On the influence of random wind stress errors on the four dimensional, mid-latitude, ocean inverse problem

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- Introduction
- Background of 4DVAR analysis
- Experiments
- Discussions

4DVAR Ocean Circulation Estimation

Cost Function

$$J[\mathbf{x}_{4D}] = \Delta\mathbf{x}_0^{T}\mathbf{B}^{-1}\Delta\mathbf{x}_0 + \Delta\boldsymbol{\tau}^{T}\mathbf{Q}_{\tau}^{-1}\Delta\boldsymbol{\tau} + \Delta\mathbf{q}^{T}\mathbf{Q}_{q}^{-1}\Delta\mathbf{q}$$
$$+\left(\mathbf{y}-\mathbf{H}\mathbf{x}_{4D}\right)^{T}\mathbf{R}^{-1}\left(\mathbf{y}-\mathbf{H}\mathbf{x}_{4D}\right)$$

Constrain

$$\mathbf{M}\mathbf{x}_{4D} = \mathbf{f}$$

- Error covariance matrix Q for wind stress error is often modeled by a diagonal matrix.
- If we model Q by a non-diagonal matrix, what is its impact to the 4DVAR ocean circulation analysis?
- If the impact is significant, what is the dynamical reason behind?

4DVAR analysis : Dynamical Constraint

Linear Ocean General Circulation Model (OGCM)

$$
\begin{aligned}
\mathbf{x}_0 &= \mathbf{x}_{\text{init}} \\
\mathbf{x}_1 &= \mathbf{M}_{1,0}\mathbf{x}_0 + \mathbf{f}_1 \\
&\vdots \\
\mathbf{x}_K &= \mathbf{M}_{K,K-1}\mathbf{x}_{K-1} + \mathbf{f}_K
\end{aligned}
$$

Linear OGCM as a Dynamical Constraint

$$
\underbrace{\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 - \mathbf{M}_{1,0}\mathbf{x}_0 \\ \vdots \\ \mathbf{x}_K - \mathbf{M}_{K,K-1}\mathbf{x}_0 \end{bmatrix}}_{\mathbf{M}\mathbf{x}_{4D}} = \underbrace{\begin{bmatrix} \mathbf{x}_{\text{init}} \\ \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_K \end{bmatrix}}_{\mathbf{f}_{4D}} + \underbrace{\begin{bmatrix} \Delta\mathbf{x}_{\text{init}} \\ \Delta\mathbf{f}_1 \\ \vdots \\ \Delta\mathbf{f}_K \end{bmatrix}}_{\Delta\mathbf{f}_{4D}}
$$

: initial error ($\Delta\mathbf{x}_{\text{init}}$)

: external forcing (boundary value) error ($\Delta\mathbf{f}_1, \ldots, \Delta\mathbf{f}_K$)

$$
\mathbf{M}\mathbf{x}_{4D} = \mathbf{f}_{4D} + \Delta\mathbf{f}_{4D}
$$

4DVAR analysis : Cost Function

Dynamical Constraints

$$\mathbf{M}\,\mathbf{x} = \mathbf{f} + \Delta\,\mathbf{f}$$

Observational Constraints

$$\mathbf{H}\,\mathbf{x} = \mathbf{y} + \Delta\,\mathbf{y}$$

Cost Function (Estimator)

$$J[\mathbf{x}] = \frac{1}{2}\Delta\,\mathbf{f}^{\mathrm{T}}\mathbf{Q}^{-1}\Delta\,\mathbf{f} + \frac{1}{2}\Delta\,\mathbf{y}^{\mathrm{T}}\mathbf{R}^{-1}\Delta\,\mathbf{y}$$

Assumptions

$\left\langle \Delta\,\mathbf{f}\,\Delta\,\mathbf{f}^{\mathrm{T}} \right\rangle = \mathbf{Q}, \quad \left\langle \Delta\,\mathbf{f} \right\rangle = 0$: model error

$\left\langle \Delta\,\mathbf{y}\,\Delta\,\mathbf{y}^{\mathrm{T}} \right\rangle = \mathbf{R}, \quad \left\langle \Delta\,\mathbf{y} \right\rangle = 0$: measurement error

4DVAR analysis : Incremental Formulation of Cost Function

Background Estimation

$$\mathbf{M}\,\mathbf{x}^{\mathrm{b}} = \mathbf{f}$$

Analysis Increment and Innovation vector

$$\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}^{\mathrm{b}}, \quad \mathbf{d} = \mathbf{y} - \mathbf{H}\,\mathbf{x}^{\mathrm{b}}$$

Dynamical Constraints in Incremental Formulation

$$\mathbf{M}\,\Delta\mathbf{x} = \Delta\mathbf{f}$$

Observational Constraints in Incremental Formulation

$$\mathbf{H}\,\Delta\mathbf{x} - \mathbf{d} = \Delta\mathbf{y}$$

Cost Function

$$2\mathrm{J}[\mathbf{x}] = \Delta\mathbf{f}^{\mathrm{T}}\mathbf{Q}^{-1}\Delta\mathbf{f} + \Delta\mathbf{y}^{\mathrm{T}}\mathbf{R}^{-1}\Delta\mathbf{y}$$

4DVAR analysis : Incremental Formulation of Cost Function

Background Estimation

$$\mathbf{M}\,\mathbf{x}^{\mathrm{b}} = \mathbf{f}$$

Analysis Increment and Innovation vector

$$\Delta\,\mathbf{x} = \mathbf{x} - \mathbf{x}^{\mathrm{b}}, \quad \mathbf{d} = \mathbf{y} - \mathbf{H}\,\mathbf{x}^{\mathrm{b}}$$

Dynamical Constraints in Incremental Formulation

$$\mathbf{M}\,\Delta\,\mathbf{x} = \Delta\,\mathbf{f}$$

Observational Constraints in Incremental Formulation

$$\mathbf{H}\,\Delta\,\mathbf{x} - \mathbf{d} = \Delta\,\mathbf{y}$$

Cost Function in Incremental Formulation

$$\begin{aligned} 2\mathrm{J}[\Delta\,\mathbf{x}] &= \Delta\,\mathbf{x}^{\mathrm{T}}\,\mathbf{M}^{\mathrm{T}}\,\mathbf{Q}^{-1}\,\mathbf{M}\,\Delta\,\mathbf{x} \\ &\quad + \left(\mathbf{H}\,\Delta\,\mathbf{x} - \mathbf{d}\right)^{\mathrm{T}}\mathbf{R}^{-1}\left(\mathbf{H}\,\Delta\,\mathbf{x} - \mathbf{d}\right) \end{aligned}$$

4DVAR analysis : Optimal Solution

4DVAR Cost Function

$$2J[\Delta \mathbf{x}_{4D}] = \Delta \mathbf{x}_{4D}^{T} \mathbf{P}^{-1} \Delta \mathbf{x}_{4D} + \left(\mathbf{H}_{4D} \Delta \mathbf{x}_{4D} - \mathbf{d}_{4D}\right)^{T} \mathbf{R}_{4D}^{-1} \left(\mathbf{H}_{4D} \Delta \mathbf{x}_{4D} - \mathbf{d}_{4D}\right)$$

$$\mathbf{P}^{-1} = \mathbf{M}^{T} \mathbf{Q}^{-1} \mathbf{M}$$

3DVAR (OI:Optimal Interpolation) Cost Function

$$2J[\Delta \mathbf{x}_{3D}] = \Delta \mathbf{x}_{3D}^{T} \mathbf{B}^{-1} \Delta \mathbf{x}_{3D} + \left(\mathbf{H}_{3D} \Delta \mathbf{x}_{3D} - \mathbf{d}_{3D}\right)^{T} \mathbf{R}_{3D}^{-1} \left(\mathbf{H}_{3D} \Delta \mathbf{x}_{3D} - \mathbf{d}_{3D}\right)$$

Optimal Solution

$$\Delta \mathbf{x}_{4D} = \mathbf{P} \mathbf{H}_{4D}^{T} \left(\mathbf{H}_{4D} \mathbf{P} \mathbf{H}_{4D}^{T} + \mathbf{R}_{4D}\right)^{-1} \mathbf{d}_{4D}$$

4DVAR analysis : Role of P-matrix and Q-matrix

Optimal Solution

$$\Delta \mathbf{x}_{4D} = \underbrace{\mathbf{P}}_{\text{model space}} \underbrace{\mathbf{H}_{4D}^{T} \left(\mathbf{H}_{4D} \mathbf{P} \mathbf{H}_{4D}^{T} + \mathbf{R}_{4D} \right)^{-1} \mathbf{d}_{4D}}_{\text{data space}}$$

- P-matrix has all information about how the data are extrapolated to its surrounding area/time.

P-matrix

$$\mathbf{P}^{-1} = \mathbf{M}^{T} \mathbf{Q}^{-1} \mathbf{M}$$

$$\mathbf{M} \mathbf{P} \mathbf{M}^{T} = \mathbf{Q}$$

- P-matrix is a function of Q-matrix.
  - ► in 4DVAR, (4D) background error covariance matrix (P-matrix) is parameterized by Q-matrix

4DVAR analysis : Two approaches to evaluate P-matrix

P-matrix defined by ensemble average

$$\mathbf{P} = \left\langle (\mathbf{x} - \mathbf{x}^{\mathrm{b}})(\mathbf{x} - \mathbf{x}^{\mathrm{b}})^{\mathrm{T}} \right\rangle$$

$$= \lim_{M \to \infty} M^{-1} \sum_{m=1}^{M} (\mathbf{x}^{(m)} - \mathbf{x}^{\mathrm{b}})(\mathbf{x}^{(m)} - \mathbf{x}^{\mathrm{b}})^{\mathrm{T}}$$

$\mathbf{M}\,\mathbf{x}^{\mathrm{b}} = \mathbf{f}$: background estimation (deterministic)

$\mathbf{M}\,\mathbf{x}^{(m)} = \mathbf{f} + \Delta\mathbf{f}^{(m)}$: perturbed solution (stochastic)

$$\left\langle \Delta\mathbf{f}\,\Delta\mathbf{f}^{\mathrm{T}} \right\rangle = \mathbf{Q}, \quad \left\langle \Delta\mathbf{f} \right\rangle = 0$$

Inverse of P-matrix

$$\mathbf{P}^{-1} = \mathbf{M}^{\mathrm{T}} \mathbf{Q}^{-1} \mathbf{M}$$

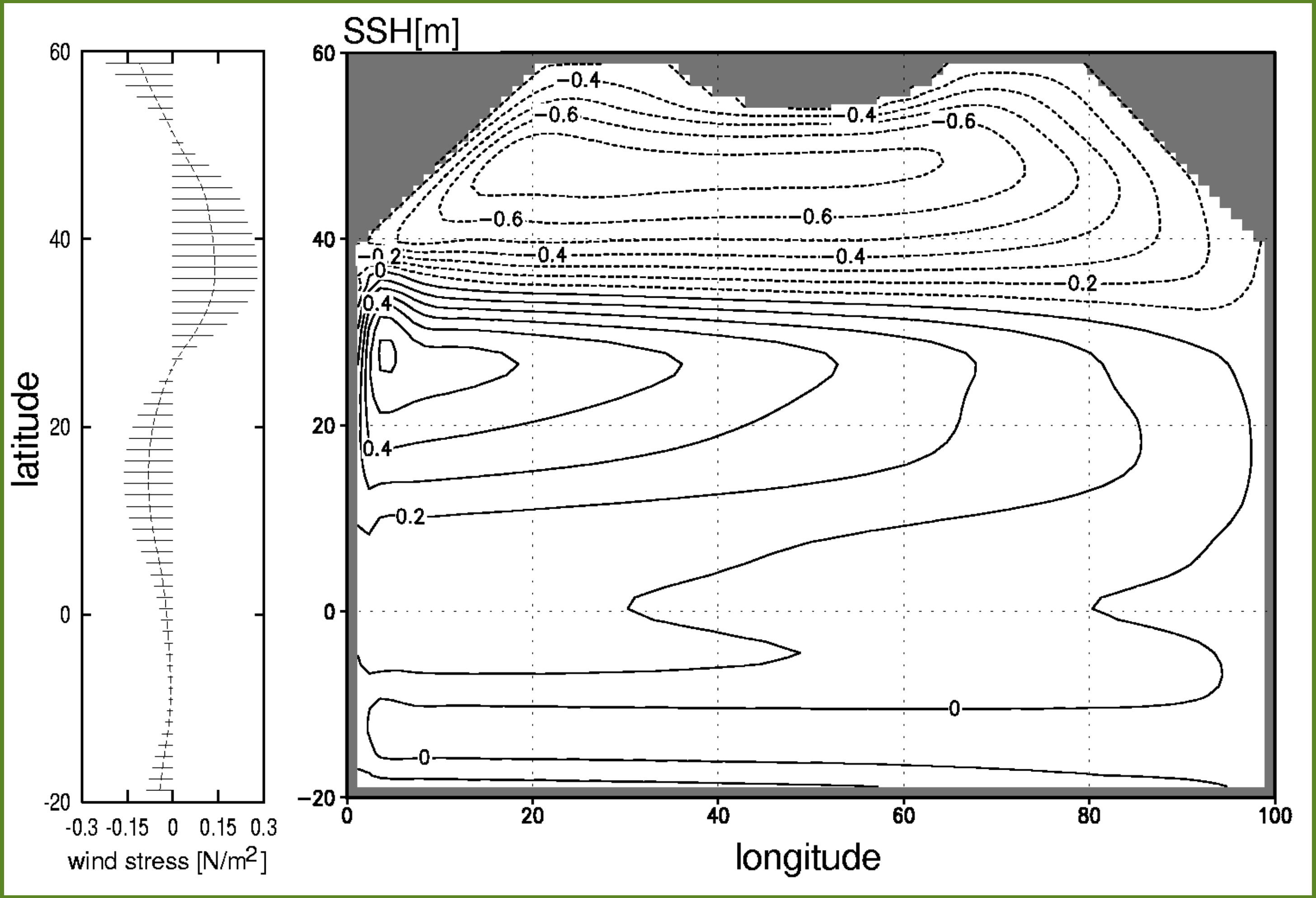
Experiments : Settings

Cost Function

$$J[\mathbf{x}_{4D}] = \Delta\boldsymbol{\tau}^{T}\mathbf{Q}_{\tau}^{-1}\Delta\boldsymbol{\tau} + (\boldsymbol{\eta} - \mathbf{H}\mathbf{x}_{4D})^{T}\mathbf{R}^{-1}(\boldsymbol{\eta} - \mathbf{H}\mathbf{x}_{4D})$$

Constrain

$$\mathbf{M}\mathbf{x}_{4D} = \mathbf{f} + \Delta\mathbf{f}$$: tangent linear ocean model



period: 1 year
resolution: 1.2 degree

$$\rightarrow \mathbf{P} = F[\mathbf{Q}_{\tau}]$$

Experiments : Wind stress error covariance model

Wind Stress Error

$$\Delta \boldsymbol{\tau}(\boldsymbol{r}, t) = (\Delta \tau_x(\boldsymbol{r}, t), 0)$$

$$\langle \Delta \tau_x(\boldsymbol{r}_1, t_1) \Delta \tau_x(\boldsymbol{r}_2, t_2) \rangle = Q_\tau(\boldsymbol{r}_1, t_1; \boldsymbol{r}_2, t_2)$$

$$\langle \Delta \tau_x(\boldsymbol{r}, t) \rangle = 0$$

Wind Stress Error Covariance Model

$$Q_\tau(\boldsymbol{r}_1, t_1; \boldsymbol{r}_2, t_2) = \sigma(\boldsymbol{r}_1)\, \rho(\tilde{\boldsymbol{r}}, \tilde{t})\, \sigma(\boldsymbol{r}_2)$$

$$\tilde{\boldsymbol{r}} = \boldsymbol{r}_1 - \boldsymbol{r}_2, \tilde{t} = t_1 - t_2$$

Wind Stress Error Correlation Model

$$\rho(\tilde{\boldsymbol{r}}, \tilde{t}) = \exp\left(-\frac{\tilde{x}^2}{L_x^2}\right) \exp\left(-\frac{\tilde{y}^2}{L_y^2}\right) \exp\left(-\frac{\tilde{t}}{L_t}\right)$$

Experiments : Wind stress error covariance model

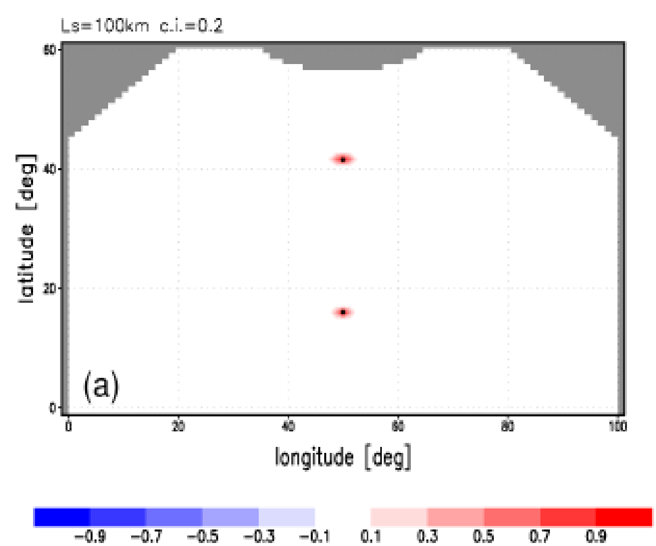
Wind Stress Error Covariance Model

$$Q_\tau(\boldsymbol{r}_1, t_1; \boldsymbol{r}_2, t_2) = \sigma(y_1)\,\rho(\tilde{\boldsymbol{r}}, \tilde{t})\,\sigma(y_2)$$

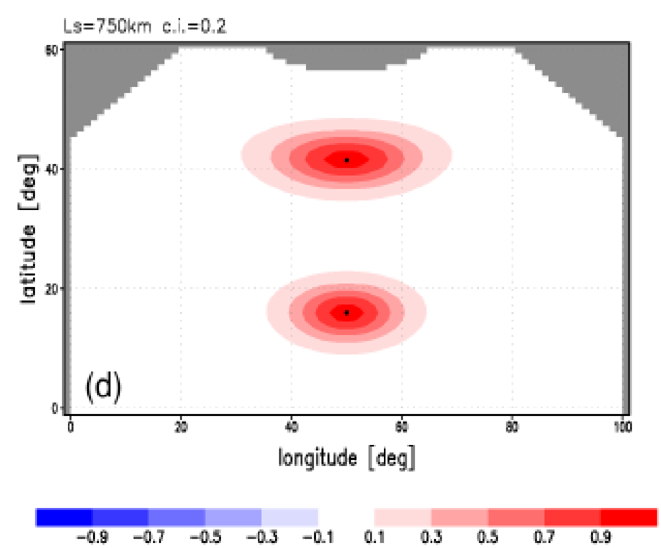
Wind Stress Error Correlation Model

$$\rho(\tilde{\boldsymbol{r}}, \tilde{t}) = \exp\left(-\frac{\tilde{x}^2}{L_x^2}\right)\exp\left(-\frac{\tilde{y}^2}{L_y^2}\right)\exp\left(-\frac{\tilde{t}}{L_t}\right)$$

$Ls = \Delta r$ (near $-$ diagonal)

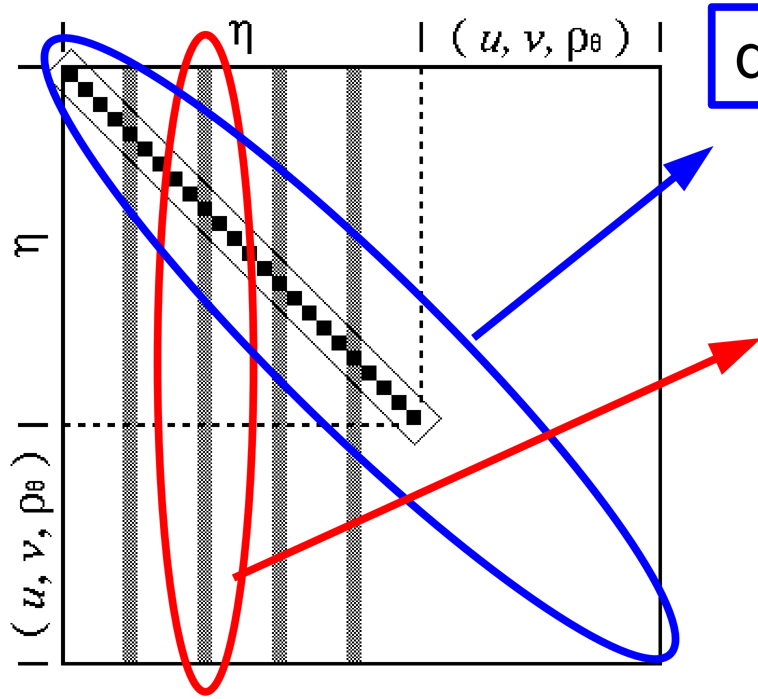


$Ls = 750$km (non $-$ diagonal)



Experiments : Subspace of P-matrix

P

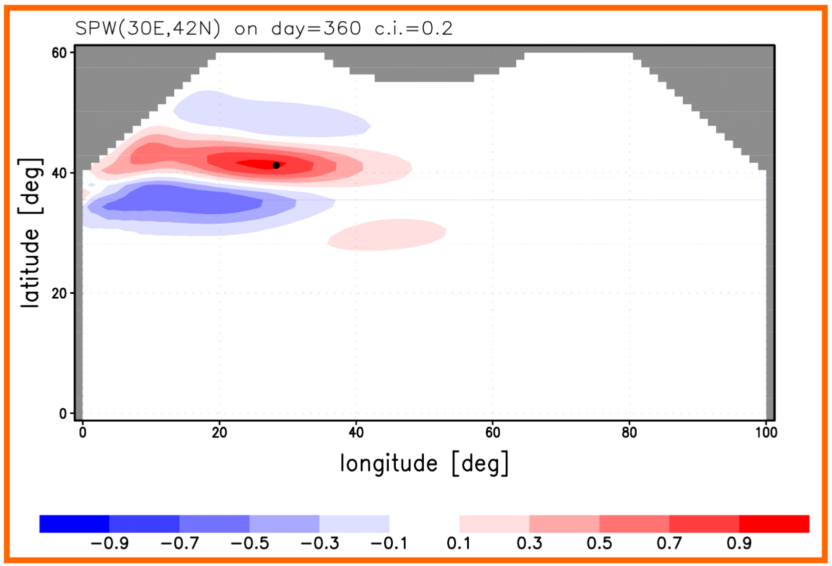


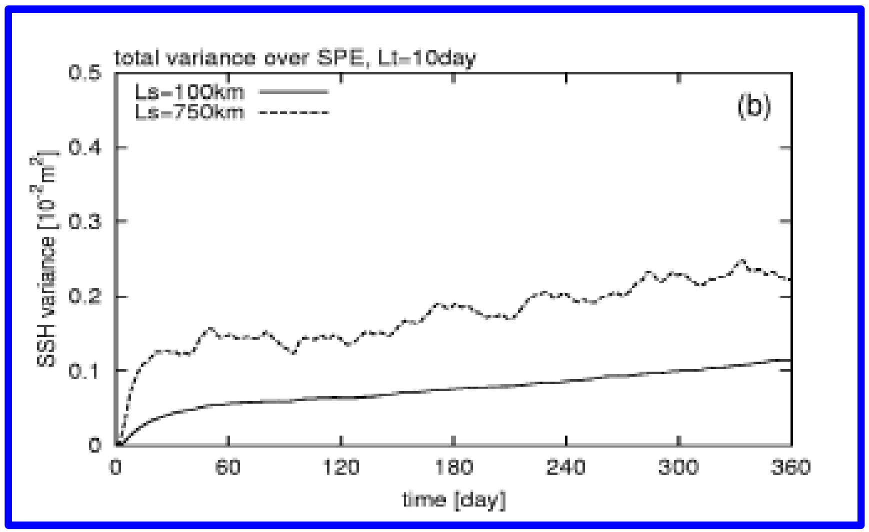
diagonal component=error variance

$$\sigma^2 = \left\langle \left(\mathbf{x} - \mathbf{x}^{\mathrm{b}} \right)^2 \right\rangle$$

column of P-matrix=representer

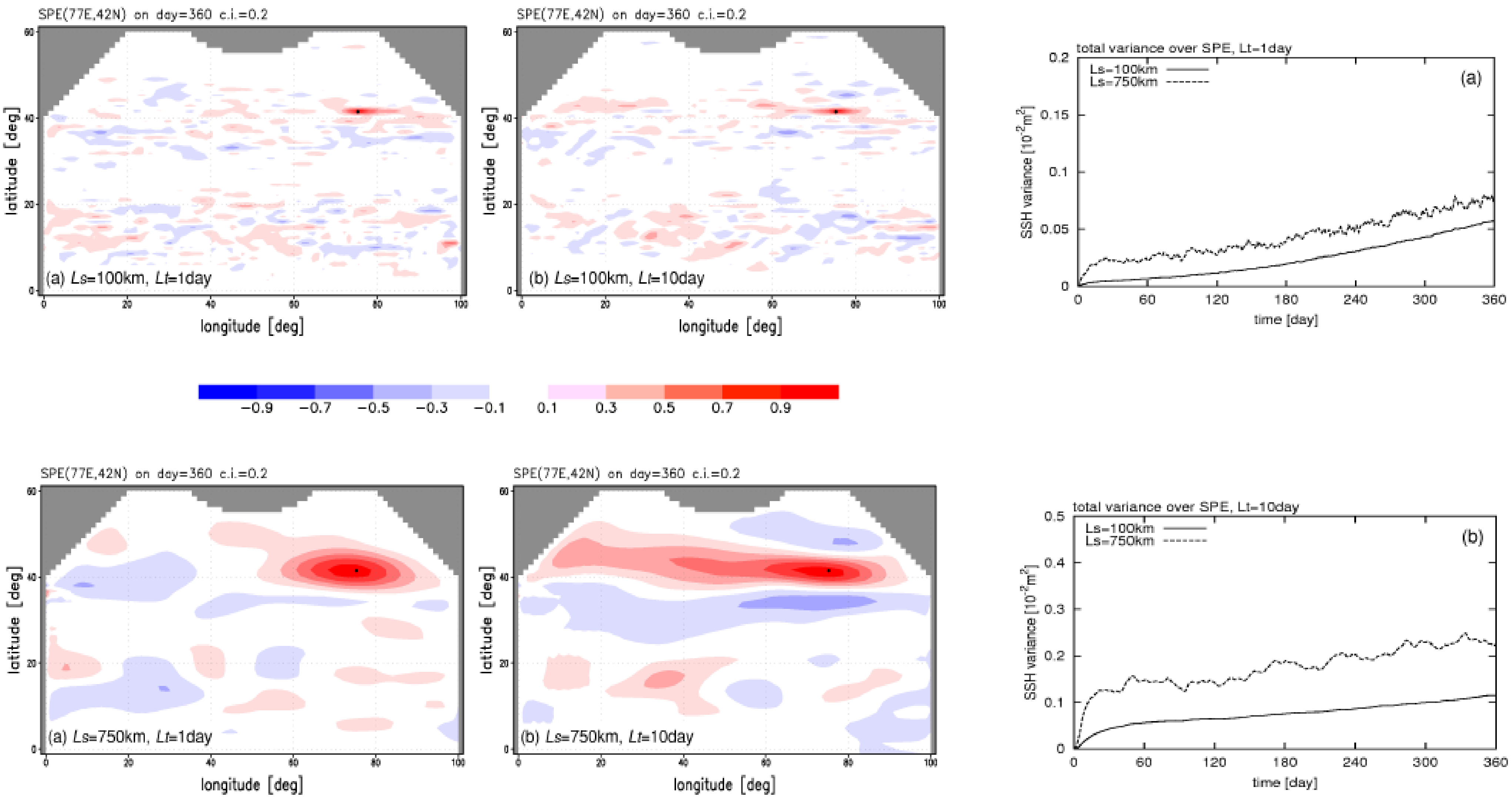
$$\mathbf{p}_m = \mathbf{P}\,\mathbf{h}_m$$





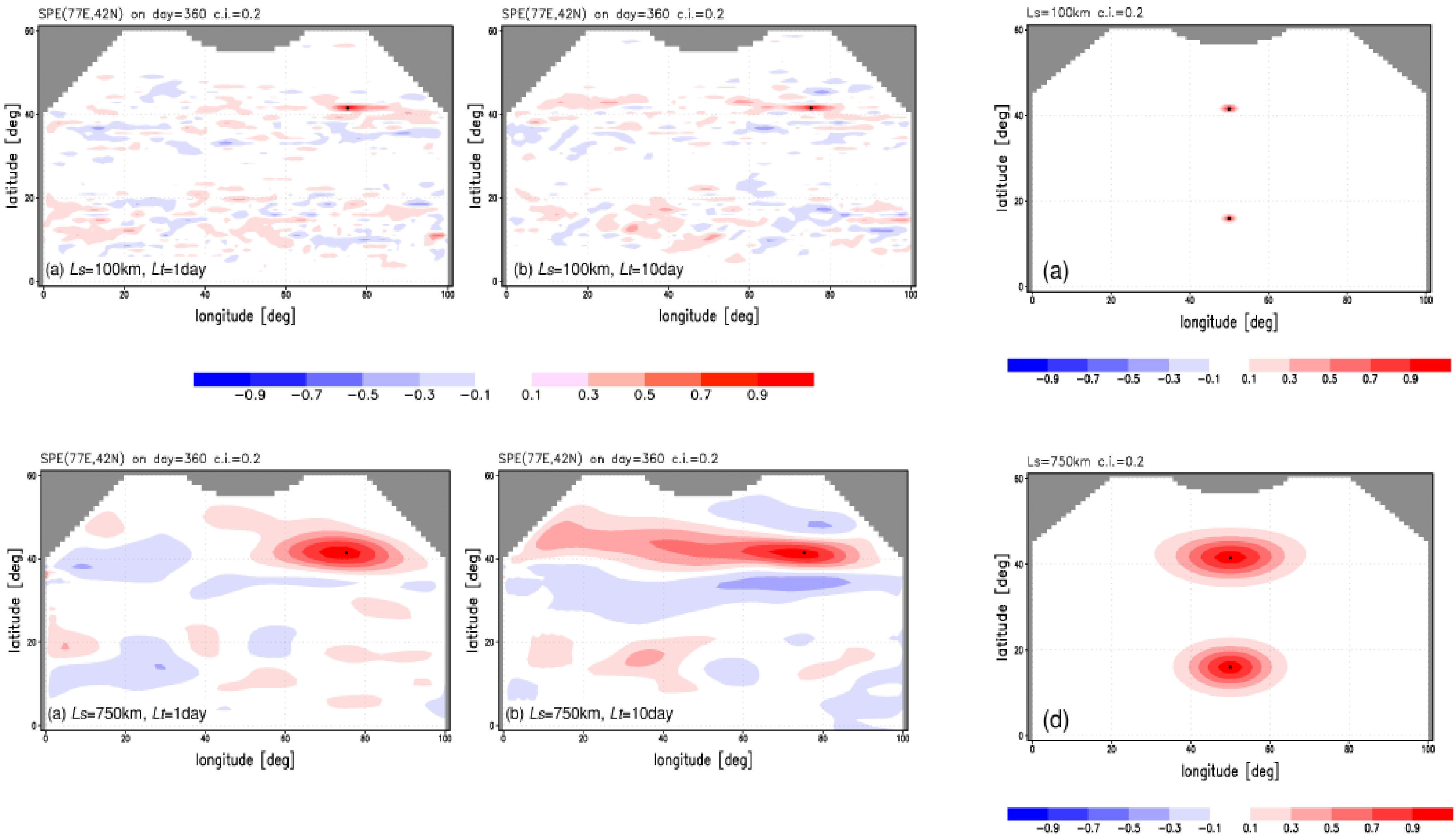
Experiments：Sensitivity of P-matrix to Q-matrix

Sensitivity to a decorrelation length scale Ls and time scale Lt



Experiments：Sensitivity of P-matrix to Q-matrix

Sensitivity to a decorrelation length scale Ls and time scale Lt



Interpretations : Wind stress curl error covariance

Wind Stress Error Covariance Model

$$Q_\tau(\boldsymbol{r}_1, t_1; \boldsymbol{r}_2, t_2) = \sigma(y_1)_x \, \rho(\tilde{\boldsymbol{r}}, \tilde{t}) \, \sigma(y_2)_x$$

wind stress error :

$$\Delta\tau_x(\mathbf{r}) = \sigma_x(y)\,\widehat{\Delta\tau}_x(\mathbf{r})$$

wind stress curl error :

$$\Delta\varsigma(\mathbf{r}) = -\frac{\partial\sigma_x(y)}{\partial y}\widehat{\Delta\tau}_x(\mathbf{r}) - \sigma_x(y)\frac{\partial\widehat{\Delta\tau}_x(\mathbf{r})}{\partial y}$$

wind stress curl error covariance :

$$Q_{\Delta\varsigma}(\mathbf{r},\mathbf{r}') = \frac{\partial\sigma_x(y)}{\partial y}\left\langle\widehat{\Delta\tau}_x(\mathbf{r})\widehat{\Delta\tau}_x(\mathbf{r}')\right\rangle\frac{\partial\sigma_x(y')}{\partial y}$$

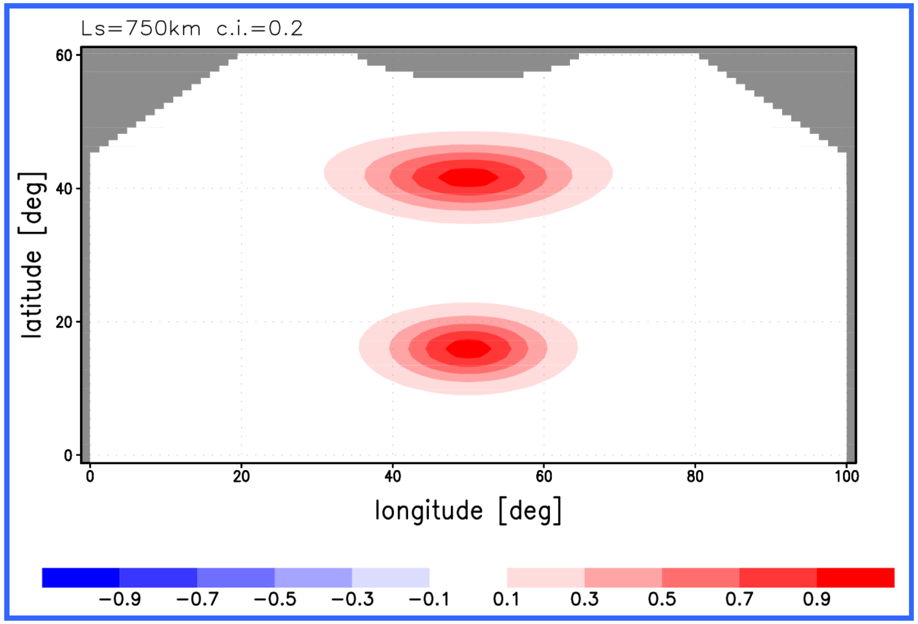
$$+\,\sigma_x(y)\left\langle\frac{\partial\widehat{\Delta\tau}_x(\mathbf{r})}{\partial y}\frac{\partial\widehat{\Delta\tau}_x(\mathbf{r}')}{\partial y}\right\rangle\sigma_x(y')$$

$$+\,\frac{\partial\sigma_x(y)}{\partial y}\left\langle\widehat{\Delta\tau}_x(\mathbf{r})\frac{\partial\widehat{\Delta\tau}_x(\mathbf{r}')}{\partial y}\right\rangle\sigma_x(y')$$

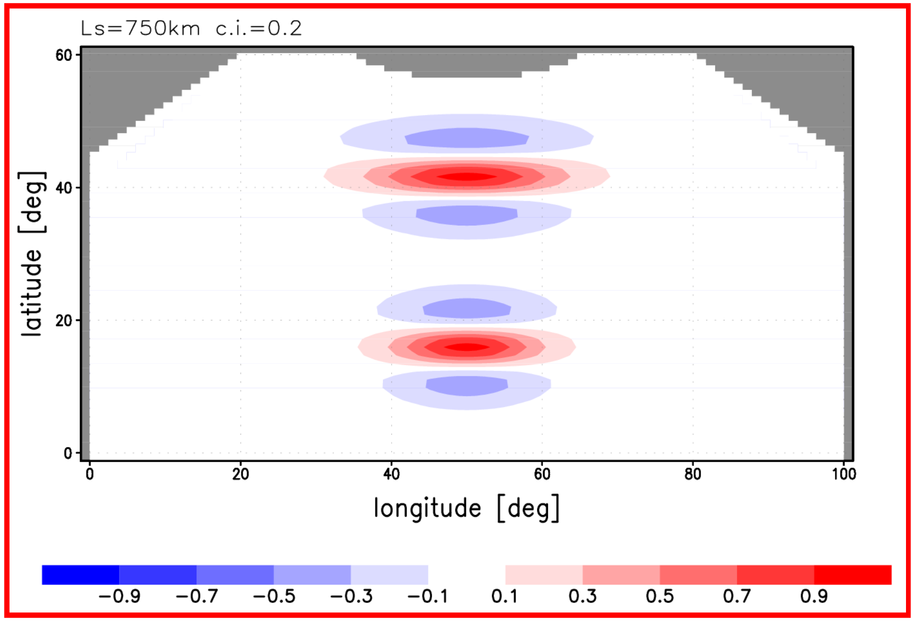
Interpretations : Wind stress curl error covariance

$$Q_{\Delta\varsigma}(\mathbf{r};\mathbf{r}') = \left(\frac{\partial\sigma_x(y)}{\partial y}\right)\rho_{\Delta\tau_x}(\mathbf{r};\mathbf{r}')\left(\frac{\partial\sigma_x(y')}{\partial y}\right) + \frac{\sqrt{2}\sigma_x(y)}{L_y}\rho_{\partial\Delta\tau_x/\partial y}(\mathbf{r};\mathbf{r}')\frac{\sqrt{2}\sigma_x(y')}{L_y}$$

$$+ \text{(cross covariance)}$$



$\rho_{\Delta\tau_x}(\mathbf{r};\mathbf{r}')$

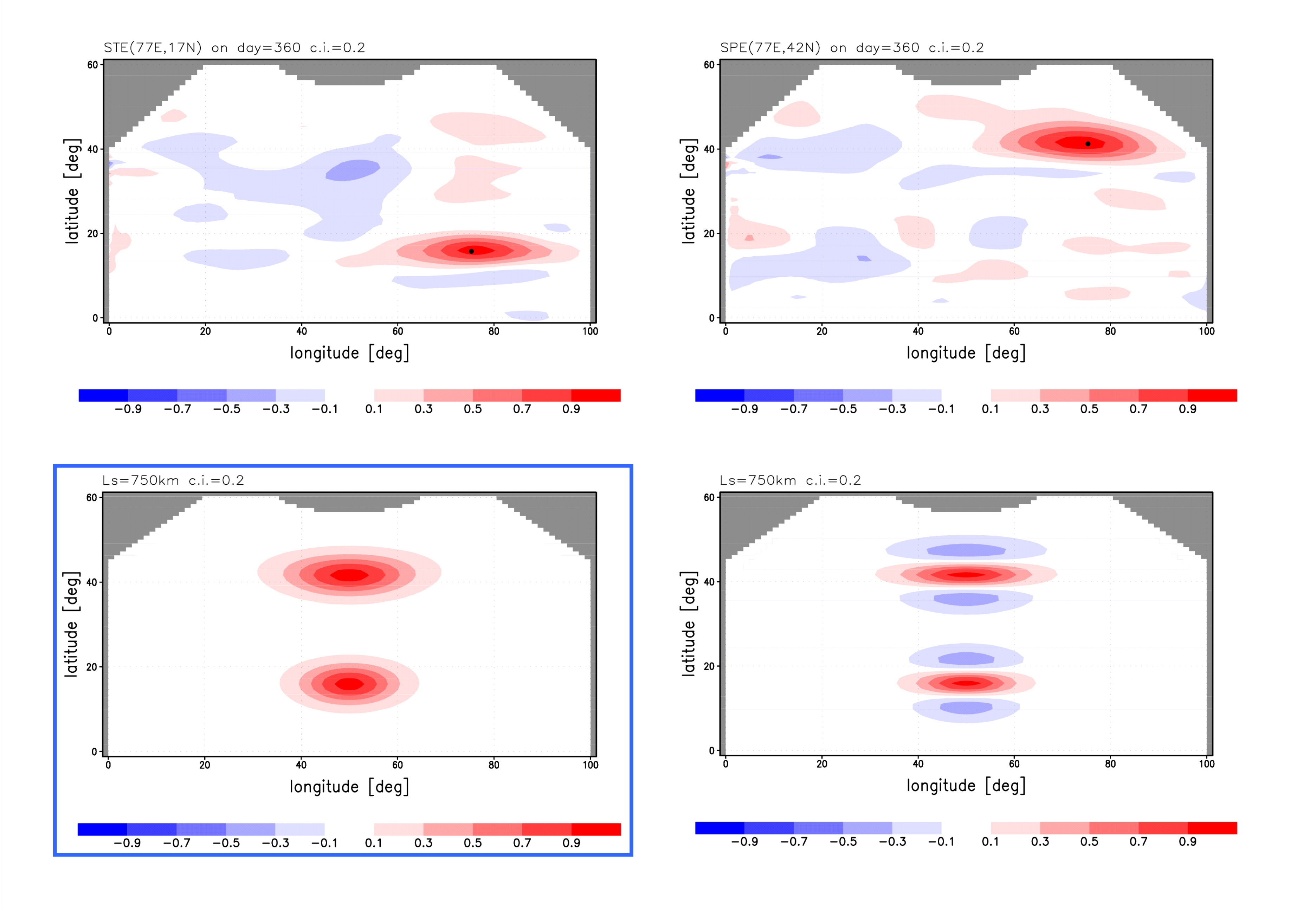


$\rho_{\partial\Delta\tau_x/\partial y}(\mathbf{r};\mathbf{r}')$

spatial structure of the correlation functions for $L_{x(y)}$=750km

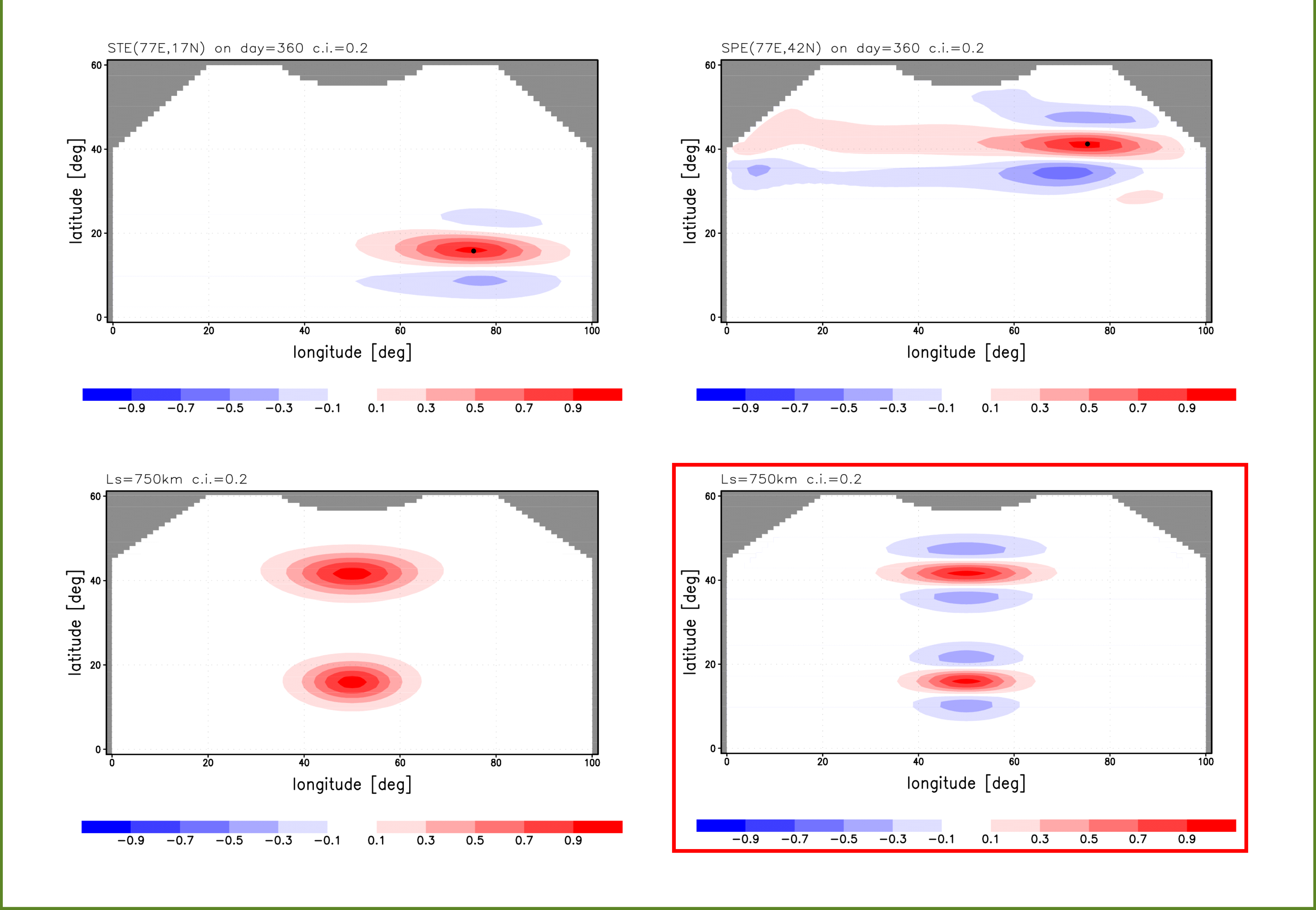
Interpretations : Wind stress curl error covariance

representer at day 360: L_s =750km, L_t =1day



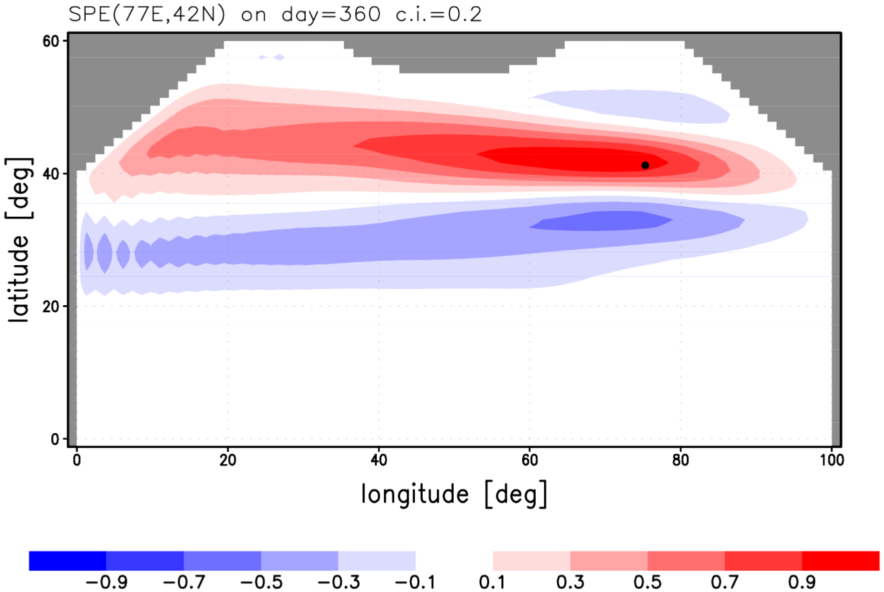
Interpretations : Wind stress curl error covariance

representer at day 360: L_s =750km, L_t =10day

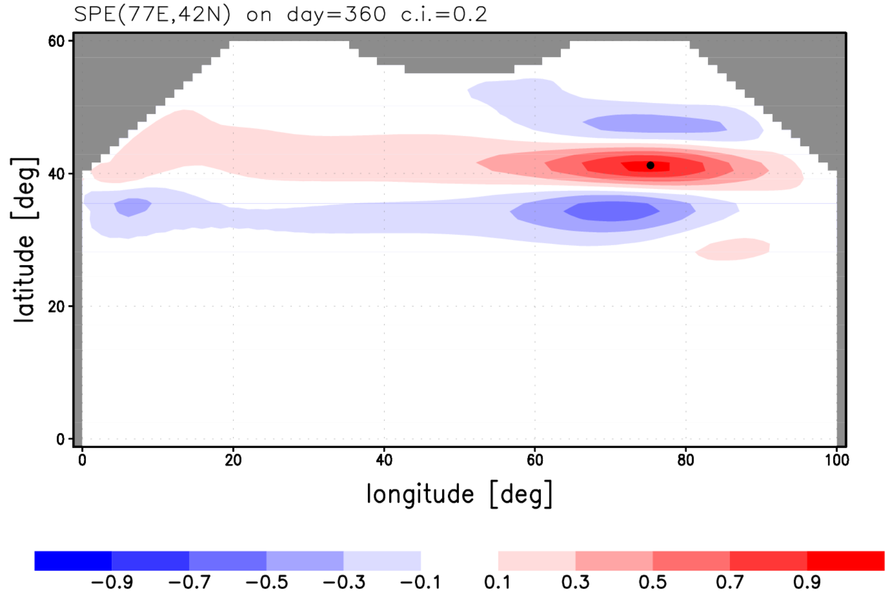


Interpretations : Role of barotropic dynamics

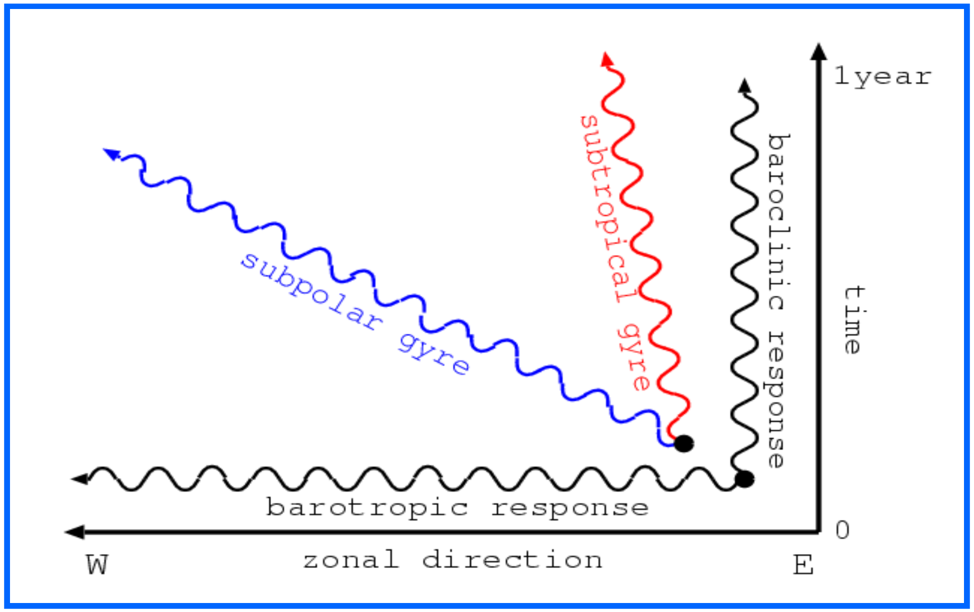
representer at day 360: L_s =750km, L_t =10day



Barotropic Run



Baroclinic Run



Summary

Impact of wind stress error covariance in the 4DVAR analysis with one year integration period was studied.

- Diagonal wind stress covariance model leads to highly localized strcture in P-matrix

- Explicit specification of wind stress error covariance in 4DVAR system leads to implicit specification of wind stress curl error covariance.

- The *a priori* model state error in the *subtropical* gyre due to a wind stress error is dominated by baroclinic response.
  The *a priori* error in the *subpolar* gyre is determined by both barotropic and baroclinic responses.
  The barotropic and baroclinic responses are near-independent