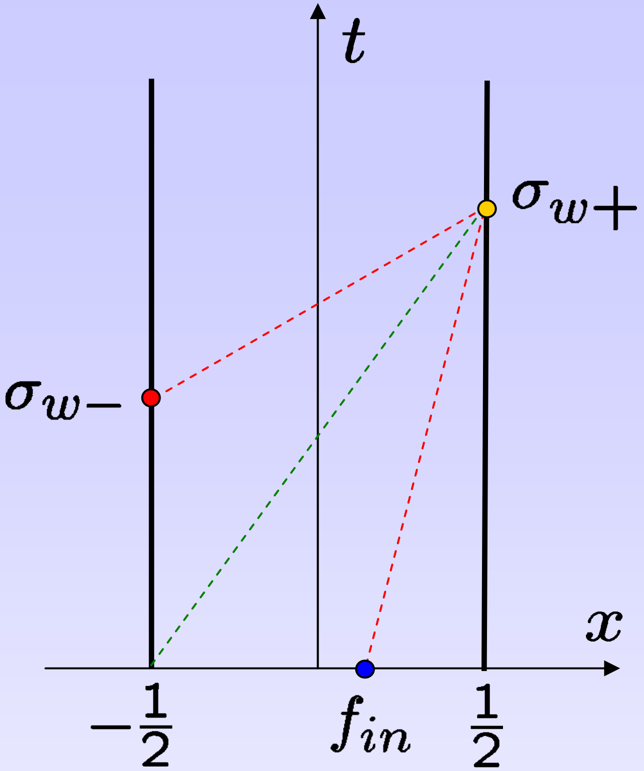
$$f_{w\pm}(t, \boldsymbol{\xi}) = \pi^{-3/2}\sigma_{w\pm}(t)\,\mathrm{exp}(-|\boldsymbol{\xi}|^2)$$

$$\sigma_{w\pm}(t) = \pm 2\sqrt{\pi}\int\int\int_{\pm\xi_x>0} \xi_x f\left(t, \pm\frac{1}{2}, \boldsymbol{\xi}\right) d\xi_x d\xi_y d\xi_z$$

Formal solution

$$f(t, x, \boldsymbol{\xi}) = \begin{cases} f_{in}(x - \xi_x t, \boldsymbol{\xi}), & \left[\frac{1}{t}\left(x - \frac{1}{2}\right) \le \xi_x \le \frac{1}{t}\left(x + \frac{1}{2}\right)\right] \\ f_{w\pm}\left(t - \frac{1}{\xi_x}\left(x \mp \frac{1}{2}\right), \boldsymbol{\xi}\right), & [\text{otherwise}] \end{cases}$$





Integral equation for $\sigma_{w\pm}$

$$\sigma_{w\pm}(t) = M_{\pm} + \int_0^t r(t-s)\,\sigma_{w\mp}(s)ds$$

$$M_{\pm} = 2\sqrt{\pi}\int\int\int_0^{1/t} \xi_x\, f_{in}\left(\pm\left(\frac{1}{2}-\xi_x t\right),\ \pm\xi_x, \xi_y, \xi_z\right)\ d\xi_x d\xi_y d\xi_z$$

$$r(t) = \frac{2}{t^3}\exp(-1/t^2)$$

Symmetric initial condition:

$$f_{in}(x,\, \xi_x,\, \xi_y,\, \xi_z) = f_{in}(-x,\, -\xi_x,\, \xi_y,\, \xi_z)$$

$$\sigma_{w+}(t) = \sigma_{w-}(t) = \sigma_w(t)$$

$$\sigma_w(t) = M(t) + \int_0^t r(t-s)\, \sigma_w(s) ds$$

Renewal equation

$$M(t) = 2\sqrt{\pi} \int \int \int_0^{1/t} \xi_x\, f_{in}\left(\frac{1}{2} - \xi_x\, t,\, \boldsymbol{\xi}\right)\, d\xi_x$$

$$r(t) = \frac{2}{t^3} \exp(-1/t^2)$$

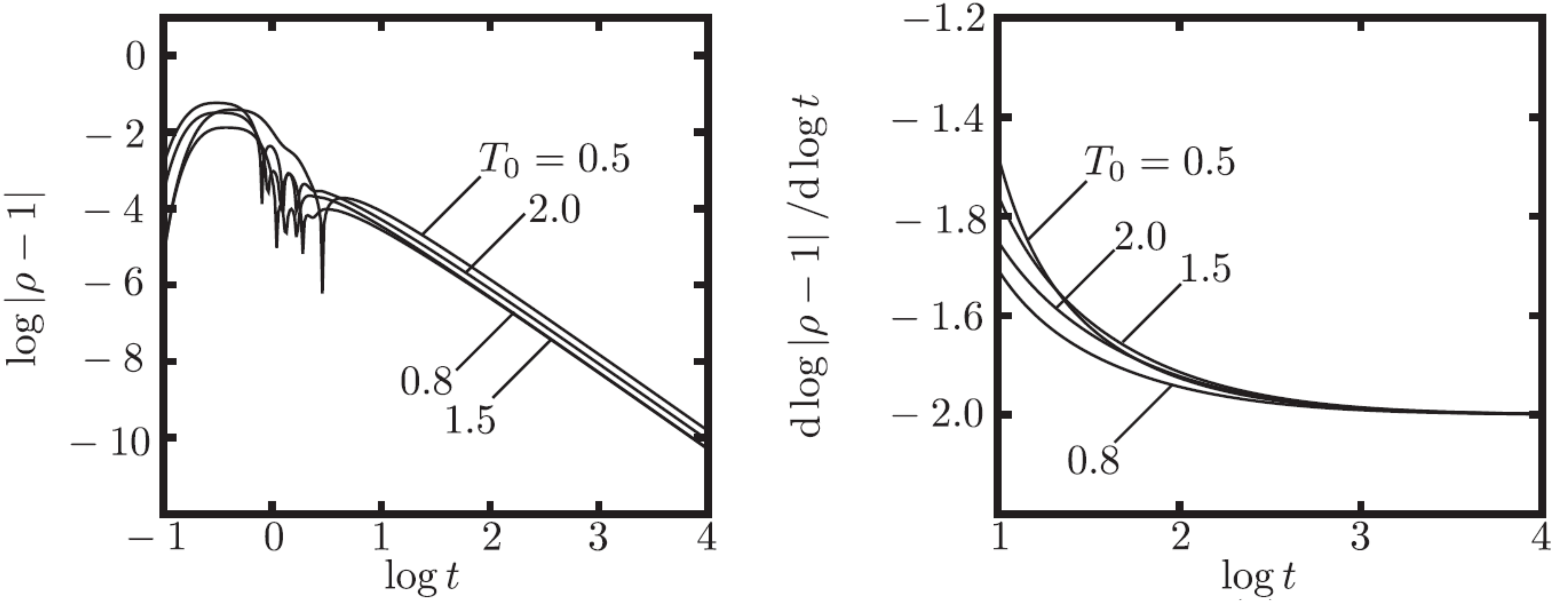
$\sigma_w(t)$ $\Longrightarrow$ $f(t,\, x,\, \boldsymbol{\xi})$ $\Longrightarrow$ Macroscopic quantities

---

Numerical result
(preliminary)

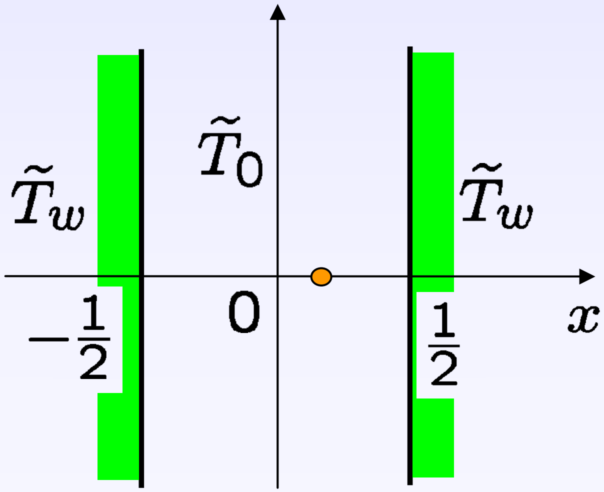
$$f_{in} = \frac{1}{(\pi T_0)^{3/2}} \exp\left(-\frac{|\boldsymbol{\xi}|^2}{T_0}\right)$$

$$(\, T_0 = \tilde{T}_0/\tilde{T}_w = 0.5,\, 0.8,\, 1.5,\, 2\,)$$

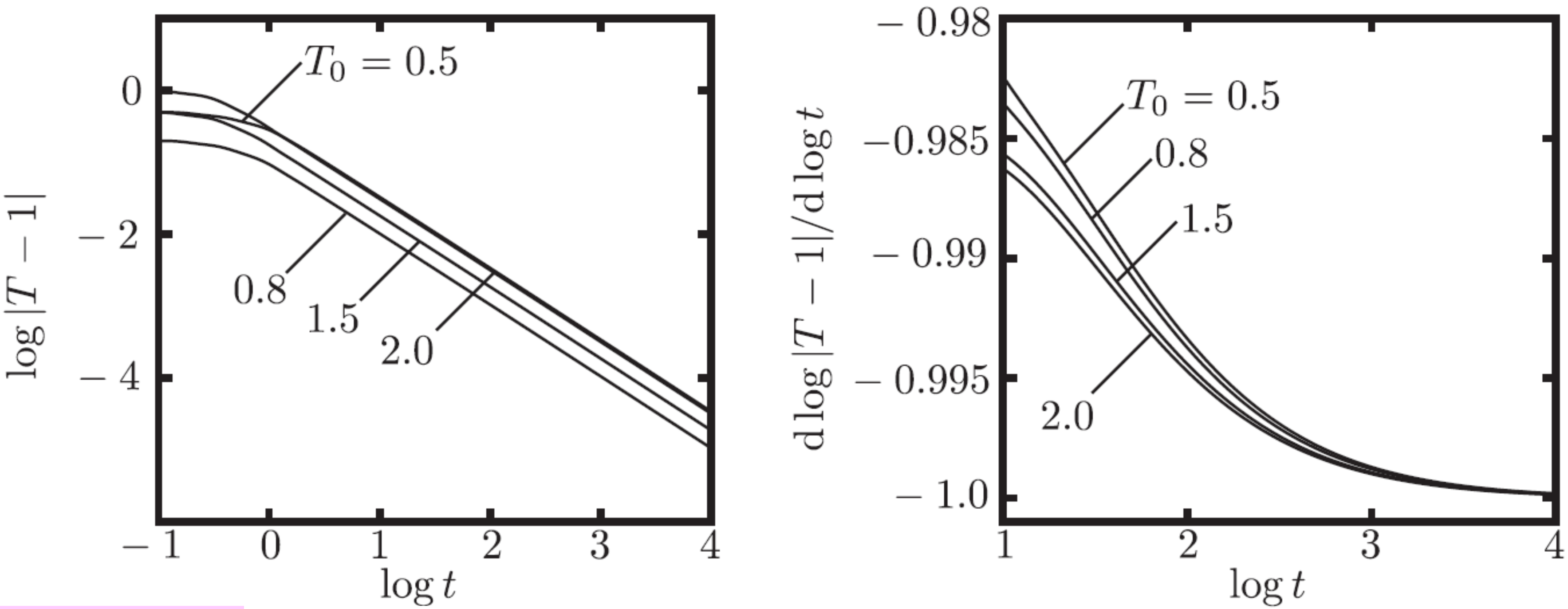


$x = 0.2$

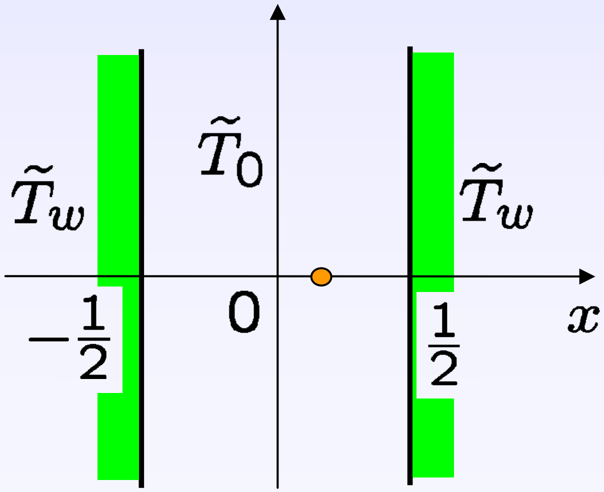
$\color{blue}{T_0} = \tilde{T}_0/\tilde{T}_w = 0.5,\ 0.8,\ 1.5,\ 2$



$$|\rho - 1| \approx \frac{C}{t^2}$$

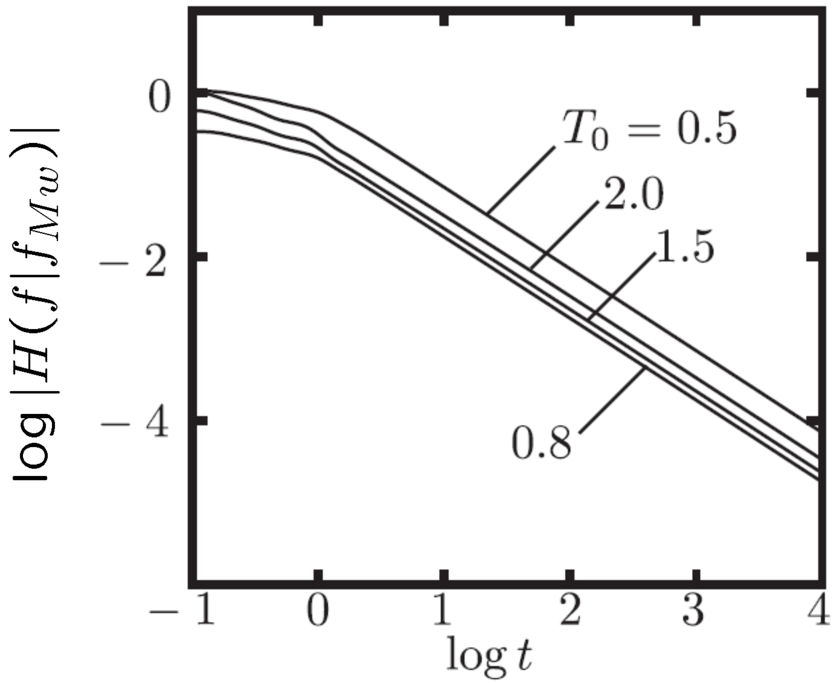


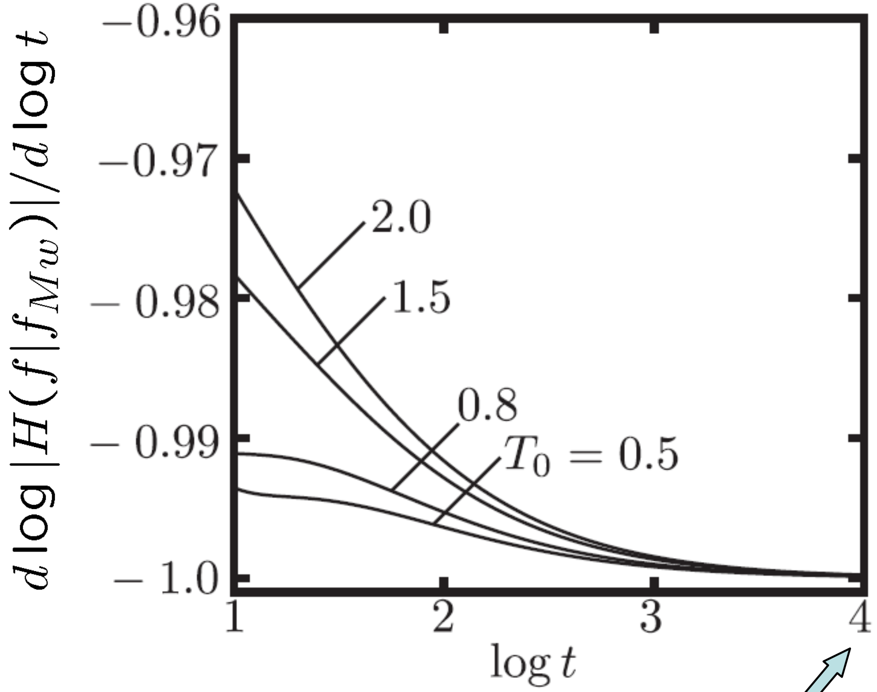
$x = 0.2$



$$T_0 = \tilde{T}_0/\tilde{T}_w = 0.5,\ 0.8,\ 1.5,\ 2$$

$$|T-1| \approx \frac{C}{t}$$

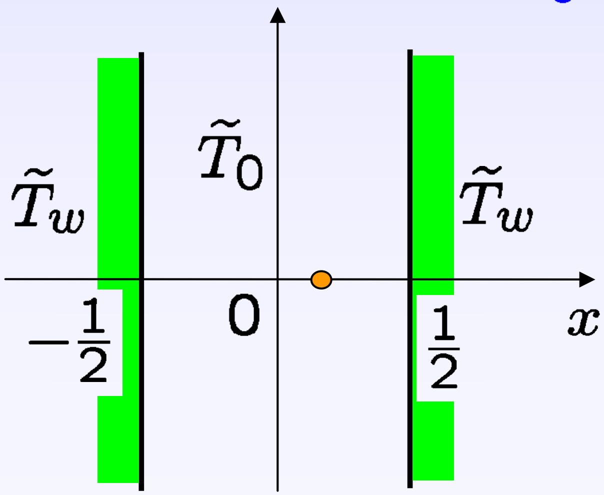




$x = 0.2$

$T_0 = \tilde{T}_0/\tilde{T}_w = 0.5,\ 0.8,\ 1.5,\ 2$

$t = 10^4$



$$|H(f|f_{Mw})| \approx \frac{C}{t}$$

$$H(f|f_{Mw}) = \int_{\mathbf{R}^3} f \ln f \, d\boldsymbol{\xi} - \int_{\mathbf{R}^3} f_{Mw} \ln f_{Mw} d\boldsymbol{\xi}$$

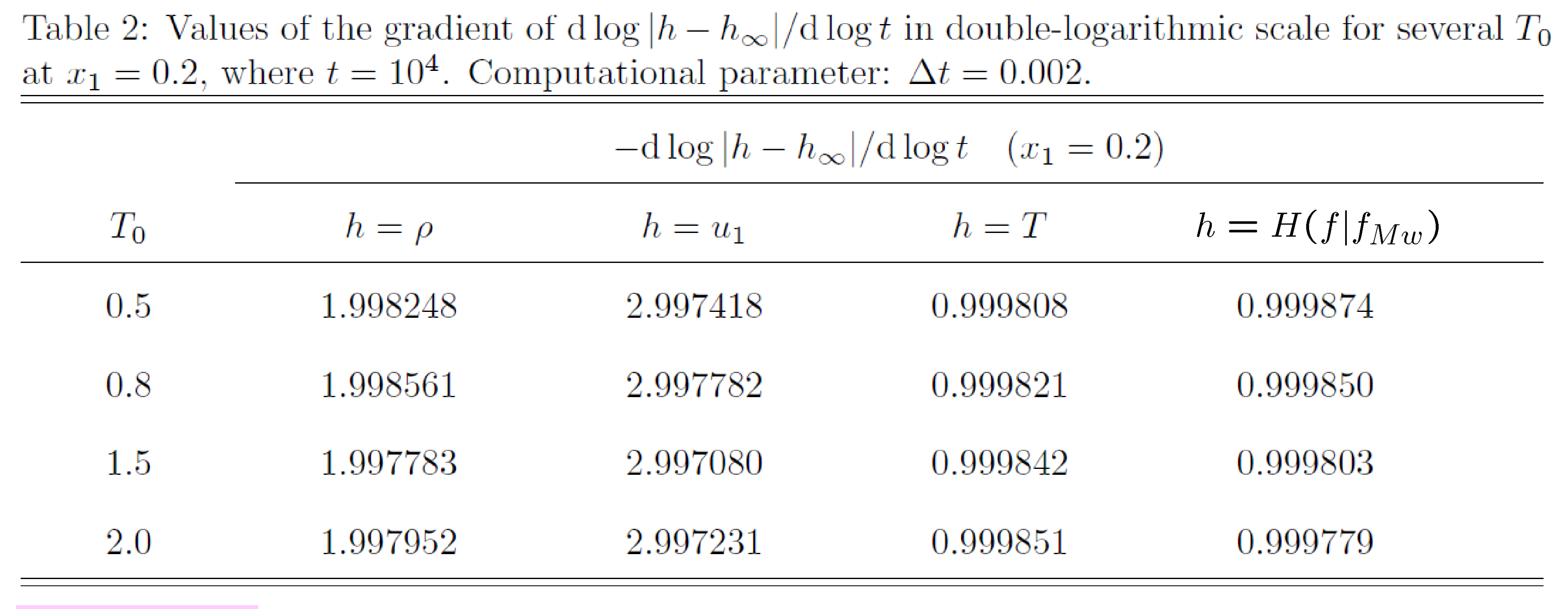


Table 2: Values of the gradient of $\mathrm{d}\log|h-h_\infty|/\mathrm{d}\log t$ in double-logarithmic scale for several T_0 at $x_1=0.2$, where $t=10^4$. Computational parameter: $\Delta t=0.002$.

	$-\mathrm{d}\log\lvert h-h_\infty\rvert/\mathrm{d}\log t \quad (x_1=0.2)$			
T_0	$h=\rho$	$h=u_1$	$h=T$	$h=H(f\lvert f_{Mw})$
0.5	1.998248	2.997418	0.999808	0.999874
0.8	1.998561	2.997782	0.999821	0.999850
1.5	1.997783	2.997080	0.999842	0.999803
2.0	1.997952	2.997231	0.999851	0.999779

$x=0.2$

Other points

$t=10^4$

$$f\to f_{Mw}:\quad O(t^{-d})\quad (d:\ \text{dimension of box})$$

$$d=1$$

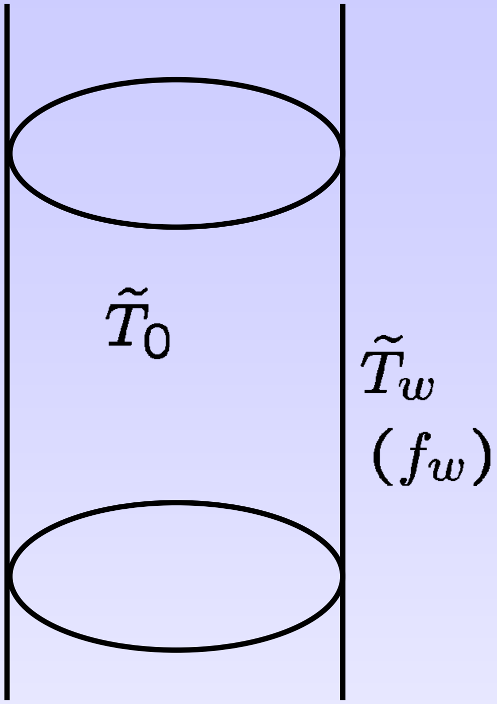
2D box ($d = 2$)

Cylinder

Circular cylinder, Cylindrical symmetry

$$f_{in} = \frac{1}{(\pi T_0)^{3/2}} \exp\left(-\frac{|\boldsymbol{\xi}|^2}{T_0}\right)$$

$$(T_0 = \tilde{T}_0/\tilde{T}_w = 0.5,\ 0.8,\ 1.5,\ 2)$$



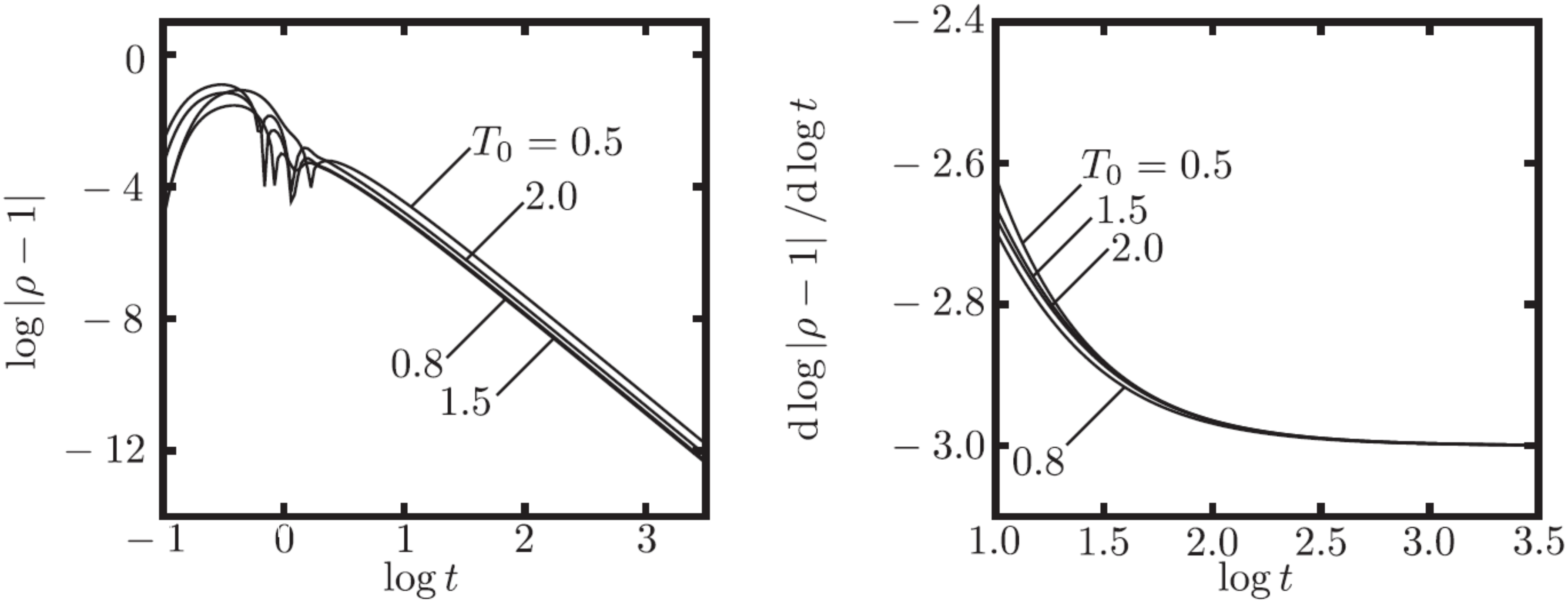
$$[\ f_w(t,\boldsymbol{\xi}) = \pi^{-3/2} \sigma_w(t) \exp(-|\boldsymbol{\xi}|^2)\ ]$$

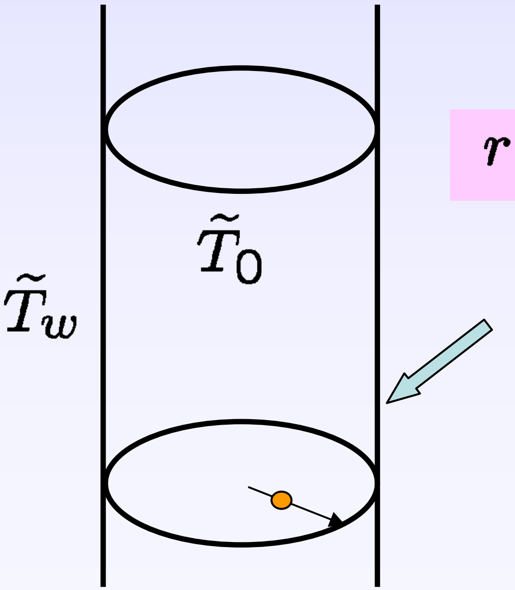
$$\sigma_w(t) = M(t) + \int_0^t r(t-s)\,\sigma_w(s)ds$$

$$M(t) = -\frac{\sqrt{\pi}}{t} \exp\left(-\frac{1}{2T_0 t^2}\right) I_1\left(-\frac{1}{2T_0 t^2}\right)$$

$$r(t) = \frac{\sqrt{\pi}}{t^4} \exp\left(-\frac{1}{2t^2}\right) \left[I_0\left(-\frac{1}{2t^2}\right) + (1+t^2) I_1\left(-\frac{1}{2t^2}\right)\right]$$

Modified Bessel functions



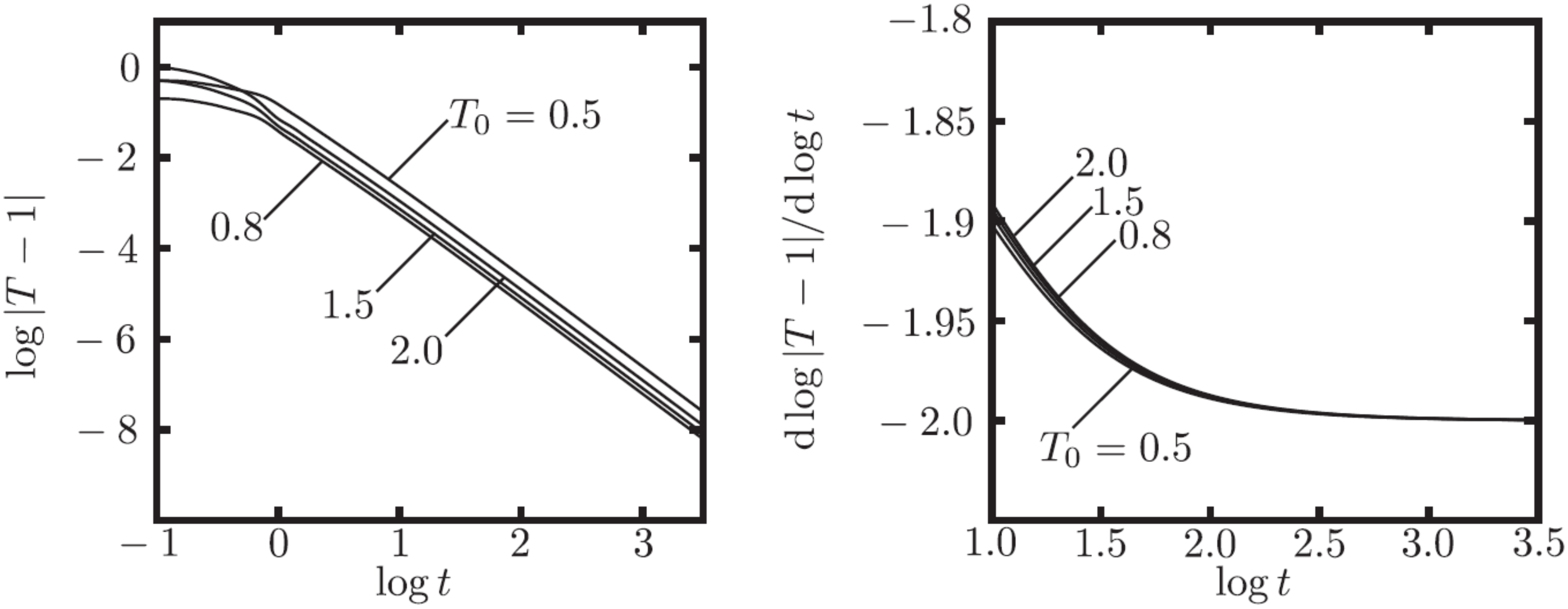


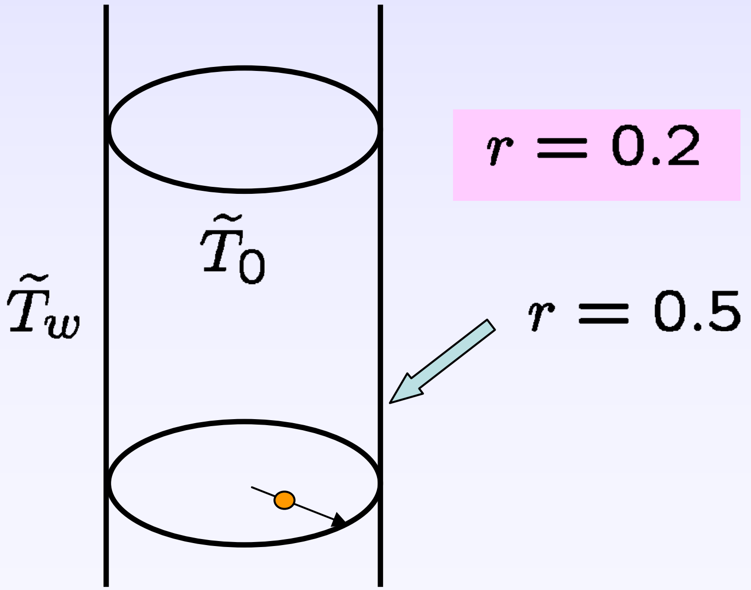
$r = 0.2$

$r = 0.5$

$T_0 = \tilde{T}_0/\tilde{T}_w = 0.5,\ 0.8,\ 1.5,\ 2$

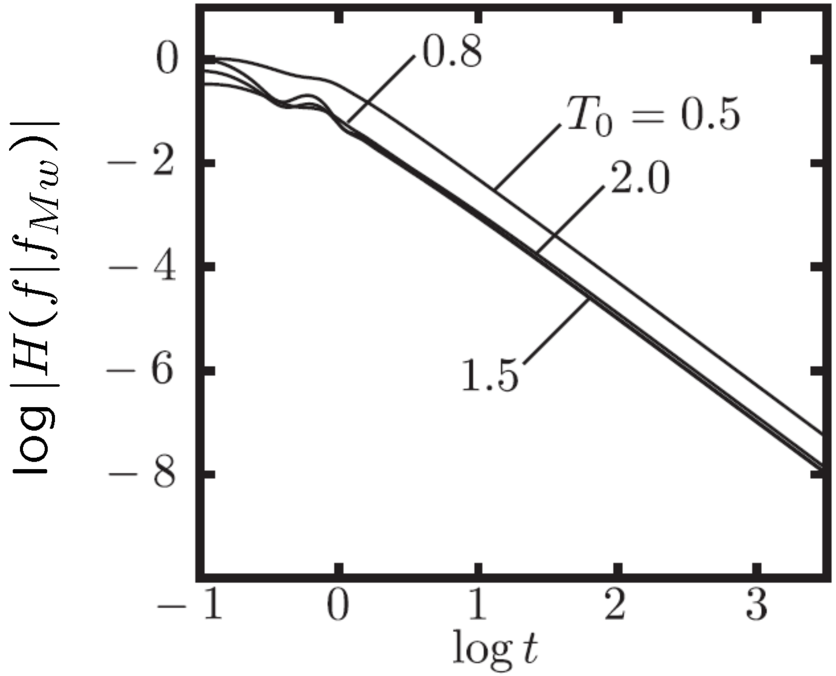
$$|\rho - 1| \approx \frac{C}{t^3}$$

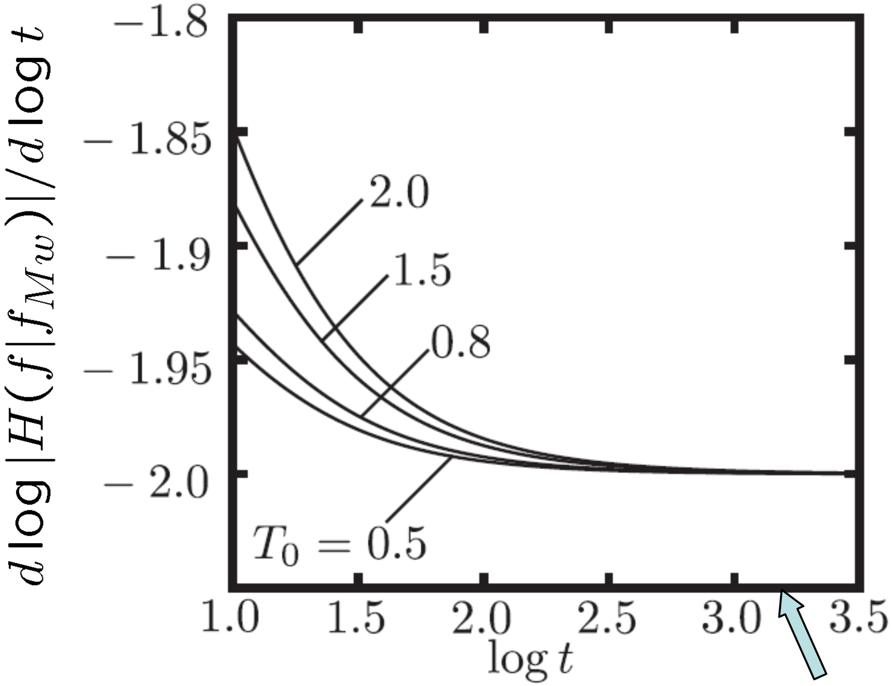




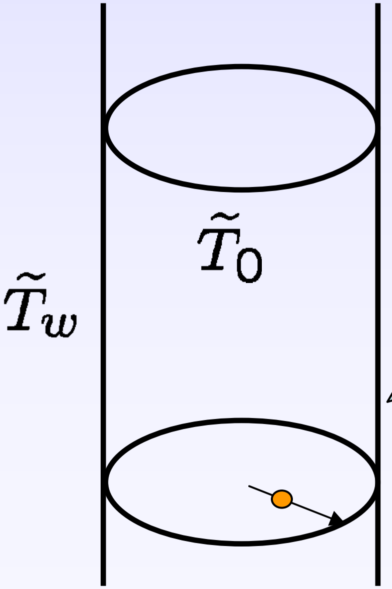
$$T_0 = \tilde{T}_0/\tilde{T}_w = 0.5,\ 0.8,\ 1.5,\ 2$$

$$|T - 1| \approx \frac{C}{t^2}$$





$t = 3.2 \times 10^4$



$r = 0.2$

$r = 0.5$

$T_0 = \tilde{T}_0/\tilde{T}_w = 0.5,\ 0.8,\ 1.5,\ 2$

$$|H(f|f_{Mw})| \approx \frac{C}{t^2}$$

$$H(f|f_{Mw}) = \int_{\mathbf{R}^3} f \ln f d\boldsymbol{\xi} - \int_{\mathbf{R}^3} f_{Mw} \ln f_{Mw} d\boldsymbol{\xi}$$

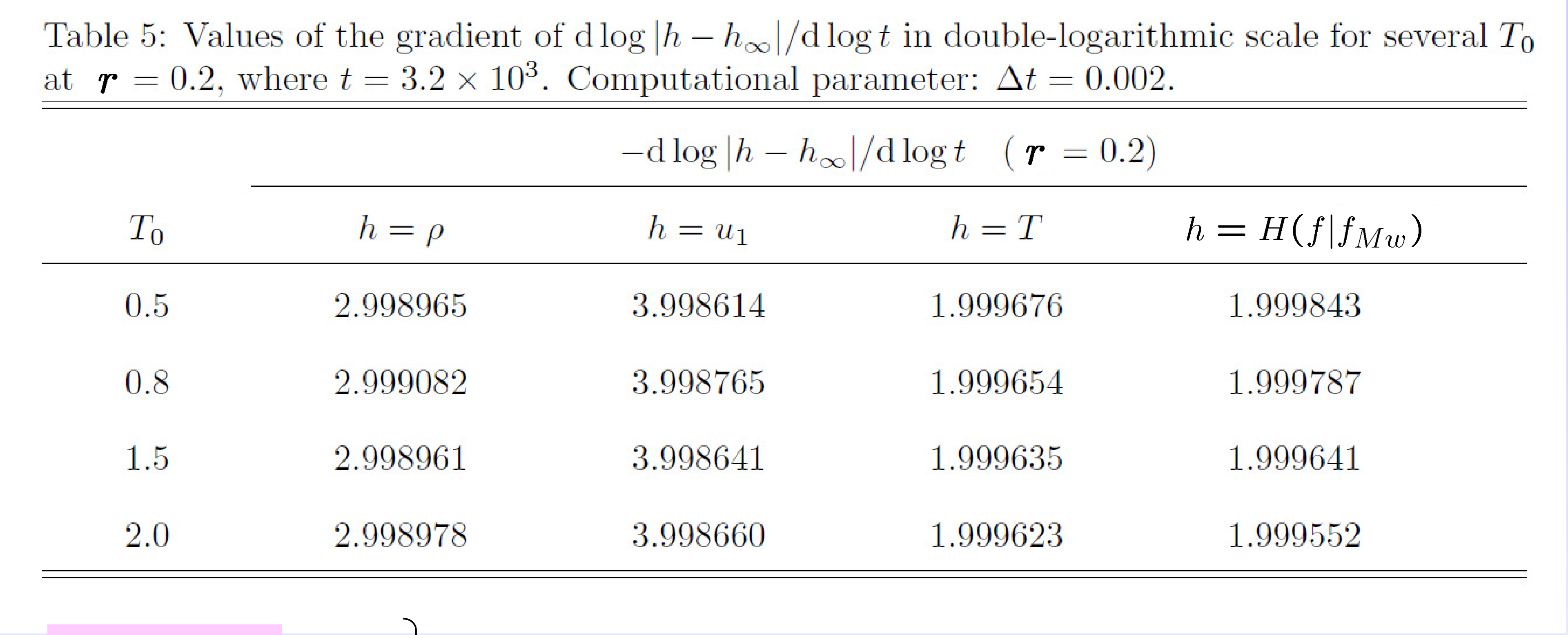


Table 5: Values of the gradient of $\mathrm{d}\log|h - h_\infty|/\mathrm{d}\log t$ in double-logarithmic scale for several T_0 at $\boldsymbol{r} = 0.2$, where $t = 3.2 \times 10^3$. Computational parameter: $\Delta t = 0.002$.

	$-\mathrm{d}\log\|h - h_\infty\|/\mathrm{d}\log t \quad (\boldsymbol{r} = 0.2)$			
T_0	$h = \rho$	$h = u_1$	$h = T$	$h = H(f\|f_{Mw})$
0.5	2.998965	3.998614	1.999676	1.999843
0.8	2.999082	3.998765	1.999654	1.999787
1.5	2.998961	3.998641	1.999635	1.999641
2.0	2.998978	3.998660	1.999623	1.999552

$$\left.\begin{array}{l}\boldsymbol{r} = 0.2 \\ \text{Other points}\end{array}\right\}$$

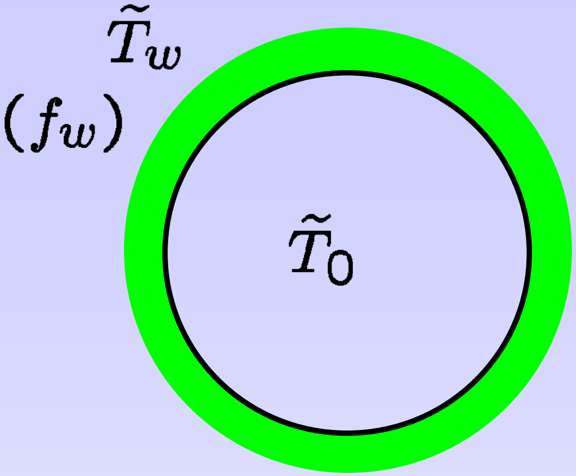
$t = 3.2 \times 10^4$

$$f \to f_{Mw} : \quad O(t^{-d}) \quad (d : \ \text{dimension of box})$$

$$d = 2$$

3D box ($d = 3$)

Sphere, Spherical symmetry



$$f_{in} = \frac{1}{(\pi T_0)^{3/2}} \exp\left(-\frac{|\boldsymbol{\xi}|^2}{T_0}\right)$$

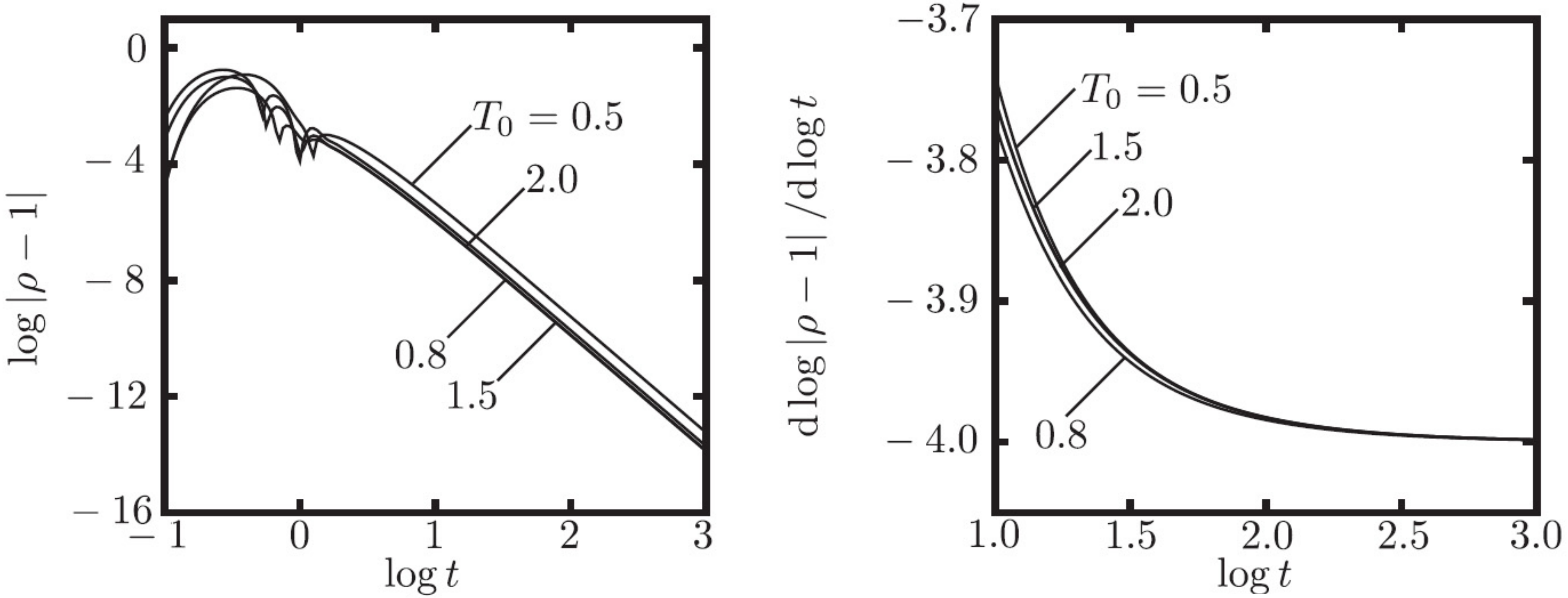
$$(T_0 = \tilde{T}_0/\tilde{T}_w = 0.5,\ 0.8,\ 1.5,\ 2)$$

$$[\ f_w(t,\boldsymbol{\xi}) = \pi^{-3/2}\sigma_w(t)\exp(-|\boldsymbol{\xi}|^2)\ ]$$

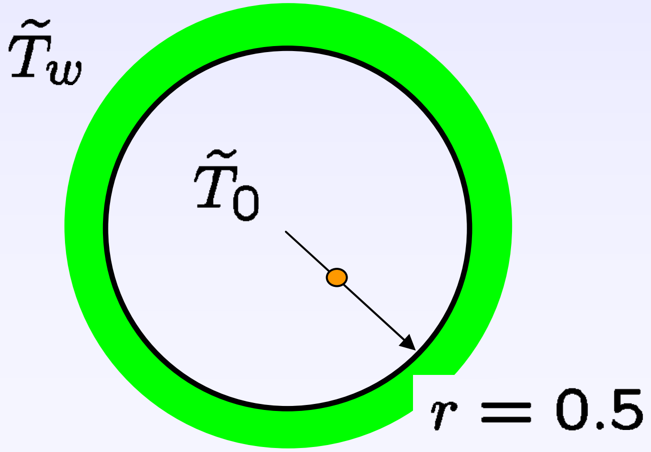
$$\sigma_w(t) = M(t) + \int_0^t r(t-s)\,\sigma_w(s)ds$$

$$M(t) = \sqrt{T_0}\left[1 - 2T_0t^2 + (1 + 2T_0t^2)\exp\left(-\frac{1}{T_0t^2}\right)\right]$$

$$r(t) = 4t - \left(4t + \frac{4}{t} + \frac{2}{t^3}\right)\exp\left(-\frac{1}{t^2}\right)$$

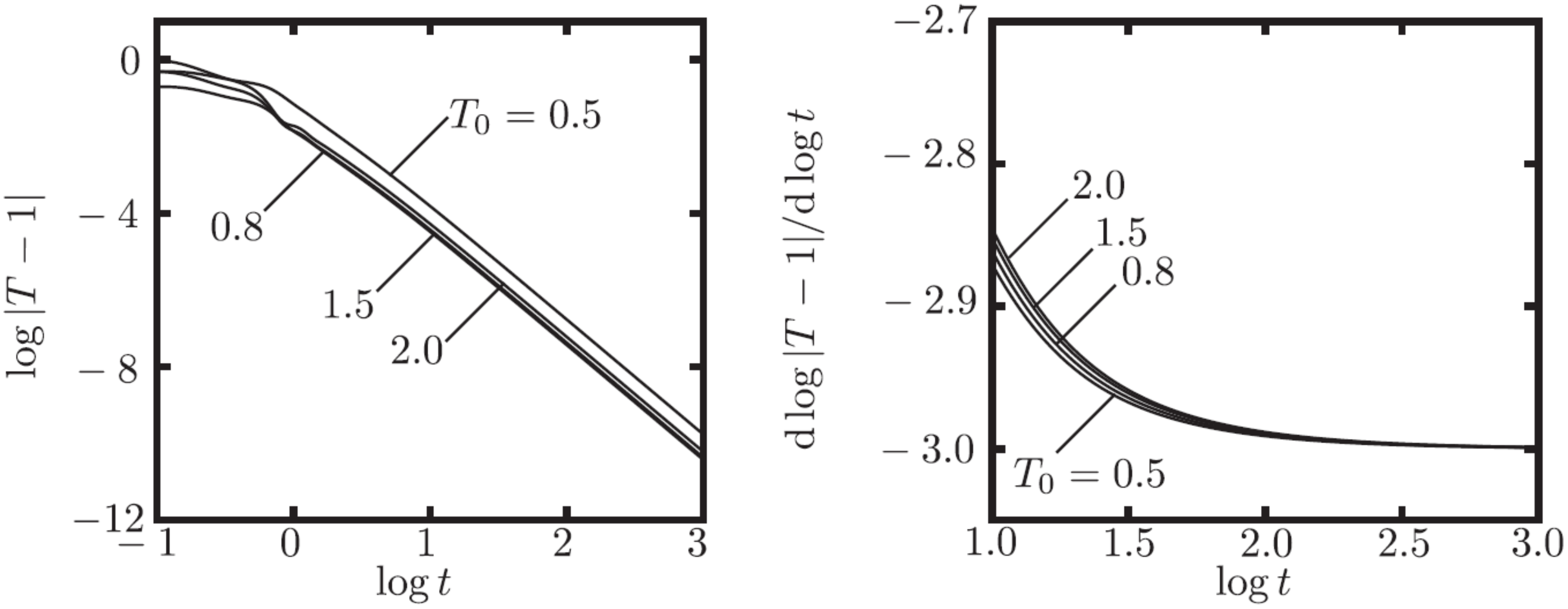


$r = 0.2$



$T_0 = \tilde{T}_0/\tilde{T}_w = 0.5,\ 0.8,\ 1.5,\ 2$

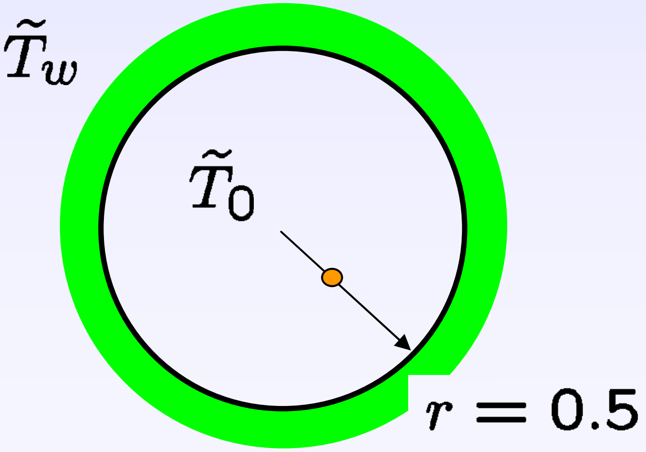
$$|\rho - 1| \approx \frac{C}{t^4}$$

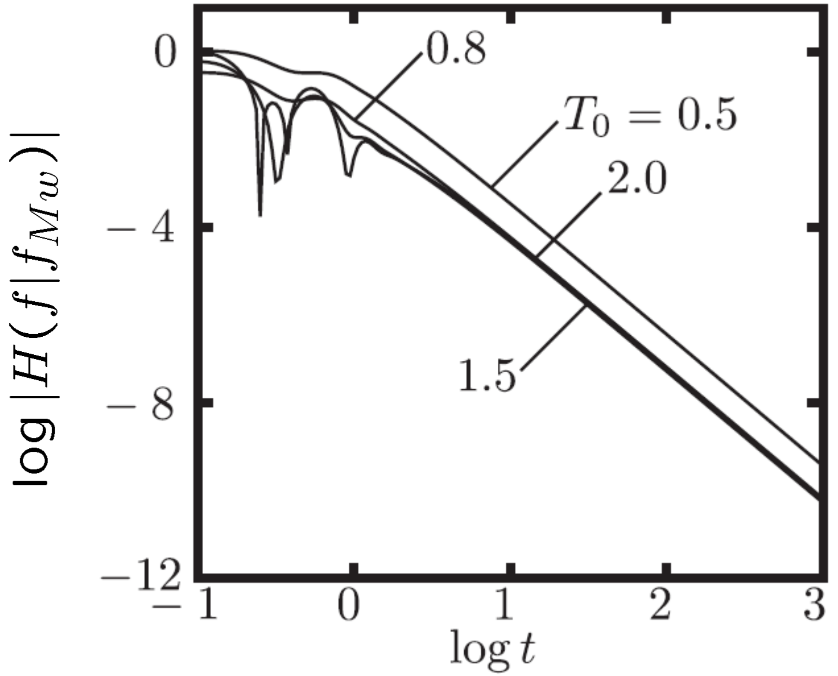


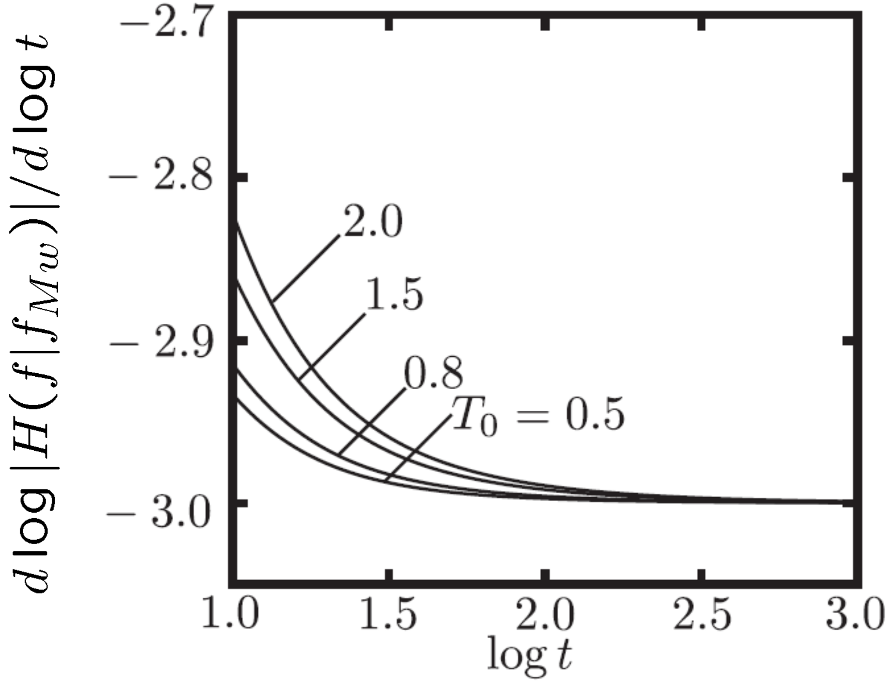
$r = 0.2$

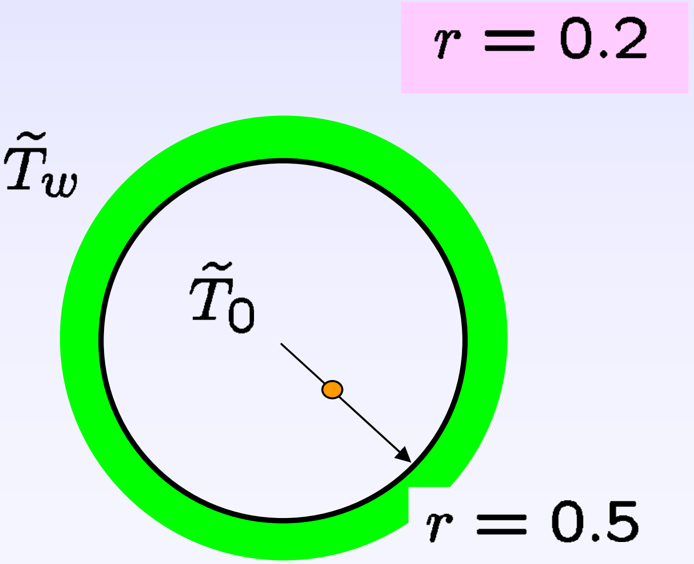
$T_0 = \tilde{T}_0/\tilde{T}_w = 0.5, 0.8, 1.5, 2$

$$|T - 1| \approx \frac{C}{t^3}$$









$$T_0 = \tilde{T}_0/\tilde{T}_w = 0.5,\ 0.8,\ 1.5,\ 2$$

$$|H(f|f_{Mw})| \approx \frac{C}{t^3}$$

$$H(f|f_{Mw}) = \int_{\mathbf{R}^3} f \ln f d\boldsymbol{\xi} - \int_{\mathbf{R}^3} f_{Mw} \ln f_{Mw} d\boldsymbol{\xi}$$

$f \to f_{Mw}$: $O(t^{-d})$ (d : dimension of box)

Numerical evidence (preliminary)

1D
2D: Circular cylinder
3D: Sphere

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Symmetric initial data