The ant in the labyrinth: random walks and percolation

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Percolation was introduced by Broadbent and Hammersley in 1957. The simplest version to describe is on the Euclidean lattice $\mathbb{Z}^d$. Let p be a fixed probability between 0 and 1. Each bond in $\mathbb{Z}^d$ is retained with probability p, and removed with probability $1-p$, independently of all the others. The percolation *cluster* containing a point x, denoted $C(x)$, consists of those points which can be reached from x by a path of retained bonds. There is a critical value $p_c \in (0,1)$ such that if $p < p_c$ then all clusters are finite, while for $p > p_c$ there is an infinite cluster.

Random walks on percolation clusters were introduced by De Gennes in 1976: he called this the problem of 'the ant in the labyrinth'. If $p = p(n, x, y)$ is the probability that a random walker ('the ant'), starting at x, is at y at time n, then p describes diffusion of heat on the cluster.

For the supercritical phase ($p > p_c$) this problem is now quite well understood, and $p(n, x, y)$ converges to a Gaussian distribution as $n \to \infty$. PDE techniques introduced by Nash in the 1950s, play an important role in some of the arguments.

The critical case $p = p_c$ is much harder, since the clusters have fractal properties. One expects that $p(n, x, x) \sim n^{-d_s/2}$, where d_s is called the *spectral dimension* of the cluster. Alexander and Orbach conjectured in 1982 that $d_s = 4/3$ in all dimensions: this has recently been proved in some high dimensional cases.