Front Speeds of Non-Coercive Hamilton-Jacobi Equations in Multiscale Media

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Fronts or interfaces in multiscale and random flows appear in many scientific areas as solutions to PDEs or free boundary problems: premixed turbulent combustion, chemical reaction fronts in liquid, algal blooms (large blobs of micro-algea) in the ocean. Two prototype scalar models for fronts are:

I. Reaction-Diffusion-Advection-Equation (RD):

$$u_t = \Delta_x u + B(\omega, x, t) \cdot \nabla_x u + f(u), \quad x \in R^N, \tag{1}$$

$f(u)$ is nonlinear reaction function; B a space-time periodic or random flow field.

II. Hamilton-Jacobi Equation (HJ):

$$u_t + H(\omega, x, t, \nabla_x u) = \kappa \, \Delta u, \quad x \in R^N, \;\; N \geq 1, \; \kappa \geq 0. \tag{2}$$

H is periodic or random in (x, t), nonlinear in $\nabla_x u$.

Examples of reactive nonlinearities:

1. $f(u) = u(1-u)$ (Kolmogorov-Petrovsky-Piskunov(KPP), Fisher, 1937)
2. $f(u) = 0\ \forall u \in [0,\theta] \cup \{1\}$, $f(u) > 0\ \forall u \in (\theta, 1)$, $f(u)$ Lipschitz continuous, combustion nonlinearity with ignition temperature θ.
3. $f(u) = u(1-u)(u-\mu)$, $\mu \in (0,1)$: bistable nonlinearity.

Examples of Hamiltonians:

1. $H(x,t,p) = |p|^2/2 + V(x)$ (classical)
2. $H(x,t,p) = |p|^2/2 + V(x) \cdot p$ (advective)
3. $H(x,t,p) = |p| + V(x) \cdot p$ (relativistic/geometric)

Consider Spatially Heterogeneous Hamilton-Jacobi Eq. (HJ) satisfying:

- H (Hamiltonian) is convex in $p \equiv \nabla_x u$, continuous in x;
- H is stationary in x (joint distribution of $H(\omega, x_1, p), \cdots, H(\omega, x_m, p)$ is invariant under translation in x);
- H is ergodic in x (events invariant under translation have probability either zero or one).

Theory of HJ homogenization requires **coercivity:**

$$H(x,p) \to +\infty, \quad \text{as } \ p \to \infty \ \text{ uniformly } \ \text{in } \ x.$$

- Periodic Media: Lions-Papanicolaou-Varadhan (1988), Evans (92);
- Quasi-periodic media and degenerate 2nd order (viscous) HJs: (Lions-Souganidis, Ishii, and references therein).
- Stationary ergodic media:

  Invisid HJs: Souganidis (99), Rezakhanlou-Tarver (00);

  Viscous HJs: Lions-Souganidis (05), Kosygina-Rezakhanlou-Varadhan (06), Schwab (08).

Non-coercive HJs:

Example 1: $H(x,p,\omega) = c_n\,|p|^n + V(x,\omega)$, $n = 2$ (classical mechanics), $n = 1$ (relativistic mechanics), potential V is an Ornstein-Uhlenbeck (OU) process, a Gaussian process with mean zero and covariance

$$\mathbf{E}[V(x)V(y)] = \frac{1}{2}\exp(-|x-y|).$$

The OU processs is unbounded in x, H is **non-coercive**.

Example 2 (advective Hamiltonian): $H(x,p,\omega) = c_n|p|^n + b(x,\omega)\cdot p$, $n = 2$ (corres. to KPP), $n = 1$ (G-equation, approximating ignition/bistable reaction fronts on hyperbolic time scale).

If $n = 1$, then for b mean-zero periodic function with amplitude above c_n, H is **non-coercive**.

Inviscid G-equation:

$$G_t + B(x) \cdot \nabla G = s_l \, |\nabla G|,$$

$s_l > 0$ (constant, laminar speed) as a small diffusion fast reaction limit of ignition/bistable reaction:

$$v_t + B(x) \cdot \nabla v = \epsilon \, \Delta v + \epsilon^{-1} \, f(v).$$

Substituting:

$$v \sim U(u(x,t)/\epsilon),$$

then equation at order $O(\epsilon^{-1})$ is:

$$(u_t + B(x) \cdot \nabla u)U' = |\nabla u|^2 U'' + f(U).$$

By traveling front identity and scaling:

$$U'' - cU' + f(U) = 0, \quad d^2 U'' - dcU' + f(U) = 0, \ \ \forall \, d > 0,$$

we have:

$$u_t + B(x) \cdot \nabla u = c|\nabla u|.$$

Over longer (diffusive) time scale $O(\epsilon^{-2})$, one adds curvature correction to G-equation, giving viscous G-equation.

Capture Front Asymptotically in Multiscale Media and Homogenization:

Scaling transform: $(x,t) \to \frac{(x,t)}{\epsilon}$, $u \to \epsilon u \equiv u^{\epsilon}$. In case of homogeneous media, front solutions are invariant.

HJ random fronts may be studied in terms of u^{ϵ} in the limit $\epsilon \downarrow 0$, where u^{ϵ} satisfies:

$$u^{\epsilon}_t + H(\omega, \frac{x}{\epsilon}, \nabla_x u^{\epsilon}) = 0, \tag{3}$$

with planar initial data: $u^{\epsilon}(x,0) = p \cdot x$.

The study of convergence of u^{ϵ} to a limiting funtion $\bar{u}(x,t)$ and the resulting limiting eqn is **Homogenization**.

Suppose $\bar{u}$ exists, then observing at $(x,t) = (0,1)$, we see:

$$\epsilon\, u(0, \frac{1}{\epsilon}) \to \bar{u}(0,1) \equiv -\bar{H}(p),$$

p dependent constant or HJ front speed ($T = \frac{1}{\epsilon}$):

$$u(0,T) \sim -\bar{H}(p)\, T.$$

Lax-Oleinik-Hopf formula:

$$u^{\epsilon}(x,t) = \inf_{y \in R^N} \left(u^{\epsilon}(y,0) + \inf_{\xi} \int_0^t L(\omega, \xi(s)/\epsilon, \xi'(s))\, ds \right), \tag{4}$$

$\xi \in W^{1,\infty}((0,t); R^N), \xi(0) = y, \xi(t) = x$, a Lipschitz path connecting y to x.

L is Lagrangian given by Legendre transform:

$$L(\omega, x, q) = \sup_{p \in R^N} (q \cdot p - H(\omega, x, p)). \tag{5}$$

L is a convex function in q.

At $x = 0$, the integral equals:

$$\int_0^t L(\omega, \xi(s)/\epsilon, -\xi'(s))\, ds, \tag{6}$$

where $\xi(0) = 0$, $\xi(t) = y$. Letting $n = 1/\epsilon$, we study the limit $S_n(y,t)/n$, where:

$$S_n(y,t) = \inf_{\xi \in W^{1,\infty}([0,nt];R^N) : \xi(0)=0, \xi(nt)=n\,y} \int_0^{nt} L(\omega, \xi(s), -\xi'(s))\, ds. \tag{7}$$

Subadditive Ergodic Theorem and Convergence of S_n/n to a Nonrandom Number.

Define:

$$S_{m,n}(y,t) = \inf_{\xi(mt)=m\,y;\xi(nt)=n\,y} \int_{mt}^{nt} L(\omega, \xi(s), -\xi'(s))\,ds. \tag{8}$$

with inf taken over all paths ξ connecting my to ny in time $(n-m)t$.

SET(Kingman): Suppose that $S_{m,n}$ are random variables satisfying:

(1) $S_{0,0} = 0$, $S_{m,n} \leq S_{m,k} + S_{k,n}$, for $m \leq k \leq n$;

(2) $\{S_{m,m+k},\ m \geq 0,\ k \geq 0\}$ equals $\{S_{m+1,m+k+1}, m \geq 0, k \geq 0\}$ in distribution;

(3) $E[S_{0,1}^{+}] < +\infty$; $\alpha_n \equiv E[S_{0,n}] < \infty$;

then:

(1)

$$\alpha = \lim_{n\to\infty} \frac{\alpha_n}{n} = \inf_{n\geq 1} \frac{\alpha_n}{n} \in [-\infty, \infty);$$

(2) $S_\infty = \lim_{n\to\infty} \frac{S_{0,n}}{n}$ exists with probability one;

(3) if $E[S_\infty] = \alpha > -\infty$, $\lim_{n\to\infty} E[|S_{0,n}/n - S_\infty|] = 0$.

Condition (1) follows directly from infemum of the path integral.

Condition (2) follows from stationarity.

Condition (3) requires further knowledge of “unboundedness” of the random Lagrangian.

If H is uniformly bounded in x (for any p), then L is uniformly bounded in x. All moment assumptions hold, SET implies the existence of the limit S_n/n with probability one. The limit is invariant under translations of the realization of L by vectors proportional to x, hence its value is almost surely nonrandom (ergodicity).

By absorbing t into n, we see that the limit is of the form: $t\bar{L}(\frac{y}{t})$, where $\bar{L}$ (homogenized Lagrangian) is convex in y/t (velocity) due to convexity of L in ξ' (velocity) and subadditivity of S_n.

The homogenization limit of HJ solution from planar data at $x = 0$ is:

$$\bar{u}(0,t) = \inf_{y}(p \cdot y + t\bar{L}(y/t)) = t \inf_{y'}(p \cdot y' + \bar{L}(y')) \equiv -t\bar{H}(p). \tag{9}$$

This further implies that in general the homogenized eqn is $\bar{u}_t + \bar{H}(\nabla_x \bar{u}) = 0$.

Uniform boundedness is however not sharp for HJ homogenization, and not true for Gaussian processes. Unboundedness is an indication of **extreme behavior of a stochastic system**.

Example I: $H = |p|^2/2 + V(x,\omega)$, suppose that V is uniformly bounded from above by V_0 (but no uniform lower bound), and that $E[|V(x)|] < \infty$. Then homogenization still holds.

Lagrangian $L(x,q) = \frac{|q|^2}{2} - V(x,\omega)$. For a linear path $\xi(s) = s\,\frac{y}{t}$, we obtain

$$S_{0,n} \le \frac{n\,|y|^2\,t}{2} - \int_0^{nt} V(\frac{s\,y}{t})\,ds.$$

Integrability of $V(x)$ implies that

$$\mathbf{E}[|S_{0,n}(t,y)|] < +\infty.$$

On the other hand, since the kinetic part of Lagrangian integral is positive, upper bound on V implies that

$$E[S_{0,n}(t,y)] \ge -n\,t\,V_0.$$

The family $S_{m,n}$ satisfies the assumptions of subadditive ergodic theorem, implying existence of the finite limit of $\frac{S_{0,n}}{n}$.

Unbounded Potentials and Running Maxima

Let V be an Ornstein-Uhlenbeck process ($N = 1$), the running maximum $V^*(y) = \sup_{u\in[0,y]} V(u)$ obeys:

Theorem 1 (Cramér)

$$\text{Prob}\,[\frac{V^*(y) - b_y}{a_y} \leq x] \to \exp(-e^{-x})$$

as $y \to \infty$, where

$$a_y^{-1} \sim b_y \sim (2\log y)^{\frac{1}{2}}$$

with $\sim$ denoting asymptotic equivalence.

Similar results hold in $N \geq 2$ (Leadbetter et al).

The theorem says that the renormalized random variables $V^*(y)$ converge in distribution to the double exponential distribution. It follows that

$$\frac{V^*(y)}{(2\log y)^{\frac{1}{2}}} \to 1, \quad \text{as } y \to \infty,$$

in probability.

Scaling $\sqrt{2\log n}$ for running max of i.i.d. sequence X_n:

$$
\begin{aligned}
Prob(\max_{1\le i\le n}\{X_1,\cdots,X_n\}\le x) &= Prob(X_1\le x, X_2\le x,\cdots,X_n\le x)\\
&= [Prob(X_1\le x)]^n
\end{aligned}
$$

which converges to zero as $n\to\infty$ for any finite x. Almost surely the running max is unbounded!

If X_1 is unit Gaussian, $x=\sqrt{2\log n}$,

$$
[Prob(X_1\le x)]^n = \left(1-(2\pi)^{-1/2}\int_{\sqrt{2\log n}}^{\infty}\exp\{-y^2/2\}\,dy\right)^n
$$

$$
\approx (1-const./n)^n,
$$

converging to a positive finite number, suggesting running max scales as $O(\sqrt{2\log n})$.

Unboundedness of V from Above and Breakdown of Homogenization

$N = 1$, minimum action integral is:

$$S_n = \inf_{u(\tau)} \int_0^{nt} [\frac{1}{2}(\frac{du}{d\tau})^2 - V(u(\tau))]\, d\tau,$$

over C^1 functions $u(\tau)$: $u(0) = 0$ and $u(nt) = nx$. The minimizing function u is monotone increasing: $\frac{du}{d\tau} \neq 0$ for all τ.

Rewrite S_n in inverse function $\tau(u)$:

$$S_n = \inf_{\tau(u)} \int_0^{nx} [\frac{1}{2}(\frac{d\tau}{du})^{-1} - V(u)\frac{d\tau}{du}]\, du,$$

over all C^1 functions $\tau(u)$: $\tau(0) = 0$ and $\tau(nx) = nt$.

Conservation of energy gives:

$$\frac{d\tau}{du} = \frac{1}{\sqrt{2[E_n - V(u)]}},$$

E_n: the total energy, $E_n > \max_{u \in [0,nx]} V(u)$.

It follows that

$$S_n = \int_0^{nx} [\sqrt{\frac{E_n - V(u)}{2}} - \frac{V(u)}{\sqrt{2(E_n - V(u))}}] \, du, \tag{10}$$

and

$$\int_0^{nx} \frac{1}{\sqrt{2(E_n - V(u))}} \, du = nt. \tag{11}$$

If $E_n > \sup_{u \in [0,nx]} V(u) + \frac{x^2}{2t^2}$, then $\int_0^{nx} \frac{1}{\sqrt{2(E_n - V(u))}} \, du < nt$, contradicting (11).
Hence $E_n \leq \sup_{u \in [0,nx]} V(u) + \frac{x^2}{2t^2}$, and

$$E_n \sim \sup_{u \in [0,nx]} V(u), \quad n \gg 1.$$

Rewrite from (10):

$$S_n = \int_0^{nx} [\sqrt{\frac{E_n - V(u)}{2}} + \frac{E_n - V(u)}{\sqrt{2(E_n - V(u))}} - \frac{E_n}{\sqrt{2(E_n - V(u))}}] \, du. \tag{12}$$

First two terms of integrand in (12) are identical. Jensen's inequality (concavity of root function):

$$\frac{1}{nx}\int_0^{nx}\sqrt{2(E_n - V(u))}\,du \quad\leq\quad \sqrt{\frac{1}{nx}\int_0^{nx} 2(E_n - V(u))\,du}$$
$$= \quad \sqrt{2E_n - 2\frac{1}{nx}\int_0^{nx} V(u)\,du},$$

where almost surely as $n \to \infty$, $\frac{1}{nx}\int_0^{nx} V(u)\,du \to 0$, by ergodicity of the Ornstein-Uhlenbeck process.

The integral of the first two terms is bounded by $3\sqrt{E_n}nx$ for large n. The integral of the third term is $-E_n nt$. So for n large:

$$-tE_n \leq \frac{1}{n}S_n \leq 3x\sqrt{E_n} - tE_n,$$

or:

$$\frac{S_n}{n} \sim -tE_n \sim -t\sup_{u\in[0,nx]} V(u) = -tV^*(nx) \to -\infty.$$

Standard homogenization limit fails. Instead, the modified limit holds in probability:

$$\frac{S_n}{n(2\log n)^{1/2}} \to -t.$$

Divergence of homogenization means that for affine data, the growth rate of HJ solutions in time is faster than linear, or front acceleration. Similar divergence occurs in $N \geq 2$.

For classical mechanics Hamiltonian, stochastic homogenization holds if and only if the potential is uniformly bounded from above.

For **advective Hamiltonian, upper bound may or may not be the criterion**.

Example IIa (H in Gradient Flows): $H(x,p) = \frac{p^2}{2} + p \cdot b(x)$, $b = \nabla U(x)$, where $U(x)$ is a scalar random vector field whose realizations are of class $\mathbf{C^2}$, p, $x \in R^N$. Such fields b include Gaussian random fields with appropriate covariance.

Corresponding Lagrangian is

$$L(x,q) = \frac{1}{2}|q - b(x)|^2 = \frac{|q|^2}{2} - q \cdot b(x) + \frac{|b(x)|^2}{2}. \tag{13}$$

For a path $\xi(s),\ 0 \le s \le n\,t$, such that $\xi(0) = 0$ and $\xi(n\,t) = n\,x$, the contribution from the second term to Lagrangian integral equals

$$\int_0^{nt} b(\xi(s)) \cdot \xi'(s)\, ds = U(nx)$$

Such a path independent term is called a null Lagrangian. Thus Lagrangian (13) leads to the same Euler-Lagrange equations of motion for the minimizing path as the Lagrangian of the potential system:

$$L_1(x,q) = \frac{|q|^2}{2} - V_1(x),$$

where $V_1(x) = -\frac{1}{2}|b|^2(x) \le 0$. By result of Example I, **homogenization holds** for such Hamiltonian of advection type, **even though flow field is unbounded** !

Example IIb (H in shear flows): $b(x) = (V_2(x'), 0)$, $x' = (x_2, \cdots, x_N)$, $0 \in R^{N-1}$.

HJ Eqn is:

$$u_t + b(x) \cdot \nabla_x u + |\nabla_x u|^2/2 = 0. \tag{14}$$

Consider a front moving in the x_1 direction in the form $u(x,t) = x_1 - \frac{1}{2}t + w(x',t)$. Then w satisfies HJ eqn:

$$w_t + |\nabla_{x'} w|^2/2 + V_2(x') = 0, \tag{15}$$

which is in the classical potential form.

If V_2 obeys the assumptions in Example I and is unbounded from above, $-w(x',t)/t$ diverges for large time t. Front speed acceleration appears due to dominance of running maxima of process V_2.

Summary of HJ Fronts in Random Media:

- **Homogenization**: behavior of the system on large scales is described by an effective nonrandom Lagrangian (or Hamiltonian). **Disorder gets averaged, extreme nature of the random media is tamed, fronts move at asymptotically constant speeds**. An elementary (and linear) analog of this phenomenon in classical probability theory is the strong **law of large numbers**.

- **Domination by finite-volume maxima of the random potential**: **Homogenization breaks down**. On an arbitrarily large scale, behavior of the system is dictated by the maximum value of the disorder on that scale. **The extreme nature of the random media prevails**. Analogue in classical probability theory is the study of extrema of stochastic sequences and processes. For example, the maximum M_n of n independent unit normal random variables behaves asymptotically as $\sqrt{2\log n}$ as $n \to \infty$.

The convergence (homogenization) and divergence (dominance by running maxima) phenomena of stochastic H-J was noted earlier in KPP (Xin 2003).

The divergence of HJ homogenization and one-sided condition of random potentials for HJ homogenization are based on E-Wehr-Xin (2008).

G-equation and Control Representation:

Suppose u is the viscosity solution of

$$u_t + V(x)\nabla u = s_l|\nabla u|, \quad u(x,0) = u_0(x), \tag{16}$$

then:

$$u(x,t) = \sup u_0(y(t)) \tag{17}$$

where the supremum is over all $y \in W^{1,\infty}([0,t];\mathbb{R}^d)$ satisfying $y(0) = x$ and the constraint

$$|y'(\tau) + V(y(\tau))| \leq s_l \tag{18}$$

for all $\tau \in [0,t]$.

Suppose $w(x,t)$ satisfies:

$$w_t + V(x)\nabla w = s_l|\nabla w| + \kappa\Delta w, \quad w(x,0) = w_0(x), \tag{19}$$

then $w(x,t)$ has the representation:

$$w(x,t) = \sup \mathbb{E}\left[u_0(Y_t)\right] \tag{20}$$

where Y_t satisfies the stochastic equation

$$dY_\tau = (\alpha_\tau - V(Y_\tau))\; d\tau + \sqrt{2\kappa}\, dW_\tau, \qquad Y_0 = x. \tag{21}$$

The supremum in (20) is over all stochastic controls α_τ which are progressively measureable with respect to the Brownian filtration, and satisfy $|\alpha_\tau| \leq s_l$.

We shall use these control formulas to analyse front speeds. The following is based on joint work with J. Nolen (2009).

G-eqn and Spatial Flow: $V = V(x)$

$$u_t + V(x) \cdot \nabla_x u = s_l |\nabla_x u|,$$

with affine initial data $u(x, 0) = p \cdot x$, where p a unit vector in R^n.

Control formula of solution is:

$$u(x, t) = \max_{\alpha} p \cdot y,$$

where $y = y(t; x)$ is the solution to the ODE with continous control α, $|\alpha| \leq s_l$:

$$y'(\tau) = -V(y(\tau)) + \alpha(\tau), \tag{22}$$

and initial data $y(0) = x$.

Gradient flow

Let $p = 1$, by ODE comparison argument, $\alpha = s_l$ is the optimal control facilitating largest velocity to the right. Solution is $u(x,t) = y$, where $y = y(t;x)$ solves:

$$y'(\tau) = -V(y(\tau)) + s_l, \tag{23}$$

with initial data $y(0) = x$. Two regimes arise.

If $\min -V(y) + s_l > 0$ for all y, closed form solution:

$$\int_x^y \frac{d\eta}{-V(\eta) + s_l} = t. \tag{24}$$

Let $t \longrightarrow +\infty$:

$$s_T = \lim_{t\to\infty} u(x,t)/t = \lim_{t\to\infty} y(t)/t = 1/E[(-V(\cdot) + s_l)^{-1}], \tag{25}$$

almost surely for a stationary ergodic process V. Bounding harmonic mean by arithmetic mean: $s_T \leq s_l - E[V]$.

Front speed slows down if $E[V] = 0$.

$-V + s_l$ changes sign:

$-V + s_l$ has discrete zeros where $-V'$ changes sign alternately. These zeros are stable and unstable equilibria of characteristic eqn (23).

Then $y(t)$ will converge to a nearest stable equilibriam from starting point x. Implying that $y(t)$ is uniformly bounded in t.

Front asymptotic speed $s_T = 0$, *front trapping* occurs!

In contract, KPP fronts slow down in gradient flows but are never trapped (Nolen-Xin, 2009).

Divergence in Shear flow $V = (v(y_2), 0)$, $(y_1, y_2) \in R^2$, $v(y_2)$ is a mean zero stationary ergodic process. Choose control as:

$$\begin{aligned} \dot{y}_1 &= -v(y_2) + s_l - s_l\, \epsilon\, (\xi - x_2) e^{-\epsilon\, s_l\, t}, \\ \dot{y}_2 &= s_l\, \epsilon\, (\xi - x_2) e^{-\epsilon\, s_l\, t}, \end{aligned}$$

where ξ is a point so that $-v(\xi) > 0$; $\epsilon \ll 1$. Then as $t \to \infty$:

$$y_2(t) = x_2 + (\xi - x_2)\,(1 - e^{-\epsilon\, s_l\, t}) \to \xi,$$

$$\begin{aligned} y_1(t) &= x_1 - \int_0^t v(y_2(\tau))\, d\tau + s_l\, t + O(1), \\ &= -v(\xi) t + s_l t + O(1), \end{aligned} \tag{26}$$

implying that $y_1/t \to -v(\xi) + s_l$.

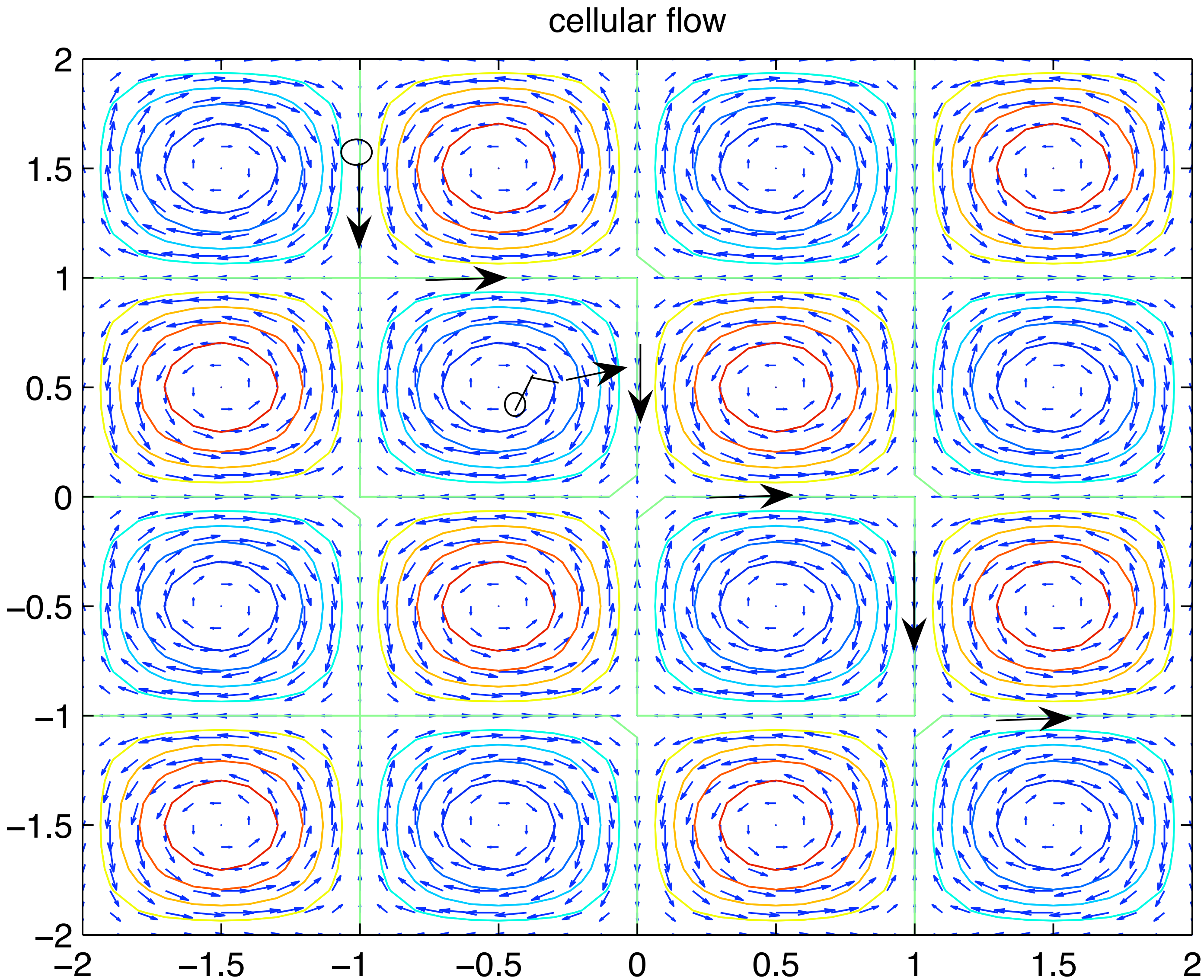
Choose ξ to reach maximum of $-v$ on $[x_2 - L, x_2 + L]$:

$$s_T \geq s_l + \sup_L \max\nolimits_{y \in [-L, L]} - v(y). \tag{27}$$

For unbounded Gaussian process v, s_T diverges as $L \to \infty$, same as in quadratic HJ.

Cell Flows (Hamiltonian Flows with Periodic Array of Vortices)

$$V(x,y) = \text{const}\ \nabla^{\perp} H,\ \ H = \sin(\pi x)\sin(\pi y).$$



A nearly optimal trajectory to maximize velocity in x is to follow the flow downward and right along separatrices, and make turns with the help of control. Without the control, the trajectory would get stuck at a saddle point!

The time to go from a small neighborhood of order $O(q)$ of a saddle point to that of the next saddle point without control is:

$$\int_q^{1-q} dx/A\sin(\pi x) = O(|\log q|/A), \quad A \gg 1,\ q \ll 1.$$

Choosing $q = s_L/(A\log A)$, the control is dominant over cell flow in $O(q)$ neighborhood of the saddle. So time to make a turn around a saddle is $O(q/s_L)$. Total travel time from saddle to saddle is $O(\log A/A)$, hence asymptotic velocity is $s_T = O(A/\log A)$.

If the starting point x is not on a separatrix, it takes finite amount of time for the control to move the particle out of the area of closed streamlines to a nearby separatrix:

$$\alpha(s) = -s_l \frac{\nabla\mathcal{H}(y(s))}{|\nabla\mathcal{H}(y(s))|} \cdot sign(\mathcal{H}(y(s)))$$

until the particle reaches a separatrix then follow the above optimal path to reach the same order of s_T.

- The enhancement $O(A/\log A)$ of inviscid G-front speed in cell flows was shown for initial starting point x on a separatrix in A. Oberman's Thesis (2001) using control representation.
- Abel, Cencini, Vergni, Vulpiani, obtained similar result using formally reduced front equation (2002).
- KPP speeds in cell flow obeys $A^{1/4}$ (Audoly-Berestycki-Pomeau 2000, Novikov-Ryzhik 2007), Ignition front speeds in cell flow behaves similarly, $O(A^p)$, $p \in (1/5, 1/4]$ (Kiselev-Ryzhik,2001).
- Lack of dissipation in standard (inviscid) G-equation may be the cause of discrepancy in speed asymptotics ?

Front Speeds of Viscous G-equation in Cellular Flows:

$$u_t + \delta V(x) \cdot \nabla u = s_l |\nabla u| + \kappa \Delta u, \tag{28}$$

where $(s_l, \kappa) \ll \delta$, with Lipschitz continuous initial data.

The traveling front solution of (28) is:

$$u = p \cdot x + H(p)\, t + w(x), \tag{29}$$

p is a unit vector in $\mathbb{R}^d$, and $w(x)$ is periodic. Equation for $(H, w(x))$:

$$H + \delta V(x) \cdot (p + \nabla_x w) = s_l |p + \nabla_x w| + \kappa \Delta_x w, \quad x \in \mathbb{T}, \tag{30}$$

where $\mathbb{T}$ is the periodic cell $[0,2] \times [0,2]$.

Eulerian view:

$$H(p) = s_l \, \langle |p + \nabla w| \rangle, \tag{31}$$

L^1 nonlinearity ! Jensen's inequality gives lower bound:

$$c^* = H(p)/|p| \geq s_l, \tag{32}$$

flow can only enhance the front speed.

Analysis of (H, w) equation shows that due to L^1:

Theorem 2 (Nolen-Xin 2009) *viscous G-front speed $H(p)$ grows slower than any positive power of δ as $\delta \gg 1$, if $s_l \ll \kappa$ independent of δ.*

In contrast, viscous quadratic HJ front speed grows at least $O(\delta^{1/4})$ (Novikov-Ryzhik 2007).

Lagrangian view: control representation for the viscous G-equation indicates that large Brownian fluctuation slows down the steering mechanism of control through the vortices. The particle trajectory has a very large probability of falling off a separatrix into rotating vortices and get temporally trapped there. Imagine: in a canoe/kayak slalom race, what a paddler has to do (control) to navigate through a 300m turbulent course to win if he constantly gets a large random kick ?!

Sketch of Proof of Upper Bound based on Cell Problem:

$$H^* + AV(x) \cdot (p + \nabla_x w) = R|p + \nabla_x w| + \Delta_x w, \quad x \in \mathbb{T}, \tag{33}$$

where $A = \delta/\kappa$ and $R = s_l/\kappa$ and $H^* = H/\kappa$. Front speed is $c^* = H^* \kappa/|p|$.

As (Novikov-Ryzhik 2007), decompose ($p = (\lambda, 0)$):

$$p \cdot x + w = \lambda\, x_1 + w = T + S \equiv \zeta, \tag{34}$$

where T is mean zero and solves the linear inhomogeneous problem:

$$\begin{aligned} &\Delta T - AV(x) \cdot \nabla T = 0, \\ &T(x_1 + 2, x_2) = T(x_1, x_2) + 2\lambda,\ T(x_1, x_2 + 2) = T(x_1, x_2), \end{aligned} \tag{35}$$

and S, a mean zero periodic function, solves the nonlinear problem:

$$\Delta S - AV(x) \cdot \nabla S = H^* - R|\nabla(T + S)|. \tag{36}$$

The linear problem (35) is well-studied and L^2 gradient estimate is:

$$C_1 A^{1/2} \lambda^2 \leq \int_{\mathbb{T}} |\nabla T|^2\, dx \leq C_2 A^{1/2} \lambda^2, \tag{37}$$

for positive constants C_1, C_2 independent of A.

Lemma: For any power $\beta > 0$, there is a constant C_β such that

$$\|\nabla T\|_{L^1(\mathbb{T})} \le C_\beta (1 + A^\beta)$$

holds for all $A > 1$.

Let $\Omega(h) = \{x \in \mathbb{T} \mid \; \mathcal{H}(x) < h\}$, $h = \epsilon^p$, $\epsilon = A^{-1}$, be the boundary region near separatrices. Then $|\Omega(h)| = O(\epsilon^p \log \epsilon)$. Cauchy-Schwarz implies for $q \in (0, p)$:

$$\int_{\Omega(h)} |\nabla T| \, dx \le C_1^{1/2} |\Omega(h)|^{1/2} \epsilon^{-1/4} \le C \epsilon^{q/2 - 1/4} \tag{38}$$

Let $D(h) = \mathbb{T} \setminus \Omega(h)$ denote the interior region, away from the separatrices. For any $r > 1$, the energy upper bound:

$$\int_{D(h)} |\nabla T|^2 \, dx \le \frac{C_r}{h} \left(\frac{\epsilon}{h^2} \right)^r \tag{39}$$

holds for a positive constant C_r.

if $h = \epsilon^p$ for any power $p < 1/2$,

$$\int_{D(\epsilon^p)} |\nabla T|^2 \, dx \leq C_p \tag{40}$$

for some constant C_p independent of $\epsilon \in (0,1)$. Thus,

$$\int_{D(\epsilon^p)} |\nabla T| \, dx \tag{41}$$

is uniformly bounded in ϵ. Combining this estimate with (38), we have for any $p < 1/2$,

$$\|\nabla T\|_{L^1(\mathbb{T})} = \int_{D(\epsilon^p)} |\nabla T| \, dx + \int_{\Omega(\epsilon^p)} |\nabla T| \, dx \leq C_p + C\epsilon^{q/2-1/4} \tag{42}$$

Choose p and q arbitrarily close to $1/2$, then for any $\beta > 0$,

$$\|\nabla T\|_{L^1(\mathbb{T})} \leq C_\beta(1 + \epsilon^{-\beta}) = C_\beta(1 + A^\beta) \tag{43}$$

This proves the lemma.

Proof of Theorem: function $\zeta = w + p \cdot x$ increases by λ across the cell in x_1. It follows:

$$
\begin{aligned}
H^*(p) \quad &= \quad R \int_{\mathbb{T}} |\nabla \zeta| \, dx/|\mathbb{T}| \quad \geq R \int_{\mathbb{T}} |\zeta_x| \, dx/|\mathbb{T}| \\
&\geq \quad R \int_{\mathbb{T}} \zeta_{x_1} \, dx/|\mathbb{T}| = R \, \lambda. \qquad (44)
\end{aligned}
$$

By Poincaré inequality and maximum principle, the ζ function in the cell $\mathbb{T}$ satisfies the upper bound:

$$\zeta \leq C[\lambda + \|\nabla \zeta\|_1], \qquad (45)$$

implying:

$$\zeta \leq C[\lambda + H^*/R] \leq C H^*/R, \qquad (46)$$

for a positive universal constant C.

The type of inequality (45) also applies to T and gives:

$$T \leq C[\lambda + \|\nabla T\|_1] \leq C_\beta A^\beta, \tag{47}$$

for all $A > 1$. Inequality (47) improves Lemma 1 of (Novikov-Ryzhik 2007) where the upper bound is $O(\lambda A^{1/4})$. It follows from (47) and (46) that

$$S \leq C(A^\beta + H^*/R). \tag{48}$$

By (31)-(34):

$$|\mathbb{T}|\, H^* = R\|\nabla(T+S)\|_1,$$

and so:

$$||\mathbb{T}|\, H^* - R\|\nabla T\|_1| \leq R\|\nabla S\|_1 \leq 2\, R\, \|\nabla S\|_2. \tag{49}$$

Gradient estimate of S: Multiplying S to (36), integrating over $\mathbb{T}$ and applying (48) gives:

$$\begin{aligned} \|\nabla S\|_2^2 &= R\int_{\mathbb{T}} S\,|\nabla(T+S)|\,dx \\ &\leq RC(A^\beta + H^*/R)\int_{\mathbb{T}} |\nabla\zeta|\,dx \\ &= C(A^\beta + H^*/R)H^*. \end{aligned} \tag{50}$$

It follows from (49) that:

$$\begin{aligned} |\mathbb{T}|\,H^* &\leq 2R\,\|\nabla S\|_2 + R\|\nabla T\|_1 \\ &\leq 2R\sqrt{C(A^\beta + H^*/R)H^*} + R\,C(1+A^\beta). \end{aligned} \tag{51}$$

Suppose $H^* > RA^\beta$. Then (51) implies

$$|\mathbb{T}|\, H^* \leq R\sqrt{2CH^*H^*/R} + R\,C(1 + A^\beta) \tag{52}$$

If $\sqrt{R} = \sqrt{s_l/\kappa}$ is sufficiently small (independently of A),

$$H^* \leq RC(1 + A^\beta).$$

Therefore, for any $\beta > 0$, there is a constant C such that

$$H^* \leq RC(1 + A^\beta), \tag{53}$$

holds for all $A \geq 1$. Consequently,

$$c^* = H/|p| = H^*\kappa/|p| \leq s_l C \left(1 + \left(\frac{\delta}{\kappa}\right)^\beta\right) \tag{54}$$

holds for $\delta > \kappa$.$\square$

Summary of noncoercive HJs and homogenization:

- **In non-coercive HJs due to unbounded random processes:** homogenization may **break down** (shear flows) or persist (gradient flows). A necessary and sufficient condition for homogenization is found for random unbounded Hamiltonian in classical mechanics, namely potential being bounded from above.

- There is **drastically different front speed asymptotics in cell flows of viscous and inviscid G-equations from KPP or ignition reaction or quadratic HJs**. The nonlinearity from L^2 to L^1 makes the key difference.

- **A tutorial presentation of reaction-diffusion, Burgers, HJ fronts in multiscale and random media is in my newly published Springer book at amazon.com for students and researchers at an affordable price $35.**

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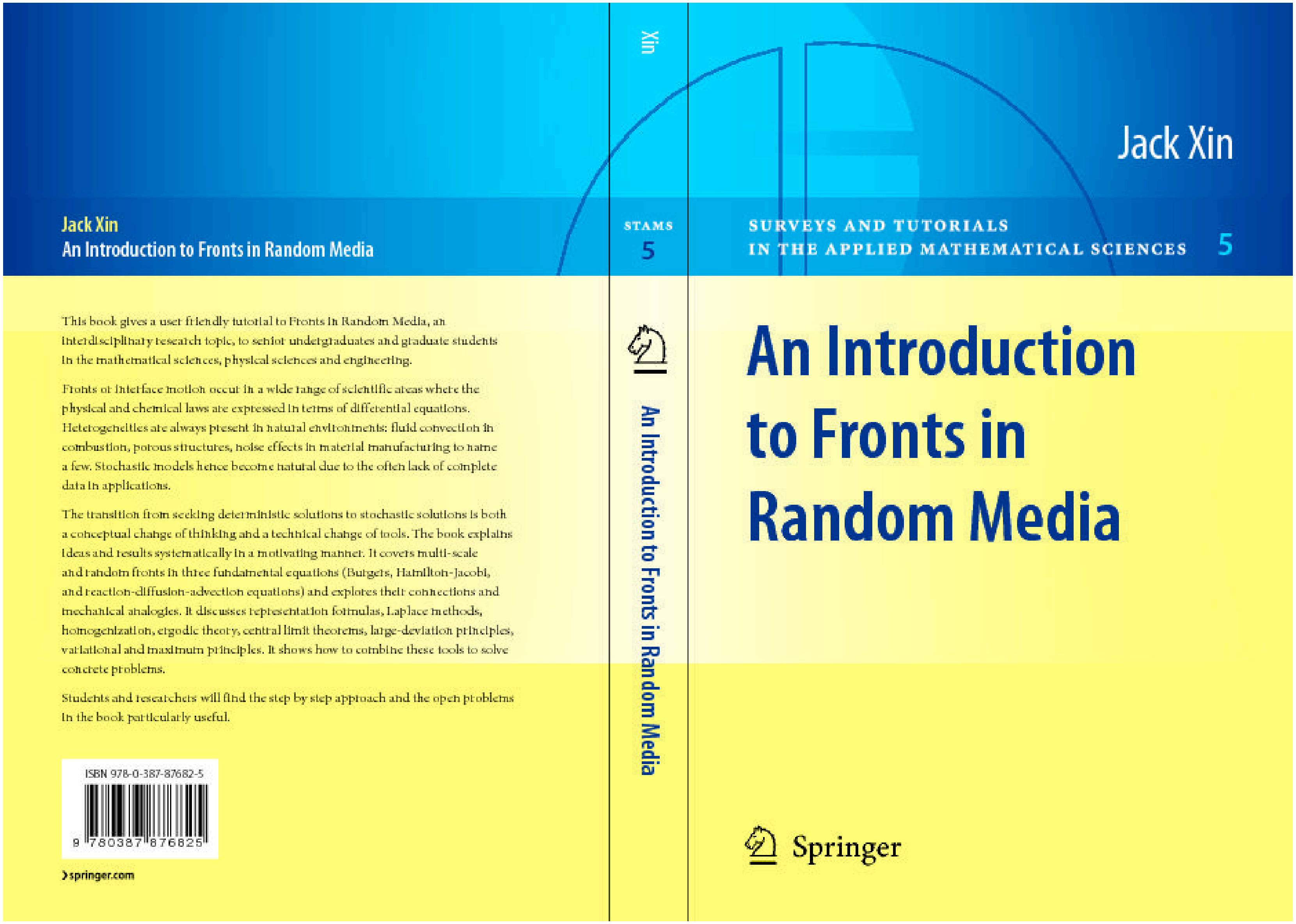

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