Traveling Fronts in Disordered Media

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Reaction-diffusion equations

We consider the reaction-diffusion equation

$$u_t = \Delta u + f(x,u)$$

on the spatial domain $D = \mathbb{R} \times \Omega$ (with $\Omega \subset \mathbb{R}^{d-1}$ bounded) and Neumann or periodic boundary conditions on ∂D.

- $u(t,x) \in [0,1]$ is the **normalized temperature** of a combusting medium or $1-u$ is a concentration of a reactant in a chemical reaction
- $f : D \times [0,1] \to [0,\infty)$ is a Lipschitz **reaction function** with $f(x,0) = f(x,1) = 0$ and **ignition temperature**

$$\theta(x) = \inf \{ u \,|\, f(x,u) > 0 \}$$

- **Ignition reaction:** $\inf_x \theta(x) > 0$
- **Positive (monostable) reaction:** $\inf_x \theta(x) = 0$
  KPP reaction: $f(x,u) \le \frac{\partial f}{\partial u}(x,0)u$

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Reaction-advection-diffusion equations

We will also consider the reaction-advection-diffusion equation

$$u_t + q(x) \cdot \nabla u = \nabla \cdot (A(x)\nabla u) + f(x,u)$$

with q divergence-free vector field (**incompressible flow**) and A uniformly elliptic (**inhomogeneous diffusion**).

Models propagation of reaction (e.g., combustion, fire). Also used in models of chemical kinetics, genetics, population dynamics.

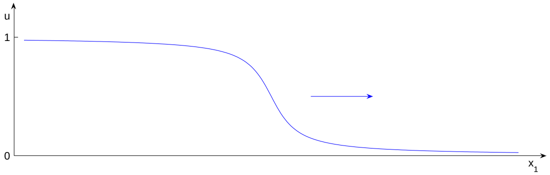
Goal: Describe long time behavior of solutions.

Transition fronts

Transition front (generalized traveling front) is a solution $u(t,x)$ that is **global in time** and satisfies for each $t \in \mathbb{R}$,

$$\lim_{x_1 \to -\infty} u(t,x) = 1 \qquad \text{and} \qquad \lim_{x_1 \to \infty} u(t,x) = 0$$

uniformly in $x' = (x_2, \dots, x_d) \in \Omega$.



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- Fronts can be attractors of general solutions of the PDE (front-like and compactly supported initial data).

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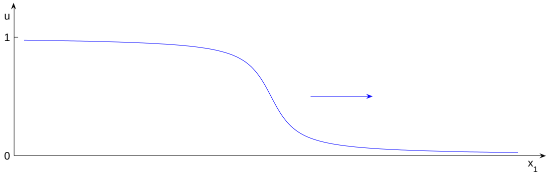

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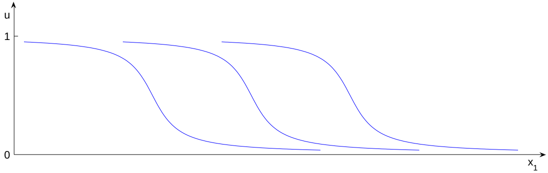
Homogeneous media: Traveling fronts

$$u_t = \Delta u + f(u)$$

A **traveling front** is a solution of the form $u(t,x) = U(x_1 - ct)$ such that $U(-\infty) = 1$ and $U(\infty) = 0$.



- Constant **profile** U and constant **speed** c
- $U'' + cU' + f(U) = 0$ gives $c > 0$ and $U' < 0$
- **Ignition reactions:** unique front speed $c_f^* > 0$
- **Positive reactions:** minimal front speed $c_f^* > 0$ and all $c \in [c_f^*, \infty)$ are achieved
- **KPP reactions:** $c_f^* = 2\sqrt{f'(0)}$ — same as for $f(u) = f'(0)u$ (Kolmogorov-Petrovskii-Piskunov)

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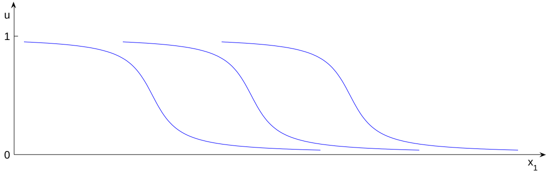

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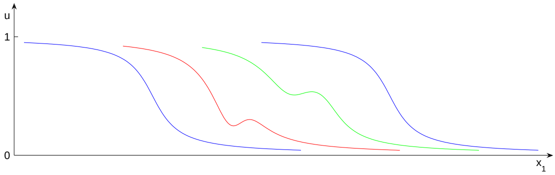
$$u_t + q(x)\cdot\nabla u = \nabla\cdot(A(x)\nabla u) + f(x,u)$$

Assume q,A,f are 1-periodic in x_1 and $\int_{[0,1]^d} q(x)dx = 0$. A **pulsating front** with speed $c > 0$ is a solution of the form $u(t,x) = U(x_1 - ct, x \bmod 1)$ such that $U(\pm\infty, x) = 0/1$.



- Time-periodic in a moving frame: $u(t+\frac{1}{c}, x+e_1) = u(t,x)$
- (U,c) solve a degenerate elliptic equation
- With mild conditions on f again unique/minimal front speed $c^*_{f,q,A} > 0$ for ignition/positive reactions (Berestycki-Hamel, Xin)

Periodic media: Pulsating fronts

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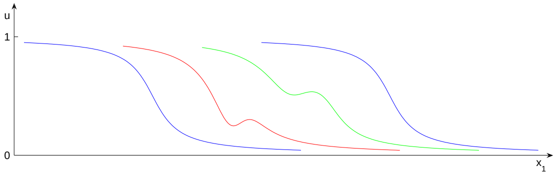

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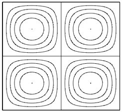
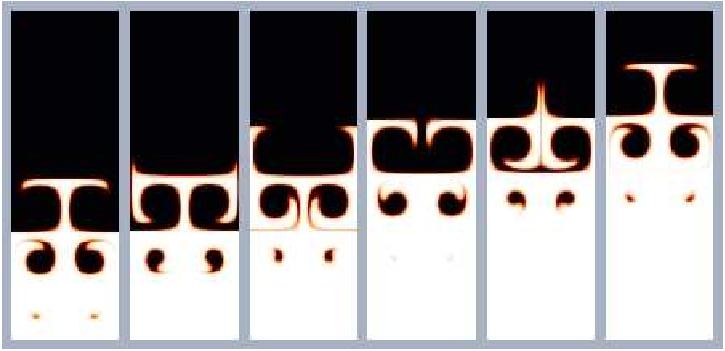
Pulsating front for a cellular flow





Fronts in 1D inhomogeneous media

In general inhomogeneous media no special forms exist.

- Nolen-Ryzhik-Mellet-Roquejoffre-Sire considered the 1D case with $f(x,u) = a(x)f_0(u)$ and a, f_0 Lipschitz:

  $$u_t = u_{xx} + a(x)f_0(u)$$

  If there are $0 < a_0 \leq a_1 < \infty$ such that $a(x) \in [a_0, a_1]$ and $\exists\theta \in (0,1)$ such that $f_0(u) > 0$ iff $u \in (\theta, 1)$, then they proved existence of a unique (right-moving) transition front, and its stability.

- The method is specialized for 1D and constant positive ignition temperature. A more robust method is needed to handle more dimensions, general q, A, and general f (non-constant, non-negative ignition temperature).

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Fronts in general inhomogeneous media

Hypotheses:

- $f(x,u)$ is Lipschitz and $f_0(u) \le f(x,u) \le f_1(u)$ for some reactions $f_0(u) \le f_1(u)$ such that f_0 is ignition (with ignition temperature $\theta > 0$) and f_1 is ignition or positive.
- $f_1'(0) < (c^*_{f_0})^2/4$ (true if f_1 ignition)
  - This is equivalent to $2\sqrt{f_1'(0)} < c^*_{f_0}$
- For some $\zeta < (c^*_{f_0})^2/4$ the function $f(x,\cdot)$ is bounded away from zero (uniformly in x) on the interval $[\alpha_f(x), 1-\varepsilon]$, with

$$\alpha_f(x) = \inf\{u \in (0,1) \,|\, f(x,u) > \zeta u\}$$

  - I.e., f cannot vanish after becoming large (except at $u = 1$)
  - This is a mild condition without which fronts might not exist: If $f(\frac{1}{2}) = 0$ and $f(u) > f(u + \frac{1}{2})$ for $u \in (0, \frac{1}{2})$, then only fronts connecting 0 and $\frac{1}{2}$ exist.

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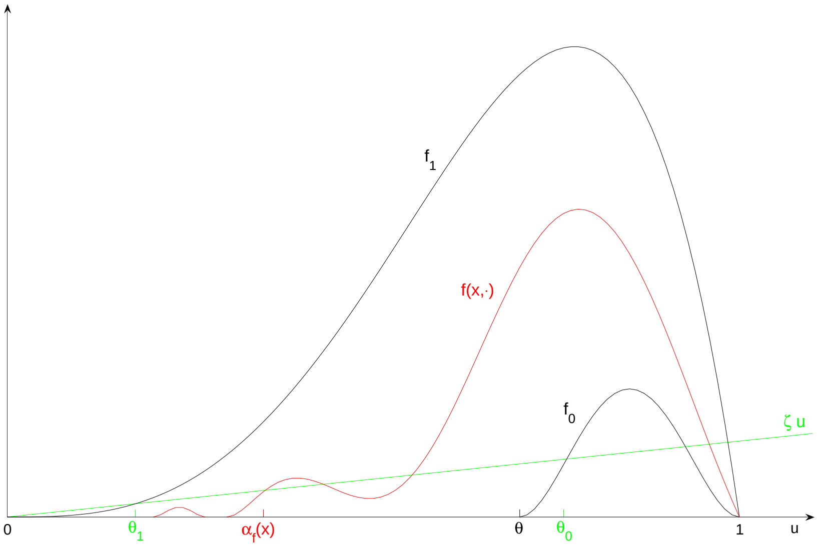

Fronts in general inhomogeneous media

Theorem

Assume the above hypotheses.
(i) There exists a transition front w_+ *for*

$$u_t = \Delta u + f(x, u)$$

moving to the right, with $(w_+)_t > 0$ *(and* w_- *moving to the left).*
(ii) If f_1 *is ignition and* f *is non-increasing in* u *on* $[1-\varepsilon, 1]$*, then these fronts are unique (up to time shifts).*
(iii) In (ii) general solutions with exponentially decaying initial data converge in L^∞_x *to time shifts of* $w_\pm$ *(global attractors). Convergence is uniform in* f *and uniformly bounded initial data.*

- Same result with periodic q, A but with $(c^*_{f_0})^2/4$ replaced by ζ_0 such that the minimal front speed for a KPP reaction with $\frac{\partial f}{\partial u}(x, 0) = \zeta_0$ is $c^*_{f_0}$.

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Remarks

- The main condition is $f_1'(0) < (c^*_{f_0})^2/4$. It guarantees that the **tail of the front is slower than the bulk**. There are examples where it is not satisfied and no fronts exist (Roquejoffre-Zlatoš).
- $f(x,\cdot)$ can be arbitrary on $(0, \alpha_f(x))$ and ignition temperature can be x-dependent.
- Uniqueness part requires f_1 ignition even for homogeneous media.
- Can be extended to periodically ondulating cylinders, but not to domains unbounded in several variables. Fronts are not unique in $\mathbb{R}^d$ ($d \geq 2$), even for homogeneous ignition reactions (and even when direction is fixed). Moreover, there are examples of non-homogenous ignition reactions in $\mathbb{R}^d$ where no fronts exist (Zlatoš).

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Front speed is not well defined in general, although the position of the front moves with speeds between $c^*_{f_0}$ and $c^*_{f_1}$.

Theorem

Assume the above hypotheses, with f_1 ignition, for a random reaction f_ω. If f_ω is stationary ergodic (with respect to translations in x_1), then there are $c_\pm \in [c^*_{f_0}, c^*_{f_1}]$ such that almost surely the random fronts $w_{\pm,\omega}$ have asymptotic speeds $c_\pm$.

Fronts in random media

Front speed is not well defined in general, although the position of the front moves with speeds between $c^*_{f_0}$ and $c^*_{f_1}$.

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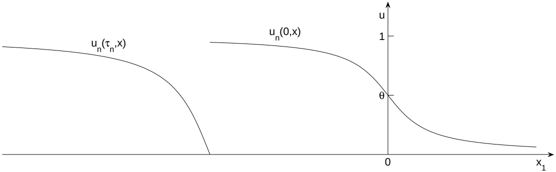
Proof: Existence of a front

Construction of front as limit of solutions u_n initially supported increasingly farther to the left: $u_n(\tau_n, x) = v(x_1 + n)$ where

- $\mathrm{supp} v = (-\infty, 0]$ and $v(-\infty) = 1$
- $v'' + f_0(v) \geq 0 \Rightarrow (u_n)_t > 0$
- $\tau_n < 0$ is such that $u_n(0, 0) = \theta$



u_n uniformly bounded in $C^{1,\delta;2,\delta} \Rightarrow \exists$ subsequence converging in $C^{1;2}_{\mathrm{loc}}(\mathbb{R} \times D)$ to some u

- u is a global solution
- Problem: to show u is a front connecting 0 and 1

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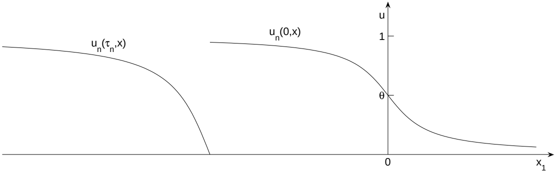

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Proof: Existence of a front

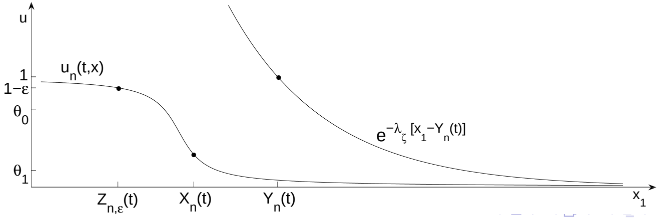
One needs to show that if

$$Z_{n,\varepsilon}(t) = \inf\left\{y \,\middle|\, u_n(t,x) \geq 1-\varepsilon \text{ whenever } x_1 \leq y\right\}$$
$$\tilde{Z}_{n,\varepsilon}(t) = \sup\left\{y \,\middle|\, u_n(t,x) \leq \varepsilon \text{ whenever } x_1 \geq y\right\}$$

then $\tilde{Z}_{n,\varepsilon}(t) - Z_{n,\varepsilon}(t)$ is uniformly bounded in n, t for every $\varepsilon > 0$.
It suffices to show that if $\lambda_\zeta = \sqrt{\zeta}$ and

$$Y_n(t) = \inf\left\{y \,\middle|\, u_n(t,x) \leq e^{-\lambda_\zeta(x_1-y)} \text{ for all } x \in D\right\}$$

then $Y_n(t) - Z_{n,\varepsilon}(t)$ is uniformly bounded in n, t for every $\varepsilon > 0$.



Proof: Existence of a front

Main idea: The tail of u_n cannot escape from the bulk due to

$$2\sqrt{f_1'(0)} < 2\sqrt{\zeta} < c_{f_0}^*$$

(when choosing $\zeta \in (f_1'(0), c_{f_0}^*)$).

Let $c_\zeta = 2\sqrt{\zeta} < c_{f_0}^*$ and $\lambda_\zeta = \sqrt{\zeta}$. Then $e^{-\lambda_\zeta(x_1 - Y_n(t_0) - c_\zeta t)}$ solves

$$u_t = \Delta u + \zeta u$$

and u_n is a subsolution where $u_n(t,x) \le \alpha_f(x)$. So define

$$X_n(t) = \sup \left\{ x_1 \,\middle|\, u_n(t,x) \ge \alpha_f(x) \text{ for some } x = (x_1, x') \right\}$$

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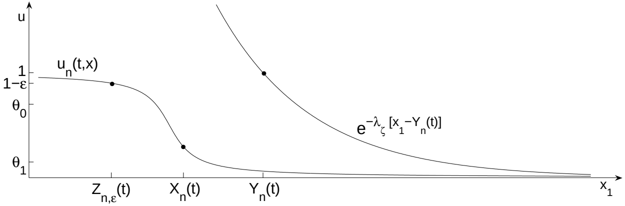

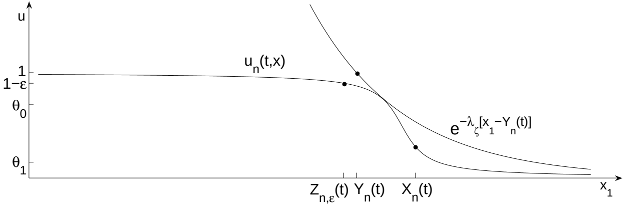

Claim 1: $Y_n'(t) \leq c_\zeta$ whenever $X_n(t) < Y_n(t)$.

Claim 2: $Z_{n,\varepsilon}(t) \geq Z_{n,\varepsilon}(t_0) + c^*_{f_0}(t - t_0 - \tau_\varepsilon)$ (Xin: with $c^*_{f_0} - \delta$).

Claim 3: $|X_n(t) - Z_{n,\varepsilon}(t)| \leq C_\varepsilon$.

Proof: Existence of a front

Proof (of $|X_n(t) - Z_{n,\varepsilon}(t)| \le C_\varepsilon$): Recall that

$$X_n(t) = \sup\left\{x_1 \,\middle|\, u_n(t,x) \ge \alpha_f(x) \text{ for some } x = (x_1, x')\right\}$$
$$Z_{n,\varepsilon}(t) = \inf\left\{y \,\middle|\, u_n(t,x) \ge 1-\varepsilon \text{ whenever } x_1 \le y\right\}$$

$(u_n)_t > 0$ and parabolic regularity give existence of uniform t_ε such that if $u_n(t,x_0) \ge \alpha_f(x_0)$, then $u_n(t+t_\varepsilon, x_0) \ge 1-\varepsilon$

- Uses: if $0 = \Delta\tilde{u} + f(x,\tilde{u})$ and $\tilde{u}(x_0) \ge \alpha_f(x_0)$, then $\tilde{u} \equiv 1$

This and additional estimates on $X_n(t)$ then give

$$Z_{n,\varepsilon}(t+t'_\varepsilon) \ge X_n(t) \qquad \text{and} \qquad X_n(t+t'_\varepsilon) - X_n(t) \le C'_\varepsilon$$

Thus $Z_{n,\varepsilon}(t) \le X_n(t) \le Z_{n,\varepsilon}(t) + C_\varepsilon$, and so u is a front.

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Proof: Uniqueness and stability of front (ignition f_1)

Main idea: If w is a front and u_n as above, then $\exists \tau_{w,n}$ such that

$$\lim_{t\to\infty} \|w(t+\tau_{w,n},x) - u_n(t,x)\|_{L^\infty_x} = 0 \qquad \textbf{uniformly in } n, w, f$$

So any two fronts are time-shifts of each other, and there is a unique front w_+.

Proof uses stability of u_n. This is obtained via construction of suitable sub- and supersolutions, using that $(u_n)_t > 0$ and also that f is non-increasing in u near $u = 0, 1$.

A similar argument with w a general (front-like) solution shows $w - u_n \to 0$ as $t \to \infty$. Since also $w_+ - u_n \to 0$ as $t \to \infty$, we have that $w_\pm$ are **global attractors** of general solutions.

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$$\lim_{t\to\infty} \|w(t+\tau_{w,n},x) - u_n(t,x)\|_{L^\infty_x} = 0 \qquad \textbf{uniformly in } n, w, f$$

So any two fronts are time-shifts of each other, and there is a unique front w_+.

Proof uses stability of u_n. This is obtained via construction of suitable sub- and supersolutions, using that $(u_n)_t > 0$ and also that f is non-increasing in u near $u = 0, 1$.

A similar argument with w a general (front-like) solution shows $w - u_n \to 0$ as $t \to \infty$. Since also $w_+ - u_n \to 0$ as $t \to \infty$, we have that $w_\pm$ are **global attractors** of general solutions.