Traveling Fronts in Disordered Media

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We consider the reaction-diffusion equation

$$u_t = \Delta u + f(x, u)$$

on the spatial domain $D = \mathbb{R} \times \Omega$ (with $\Omega \subset \mathbb{R}^{d-1}$ bounded) and Neumann or periodic boundary conditions on $\partial D$.

- $u(t, x) \in [0, 1]$ is the normalized temperature of a combusting medium or $1 - u$ is a concentration of a reactant in a chemical reaction.

- $f : D \times [0, 1] \to [0, \infty)$ is a Lipschitz reaction function with $f(x, 0) = f(x, 1) = 0$ and ignition temperature

$$\theta(x) = \inf \{ u \mid f(x, u) > 0 \}$$

- Ignition reaction: $\inf_x \theta(x) > 0$

- Positive (monostable) reaction: $\inf_x \theta(x) = 0$

- KPP reaction: $f(x, u) \leq \frac{\partial f}{\partial u}(x, 0) u$
Reaction-diffusion equations

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We will also consider the reaction-advection-diffusion equation

$$u_t + q(x) \cdot \nabla u = \nabla \cdot (A(x) \nabla u) + f(x, u)$$

with $q$ divergence-free vector field (incompressible flow) and $A$ uniformly elliptic (inhomogeneous diffusion).

Models propagation of reaction (e.g., combustion, fire). Also used in models of chemical kinetics, genetics, population dynamics.

**Goal:** Describe long time behavior of solutions.
Transition front (generalized traveling front) is a solution $u(t, x)$ that is **global in time** and satisfies for each $t \in \mathbb{R}$,

$$\lim_{x_1 \to -\infty} u(t, x) = 1 \quad \text{and} \quad \lim_{x_1 \to \infty} u(t, x) = 0$$

uniformly in $x' = (x_2, \ldots, x_d) \in \Omega$.

- This front moves to the right. Also a front moving left.
- Fronts can be attractors of general solutions of the PDE (front-like and compactly supported initial data).
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Homogeneous media: Traveling fronts

\[ u_t = \Delta u + f(u) \]

A **traveling front** is a solution of the form \( u(t, x) = U(x_1 - ct) \) such that \( U(-\infty) = 1 \) and \( U(\infty) = 0 \).

- **Constant profile** \( U \) and constant speed \( c \)
- \( U'' + cU' + f(U) = 0 \) gives \( c > 0 \) and \( U' < 0 \)
- **Ignition reactions:** unique front speed \( c_f^* > 0 \)
- **Positive reactions:** minimal front speed \( c_f^* > 0 \) and all \( c \in [c_f^*, \infty) \) are achieved
- **KPP reactions:** \( c_f^* = 2\sqrt{f'(0)} \) — same as for \( f(u) = f'(0)u \) (Kolmogorov-Petrovskii-Piskunov)
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Periodic media: Pulsating fronts

\[ u_t + q(x) \cdot \nabla u = \nabla \cdot (A(x)\nabla u) + f(x, u) \]

Assume \( q, A, f \) are 1-periodic in \( x_1 \) and \( \int_{[0, 1]} q(x) \, dx = 0 \). A **pulsating front** with speed \( c > 0 \) is a solution of the form \( u(t, x) = U(x_1 - ct, x \mod 1) \) such that \( U(\pm \infty, x) = 0/1 \).

- Time-periodic in a moving frame: \( u(t + \frac{1}{c}, x + e_1) = u(t, x) \)
- \( (U, c) \) solve a degenerate elliptic equation
- With mild conditions on \( f \) again unique/minimal front speed \( c^*_f, q, A > 0 \) for ignition/positive reactions (Berestycki-Hamel, Xin)
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Pulsating front for a cellular flow

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Traveling Fronts in Disordered Media
In general inhomogeneous media no special forms exist.

- Nolen-Ryzhik-Mellet-Roquejoffre-Sire considered the 1D case with $f(x, u) = a(x)f_0(u)$ and $a, f_0$ Lipschitz:

$$u_t = u_{xx} + a(x)f_0(u)$$

If there are $0 < a_0 \leq a_1 < \infty$ such that $a(x) \in [a_0, a_1]$ and $\exists \theta \in (0, 1)$ such that $f_0(u) > 0$ iff $u \in (\theta, 1)$, then they proved existence of a unique (right-moving) transition front, and its stability.

- The method is specialized for 1D and constant positive ignition temperature. A more robust method is needed to handle more dimensions, general $q, A$, and general $f$ (non-constant, non-negative ignition temperature).
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Hypotheses:

- $f(x, u)$ is Lipschitz and $f_0(u) \leq f(x, u) \leq f_1(u)$ for some reactions $f_0(u) \leq f_1(u)$ such that $f_0$ is ignition (with ignition temperature $\theta > 0$) and $f_1$ is ignition or positive.
- $f'_1(0) < (c_{f_0}^*)^2/4$ (true if $f_1$ ignition)
  - This is equivalent to $2 \sqrt{f'_1(0)} < c_{f_0}^*$
  - For some $\zeta < (c_{f_0}^*)^2/4$ the function $f(x, \cdot)$ is bounded away from zero (uniformly in $x$) on the interval $[\alpha_f(x), 1 - \varepsilon]$, with

  $$\alpha_f(x) = \inf\{u \in (0, 1) \mid f(x, u) > \zeta u\}$$

  - I.e., $f$ cannot vanish after becoming large (except at $u = 1$)
  - This is a mild condition without which fronts might not exist:
    If $f(\frac{1}{2}) = 0$ and $f(u) > f(u + \frac{1}{2})$ for $u \in (0, \frac{1}{2})$, then only fronts connecting $0$ and $\frac{1}{2}$ exist.
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Fronts in general inhomogeneous media

\[ f(x, \cdot) \]

\[ \alpha_i(x) \]

\[ f_1 \]

\[ f_0 \]

\[ \zeta u \]
Theorem

Assume the above hypotheses.
(i) There exists a transition front $w_+$ for

$$u_t = \Delta u + f(x, u)$$

moving to the right, with $(w_+)_t > 0$ (and $w_-$ moving to the left).
(ii) If $f_1$ is ignition and $f$ is non-increasing in $u$ on $[1 - \varepsilon, 1]$, then these fronts are unique (up to time shifts).
(iii) In (ii) general solutions with exponentially decaying initial data converge in $L^\infty_x$ to time shifts of $w_\pm$ (global attractors). Convergence is uniform in $f$ and uniformly bounded initial data.

Same result with periodic $q, A$ but with $(c_{f_0}^*)^2/4$ replaced by $\zeta_0$ such that the minimal front speed for a KPP reaction with $\frac{\partial f}{\partial u}(x, 0) = \zeta_0$ is $c_{f_0}^*$. 

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- Same result with periodic $q, A$ but with $(c_{f_0}^*)^2 / 4$ replaced by $\zeta_0$ such that the minimal front speed for a KPP reaction with $\frac{\partial f}{\partial u}(x, 0) = \zeta_0$ is $c_{f_0}^*$.
Remarks

- The main condition is $f'_1(0) < (c^*_f)^2/4$. It guarantees that the tail of the front is slower than the bulk. There are examples where it is not satisfied and no fronts exist (Roquejoffre-Zlatoš).
- $f(x, \cdot)$ can be arbitrary on $(0, \alpha_f(x))$ and ignition temperature can be $x$-dependent.
- Uniqueness part requires $f_1$ ignition even for homogeneous media.
- Can be extended to periodically ondulating cylinders, but not to domains unbounded in several variables. Fronts are not unique in $\mathbb{R}^d$ ($d \geq 2$), even for homogeneous ignition reactions (and even when direction is fixed). Moreover, there are examples of non-homogenous ignition reactions in $\mathbb{R}^d$ where no fronts exist (Zlatoš).
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Front speed is not well defined in general, although the position of the front moves with speeds between $c_{f_0}^*$ and $c_{f_1}^*$. 

**Theorem**

Assume the above hypotheses, with $f_1$ ignition, for a random reaction $f_ω$. If $f_ω$ is stationary ergodic (with respect to translations in $x_1$), then there are $c_± \in [c_{f_0}^*, c_{f_1}^*]$ such that almost surely the random fronts $w_{±,ω}$ have asymptotic speeds $c_±$. 
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Traveling Fronts in Disordered Media
Proof: Existence of a front

Construction of front as limit of solutions $u_n$ initially supported increasingly farther to the left: $u_n(\tau_n, x) = v(x_1 + n)$ where

- $\text{supp } v = (-\infty, 0]$ and $v(-\infty) = 1$
- $v'' + f_0(v) \geq 0 \Rightarrow (u_n)_t > 0$
- $\tau_n < 0$ is such that $u_n(0, 0) = \theta$

$u_n$ uniformly bounded in $C^{1,\delta;2,\delta} \Rightarrow \exists$ subsequence converging in $C^{1;2}_{\text{loc}}(\mathbb{R} \times D)$ to some $u$

- $u$ is a global solution
- Problem: to show $u$ is a front connecting 0 and 1
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One needs to show that if

\[ Z_{n,\varepsilon}(t) = \inf \{ y \mid u_n(t, x) \geq 1 - \varepsilon \text{ whenever } x_1 \leq y \} \]
\[ \tilde{Z}_{n,\varepsilon}(t) = \sup \{ y \mid u_n(t, x) \leq \varepsilon \text{ whenever } x_1 \geq y \} \]

then \( \tilde{Z}_{n,\varepsilon}(t) - Z_{n,\varepsilon}(t) \) is uniformly bounded in \( n, t \) for every \( \varepsilon > 0 \).

It suffices to show that if \( \lambda_\zeta = \sqrt{\zeta} \) and

\[ Y_n(t) = \inf \{ y \mid u_n(t, x) \leq e^{-\lambda_\zeta (x_1 - y)} \text{ for all } x \in D \} \]

then \( Y_n(t) - Z_{n,\varepsilon}(t) \) is uniformly bounded in \( n, t \) for every \( \varepsilon > 0 \).
Main idea: The tail of $u_n$ cannot escape from the bulk due to

$$2\sqrt{f'_1(0)} < 2\sqrt{\zeta} < c^*_f$$

(when choosing $\zeta \in (f'_1(0), c^*_f)$).

Let $c_\zeta = 2\sqrt{\zeta} < c^*_f$ and $\lambda_\zeta = \sqrt{\zeta}$. Then $e^{-\lambda_\zeta (x_1 - Y_n(t_0) - c_\zeta t)}$ solves

$$u_t = \Delta u + \zeta u$$

and $u_n$ is a subsolution where $u_n(t, x) \leq \alpha_f(x)$. So define

$$X_n(t) = \sup \{ x_1 \mid u_n(t, x) \geq \alpha_f(x) \text{ for some } x = (x_1, x') \}$$
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Claim 1: $Y'_n(t) \leq c_\zeta$ whenever $X_n(t) < Y_n(t)$.

Claim 2: $Z_{n,\varepsilon}(t) \geq Z_{n,\varepsilon}(t_0) + c^*_f(t - t_0 - \tau_\varepsilon)$ (Xin: with $c^*_f - \delta$).

Claim 3: $|X_n(t) - Z_{n,\varepsilon}(t)| \leq C_\varepsilon$. 

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Proof (of $|X_n(t) - Z_{n,\varepsilon}(t)| \leq C_\varepsilon$): Recall that

\begin{align*}
X_n(t) &= \sup \{ x_1 \mid u_n(t, x) \geq \alpha_f(x) \text{ for some } x = (x_1, x') \} \\
Z_{n,\varepsilon}(t) &= \inf \{ y \mid u_n(t, x) \geq 1 - \varepsilon \text{ whenever } x_1 \leq y \}
\end{align*}

$(u_n)_t > 0$ and parabolic regularity give existence of uniform $t_\varepsilon$ such that if $u_n(t, x_0) \geq \alpha_f(x_0)$, then $u_n(t + t_\varepsilon, x_0) \geq 1 - \varepsilon$

- Uses: if $0 = \Delta \tilde{u} + f(x, \tilde{u})$ and $\tilde{u}(x_0) \geq \alpha_f(x_0)$, then $\tilde{u} \equiv 1$

This and additional estimates on $X_n(t)$ then give

\begin{align*}
Z_{n,\varepsilon}(t + t_\varepsilon') &\geq X_n(t) \quad \text{and} \quad X_n(t + t_\varepsilon') - X_n(t) \leq C'_\varepsilon
\end{align*}

Thus $Z_{n,\varepsilon}(t) \leq X_n(t) \leq Z_{n,\varepsilon}(t) + C_\varepsilon$, and so $u$ is a front.
Proof: Existence of a front

Proof (of $|X_n(t) - Z_{n,\varepsilon}(t)| \leq C_\varepsilon$): Recall that

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\[ Z_{n,\varepsilon}(t) = \inf \{ y \mid u_n(t, x) \geq 1 - \varepsilon \text{ whenever } x_1 \leq y \} \]

$(u_n)_t > 0$ and parabolic regularity give existence of uniform $t_\varepsilon$ such that if $u_n(t, x_0) \geq \alpha_f(x_0)$, then $u_n(t + t_\varepsilon, x_0) \geq 1 - \varepsilon$

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Thus $Z_{n,\varepsilon}(t) \leq X_n(t) \leq Z_{n,\varepsilon}(t) + C_\varepsilon$, and so $u$ is a front.
Proof: Uniqueness and stability of front (ignition $f_1$)

**Main idea:** If $w$ is a front and $u_n$ as above, then $\exists \tau_{w,n}$ such that

$$\lim_{t \to \infty} \|w(t + \tau_{w,n}, x) - u_n(t, x)\|_{L^\infty_x} = 0 \quad \text{uniformly in } n, w, f$$

So any two fronts are time-shifts of each other, and there is a unique front $w_+$.

Proof uses stability of $u_n$. This is obtained via construction of suitable sub- and supersolutions, using that $(u_n)_t > 0$ and also that $f$ is non-increasing in $u$ near $u = 0, 1$.

A similar argument with $w$ a general (front-like) solution shows $w - u_n \to 0$ as $t \to \infty$. Since also $w_+ - u_n \to 0$ as $t \to \infty$, we have that $w_\pm$ are global attractors of general solutions.
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\lim_{t \to \infty} \| w(t + \tau_{w,n}, x) - u_n(t, x) \|_{L_x^\infty} = 0 \quad \text{uniformly in } n, w, f
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