Point lattices with a fine-grained nesting property as experimental designs for multi-fidelity computer codes

SFU-UBC Seminar

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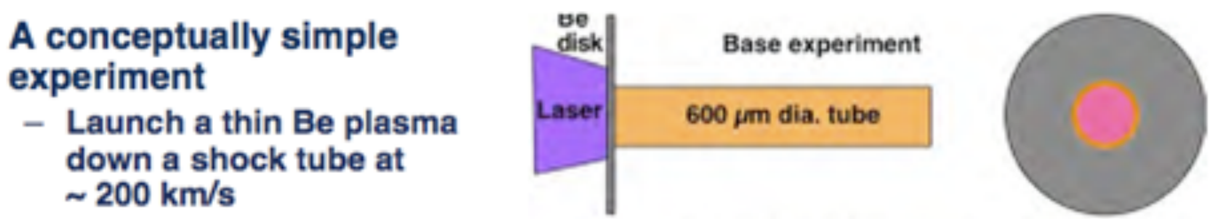
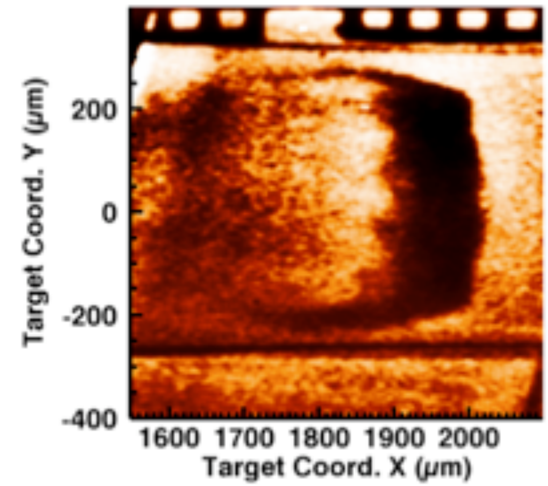
Radiative shock

- A radiative shock is a shock wave that is sufficiently fast that energy balance requires that it radiate energy away
- Radiation travels out in front of the shock
- The physics is relevant to astrophysics (e.g. study of supernovae and solar wind phenomena)

Radiative shock experiment





- A radiative shock is a wave in which both hydrodynamic and radiation transport physics play a significant role in the shock's propagation and structure.

Radiative shock

- Have additional dynamics of the experiment to worry about
- The radiation interacts with the wall of the tube, leading to ablation of the plastic
- Causes the generation of a second shock (the "wall shock") that interacts with the primary shock

Important feature of the computer code

- Physicists have built simulators for the system – *CRASH Code*
- Computer codes are deterministic
- Impacts the design and analysis strategy:
  - Use a Gaussian process model (Sacks et al., 1989)

A new statistical model for combining outputs from multi-fidelity simulators

- Have simulations from 1-D and 2-D models
- 2-D models runs come at a higher computational cost
- Would like to *use all simulations, and experiments, to make predictions*

A new statistical model for combining outputs from multi-fidelity simulators

- Have simulations from 1-D and 2-D models
- 2-D models runs come at a higher computational cost
- Would like to *use all simulations, and experiments, to make predictions*
- 1-D CRASH Simulations
  - 1024 simulations
  - Experiment variables: Be thickness, Laser energy, Xe fill pressure, Observation time
  - Calibration parameters: Electron flux limiter, Laser energy scale factor
- 2-D CRASH Simulations
  - 104 simulations
  - Experiment variables: Be thickness, Laser energy, Xe fill pressure, Observation time
  - Calibration parameters: Electron flux limiter, Wall opacity, Be gamma

A new statistical model for combining outputs from multi-fidelity simulators

- Different levels of code have a shared component and a discrepancy

$$\begin{aligned} y_2(x_f) &= \boxed{\eta(x_f)} + \delta(x_f) \\ y_1(x_f) &= \boxed{\eta(x_f)} \end{aligned}$$

Gaussian Process Models

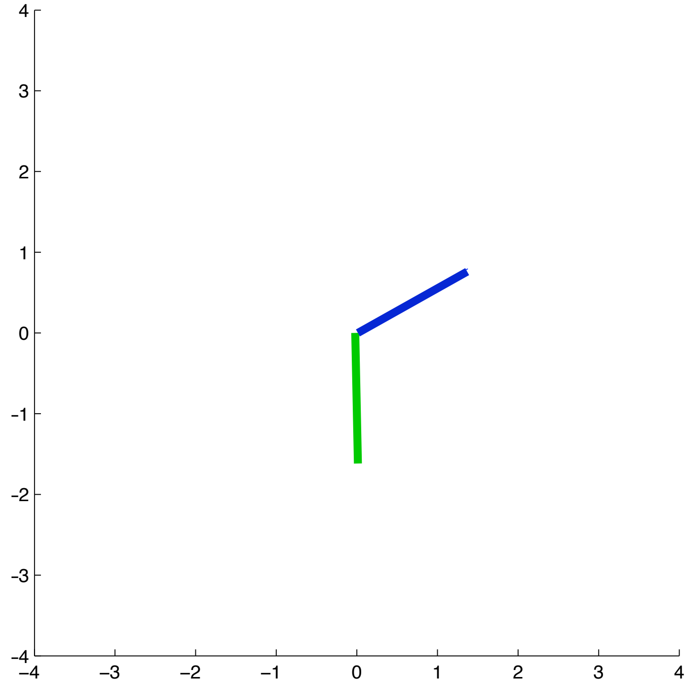
- **Idea:** Sample x_f of the codes with designs $X_2 \subset X_1$ that contain each other to enable direct observation of δ

Designing nested point lattices

- Background on point lattices and subsampling
- Construction of lattice and nesting scheme
- Results: New lattices

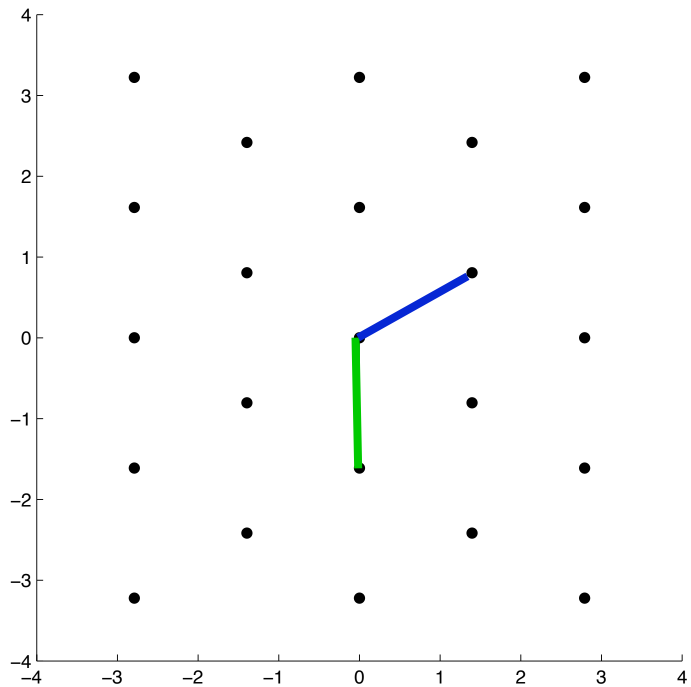
Point lattices

- Definition via basis $\mathbf{R}$



Point lattices

- Definition via basis $\{\mathbf{R}k : k \in \mathbb{Z}^n\}$



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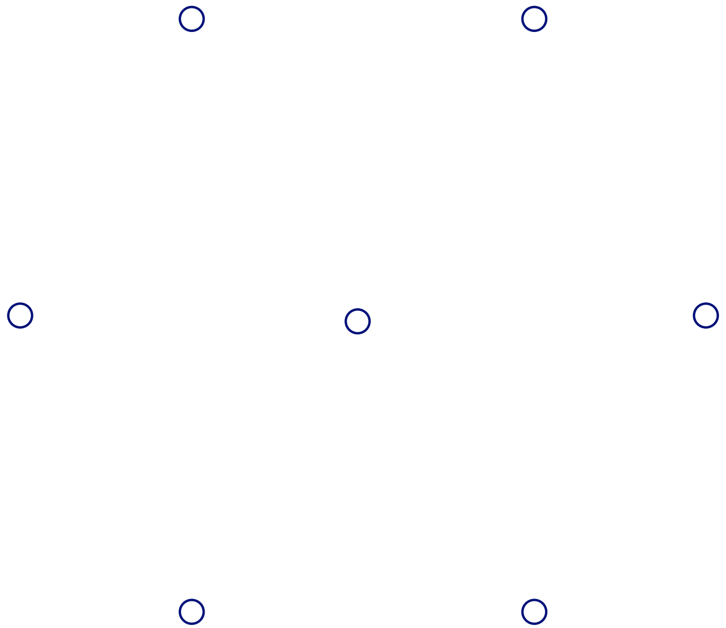
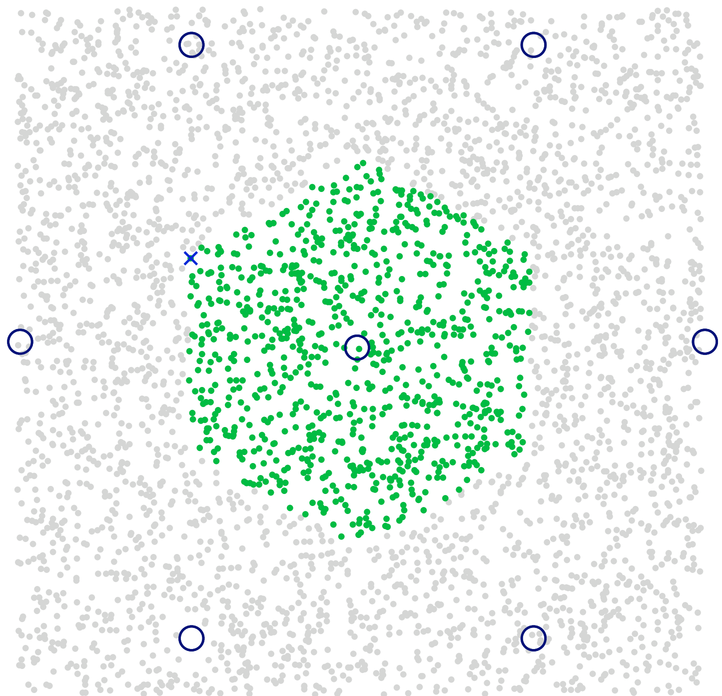
Point lattices

- Definition via basis $\{\mathbf{R}k : k \in \mathbb{Z}^n\}$
- Discrete subgroup of $\mathbb{R}^n$, e.g. periodic tiling, has lattice basis
- Shift-invariant neighbourhood leads to one Voronoi polytope $\mathcal{V}$ around any point

Lattice properties of $\mathcal{V}$



Lattice properties of $\mathcal{V}$



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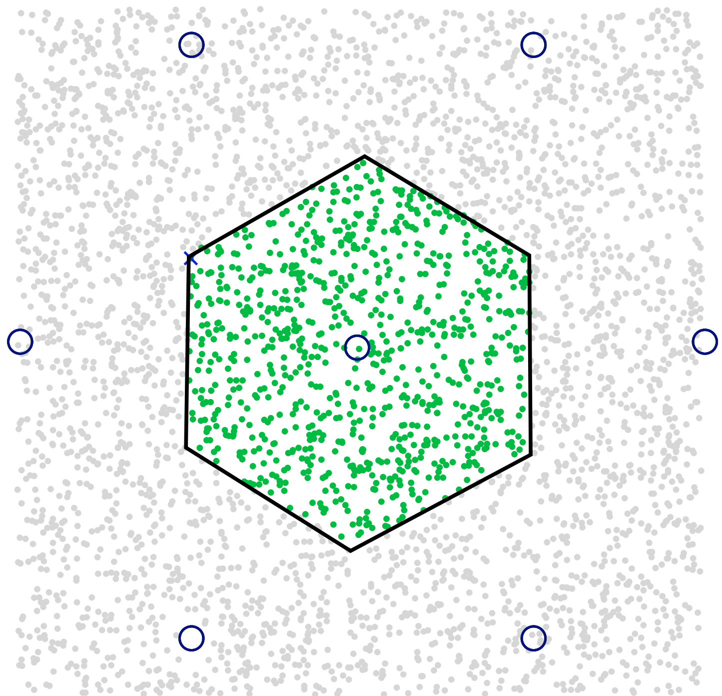

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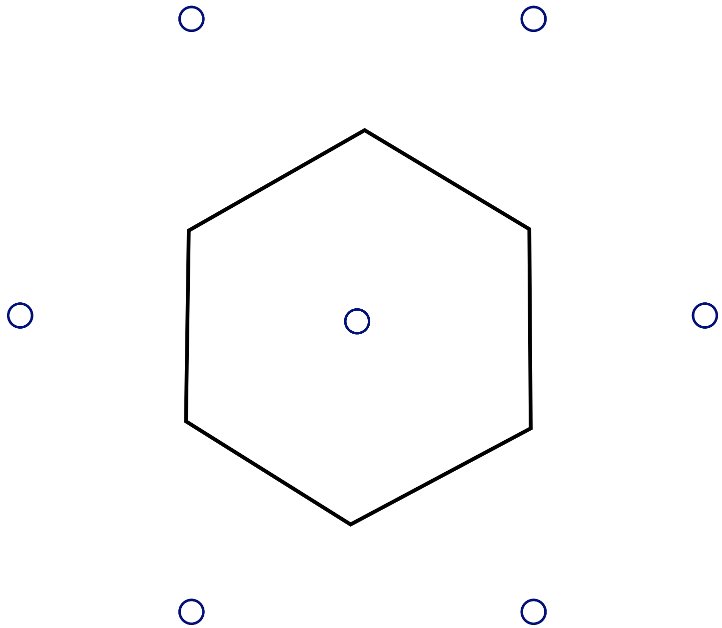

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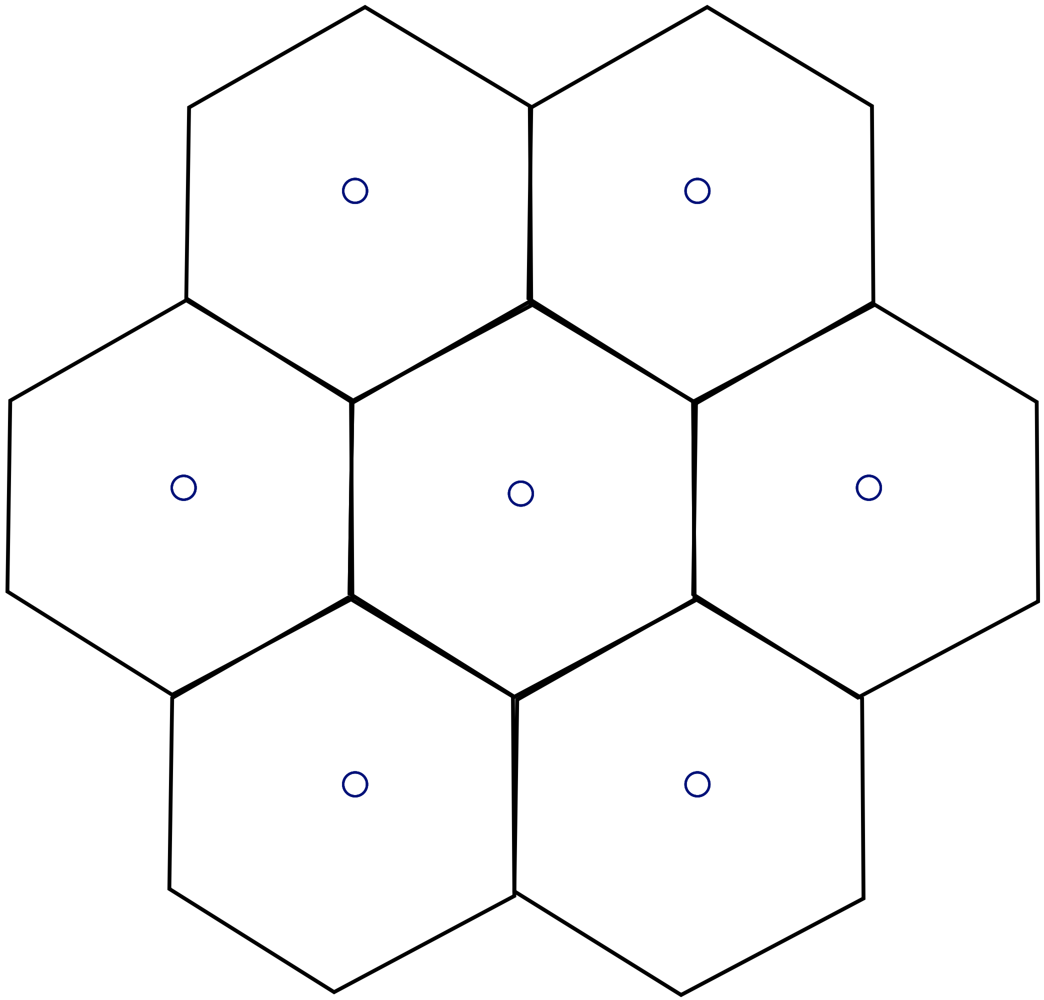

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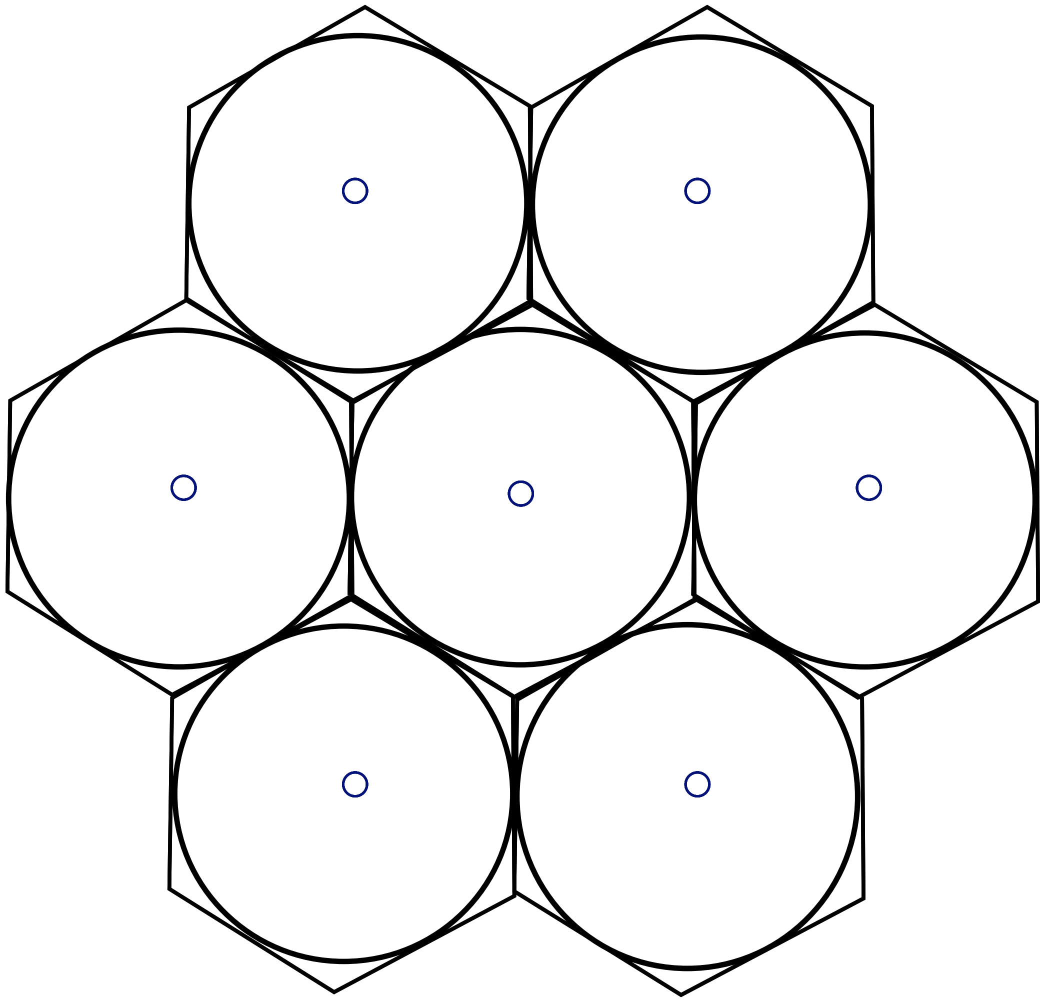

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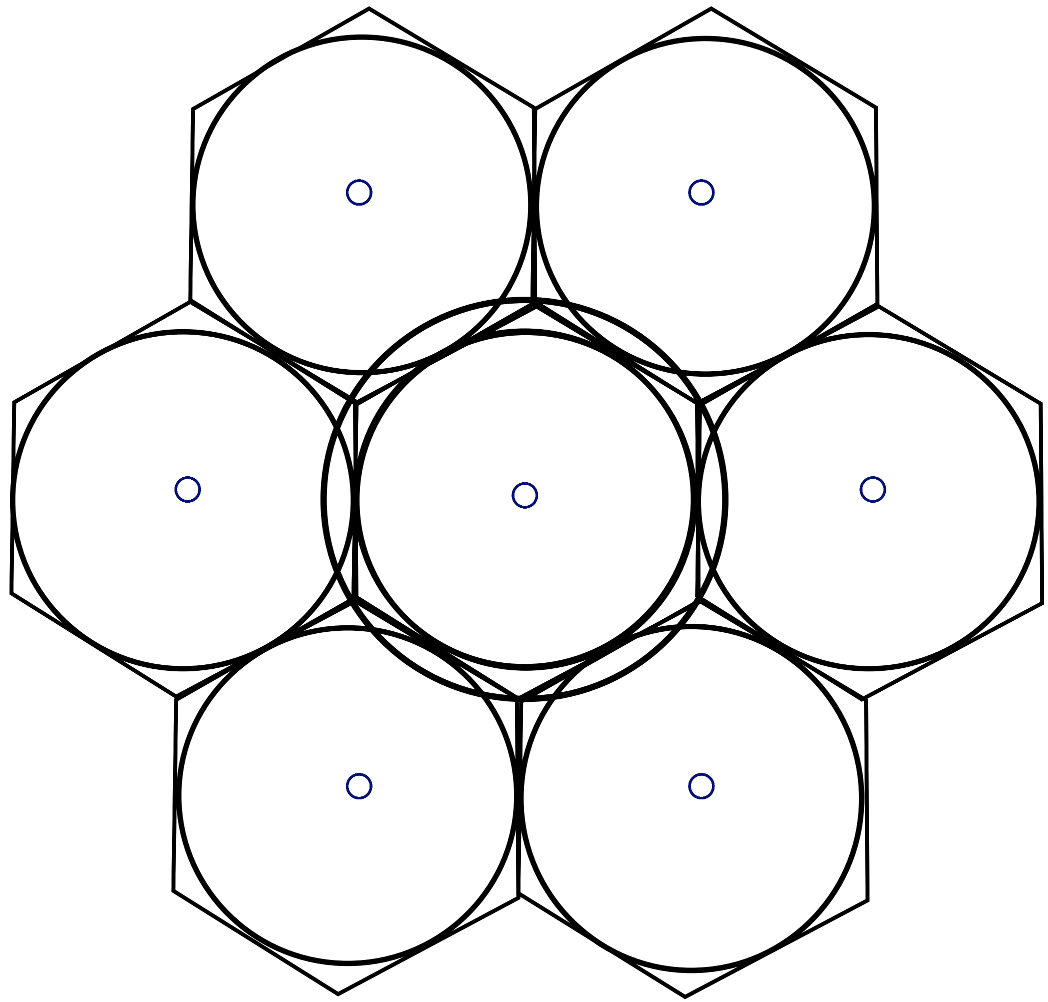

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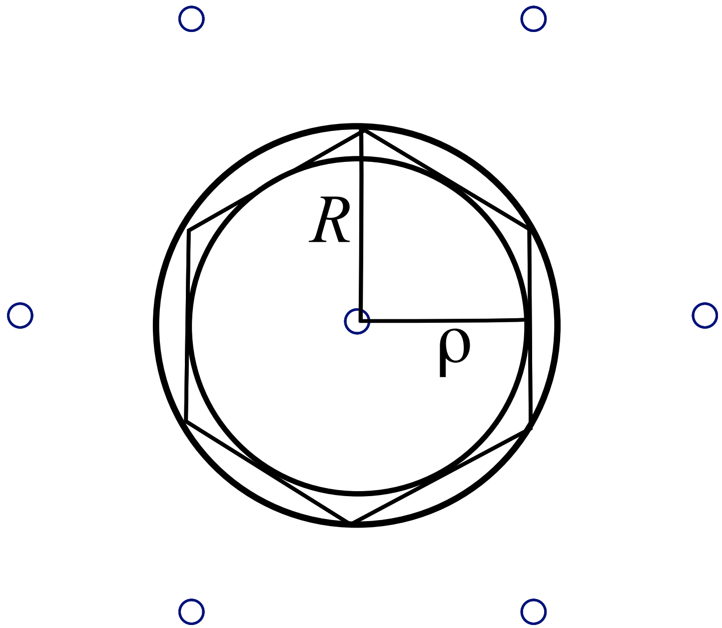
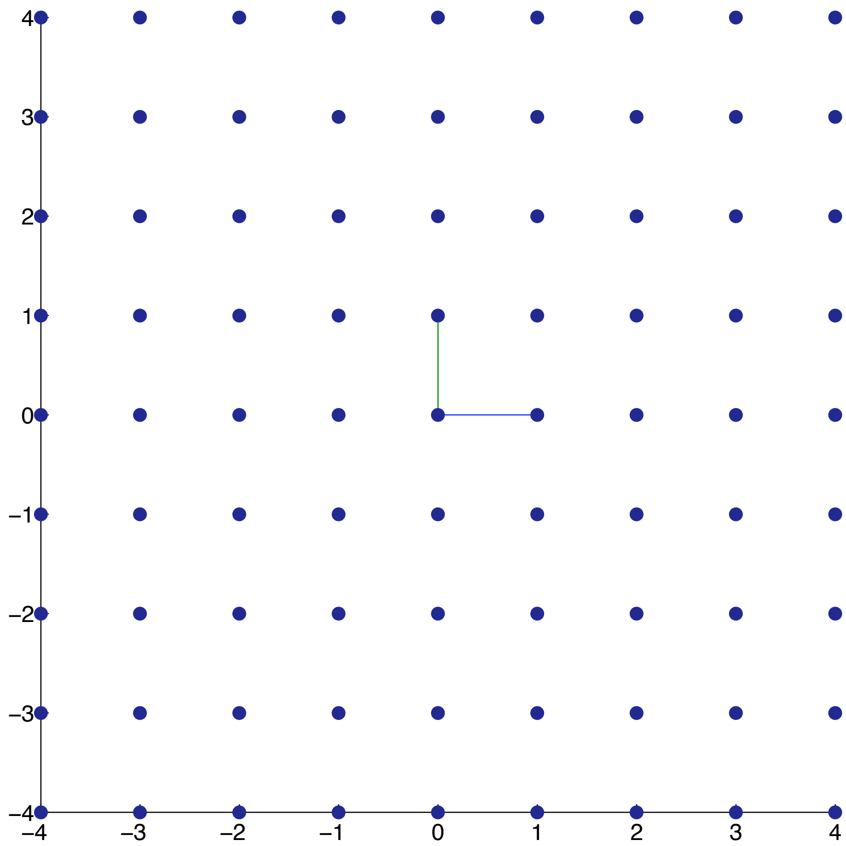


Subsampling via dilation

- Lattice with basis $\mathbf{RK}$ for $\mathbf{K} \in \mathbb{Z}^{n \times n}$
  is dilated by factor $|\det \mathbf{K}| = \alpha^n = \delta \in \mathbb{Z}^+$

$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



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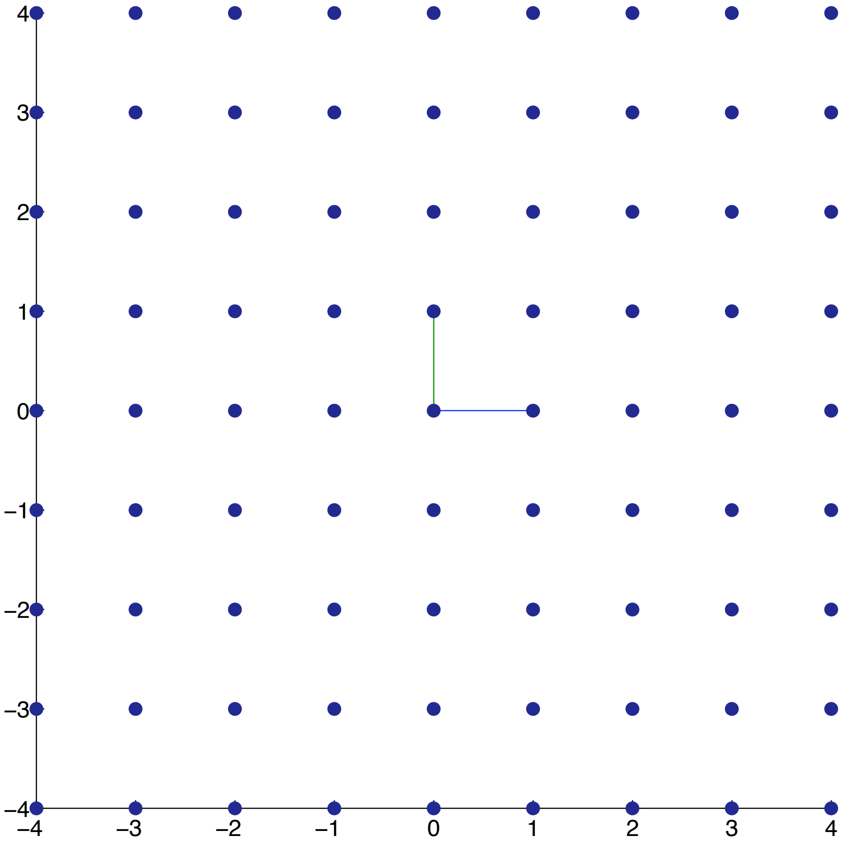
$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2\mathbf{I} \qquad \det \mathbf{K} = 2^n = 4$$



dyadic subsampling

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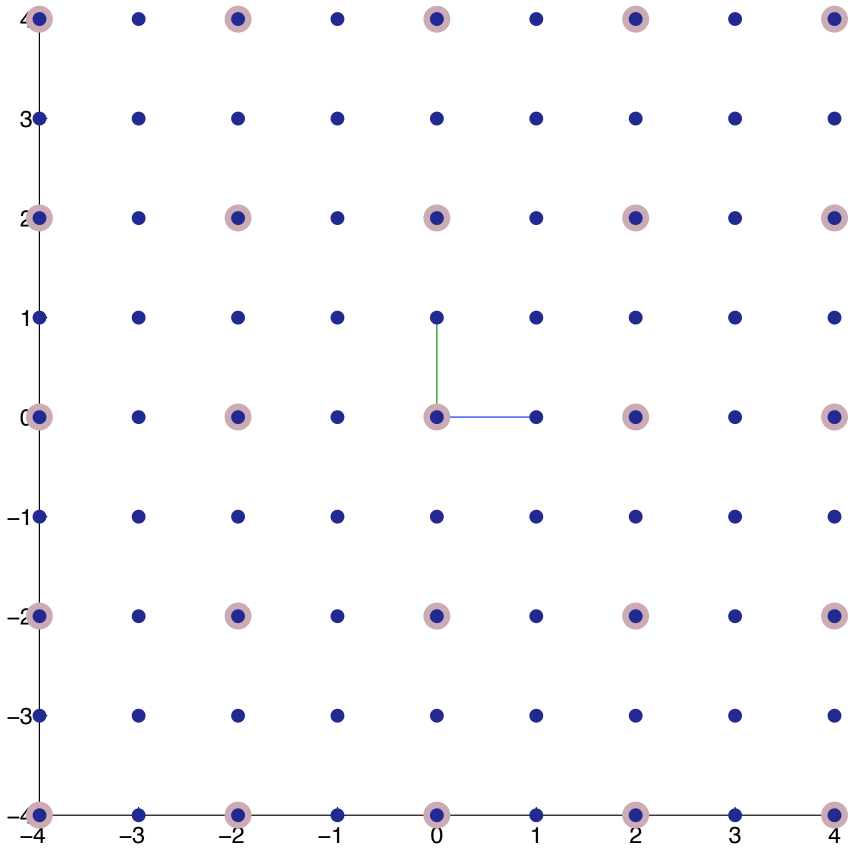

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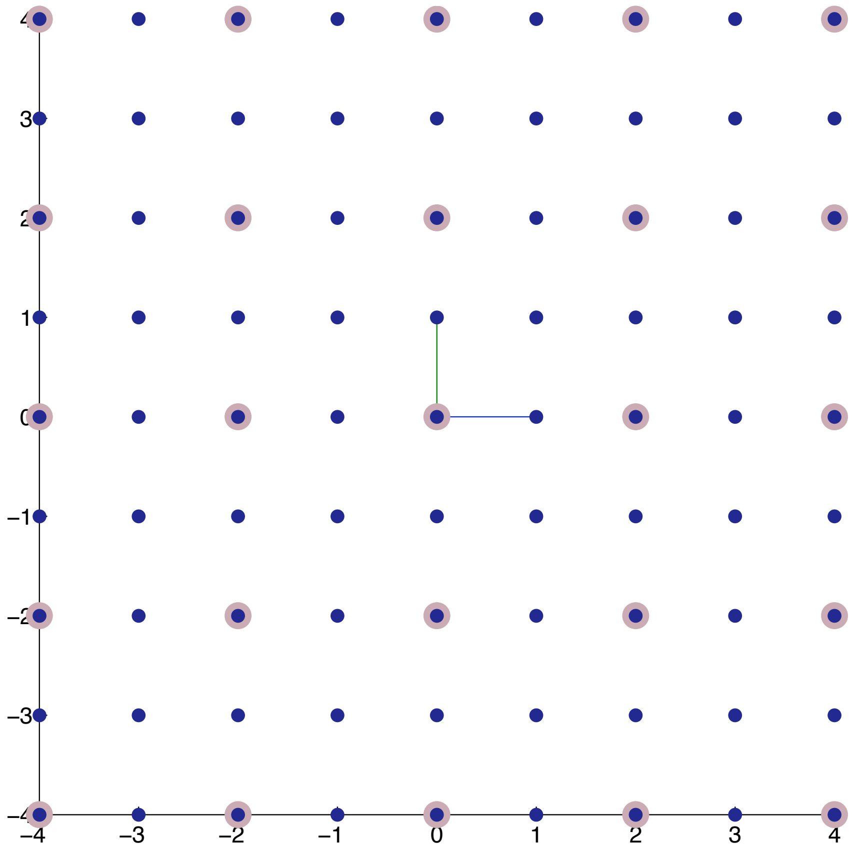

Reduction factor is exponential in n

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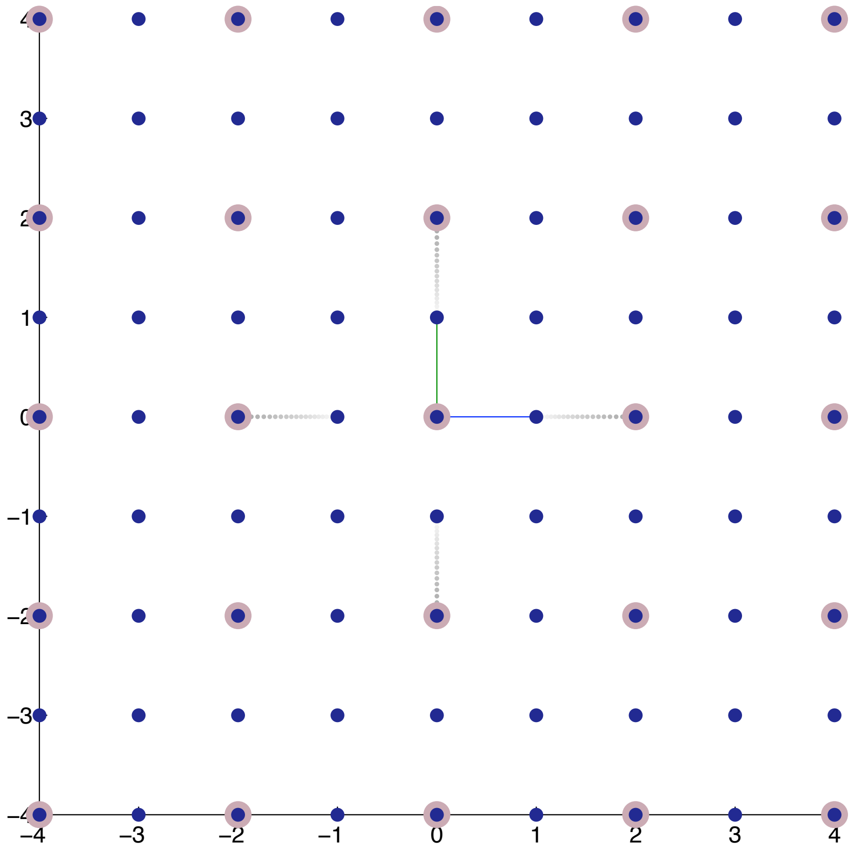

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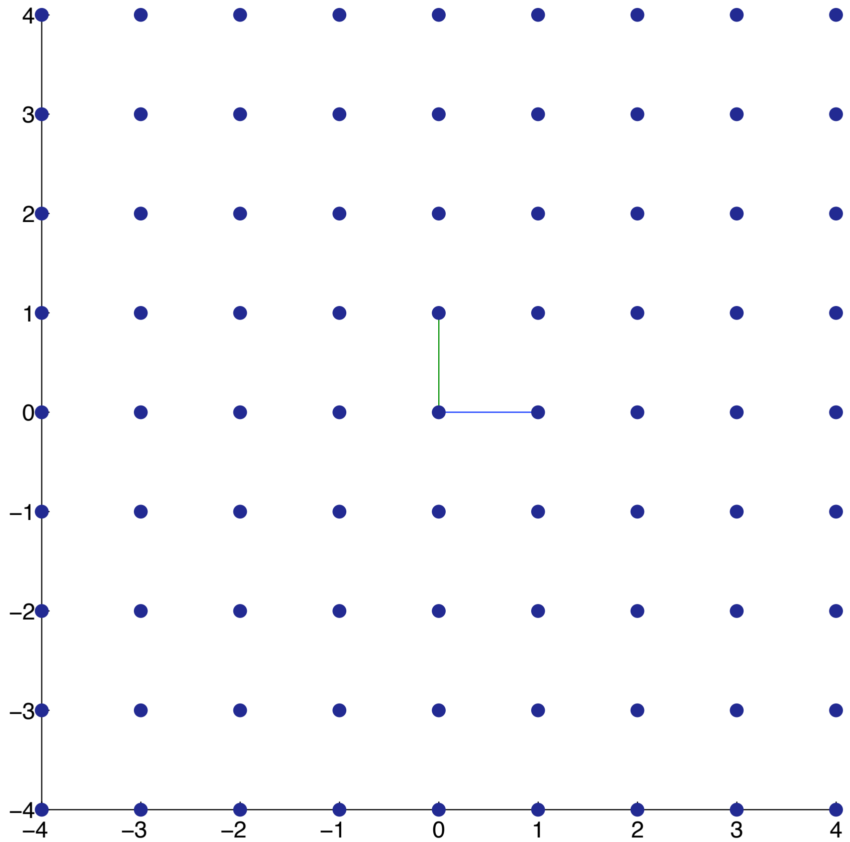

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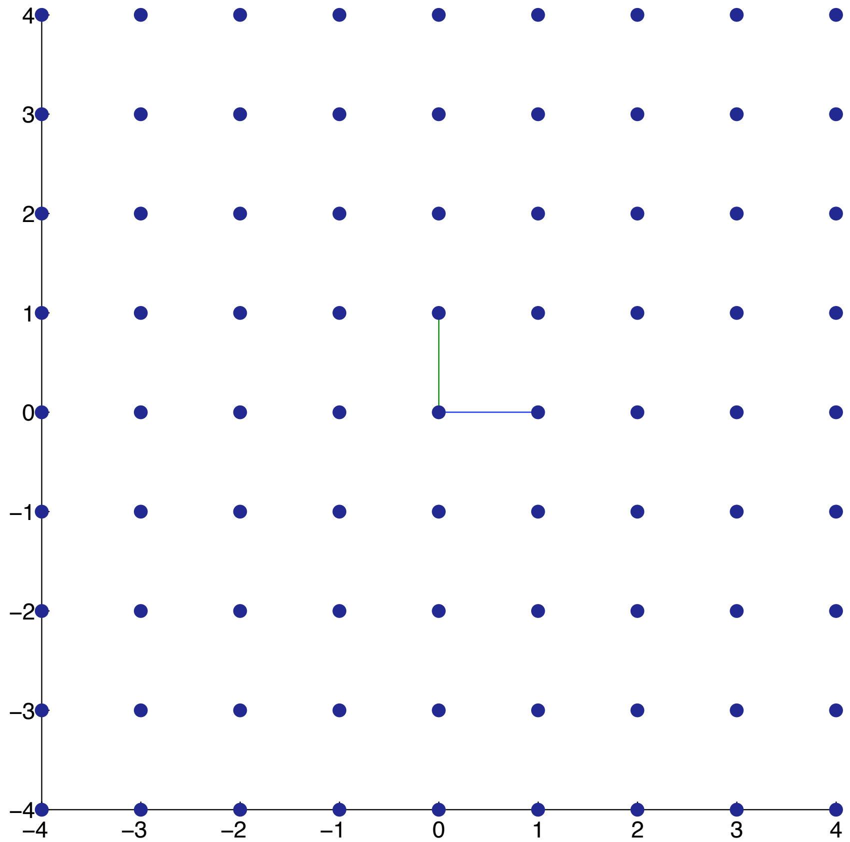

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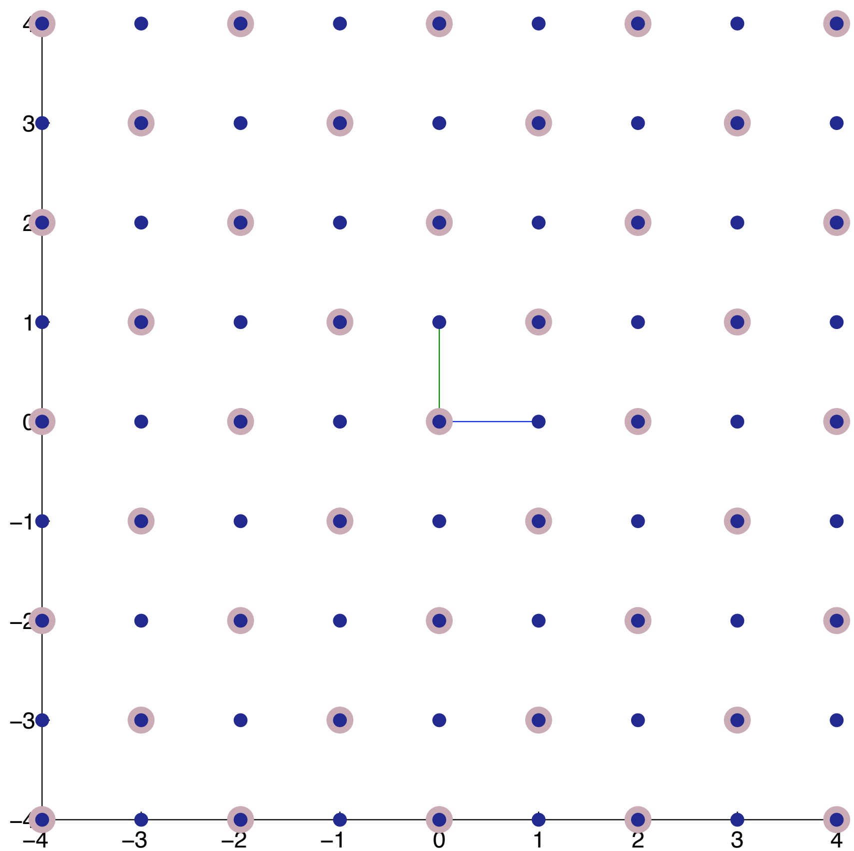

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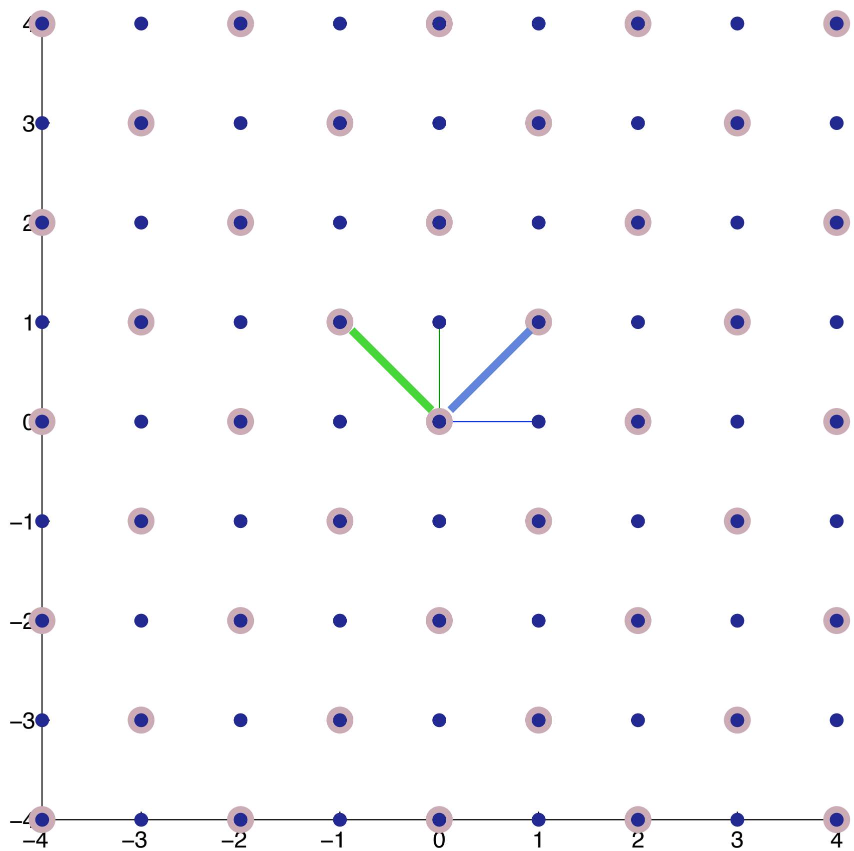

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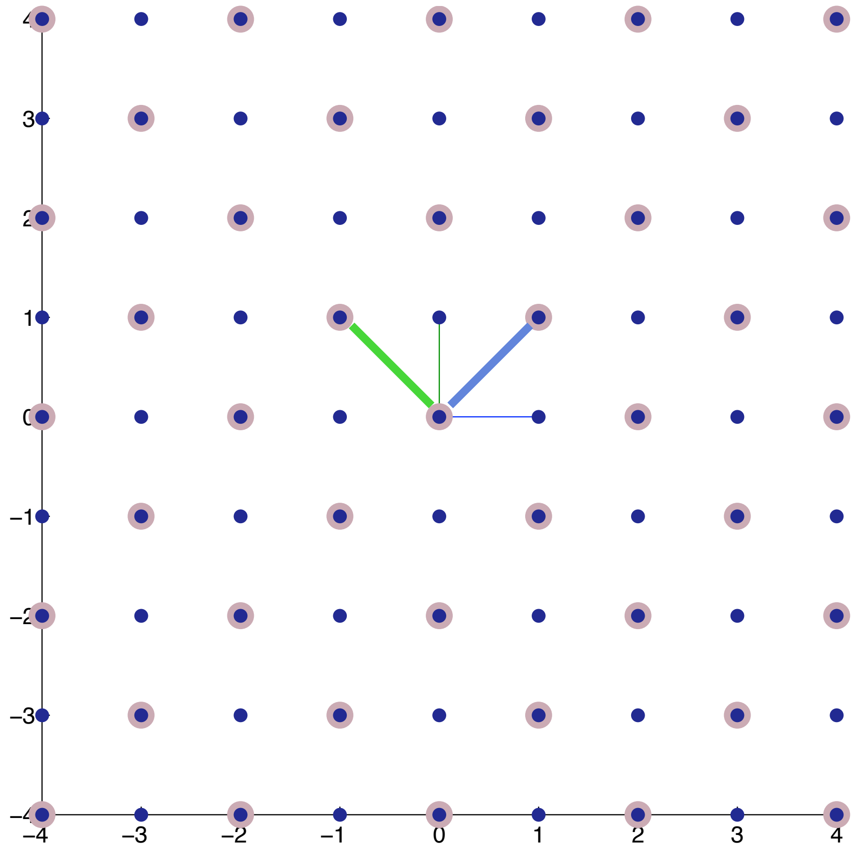

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This low rate dilation does not exist for integer lattices with $n > 2$

[Van De Ville, Blu, Unser, SPL 05]

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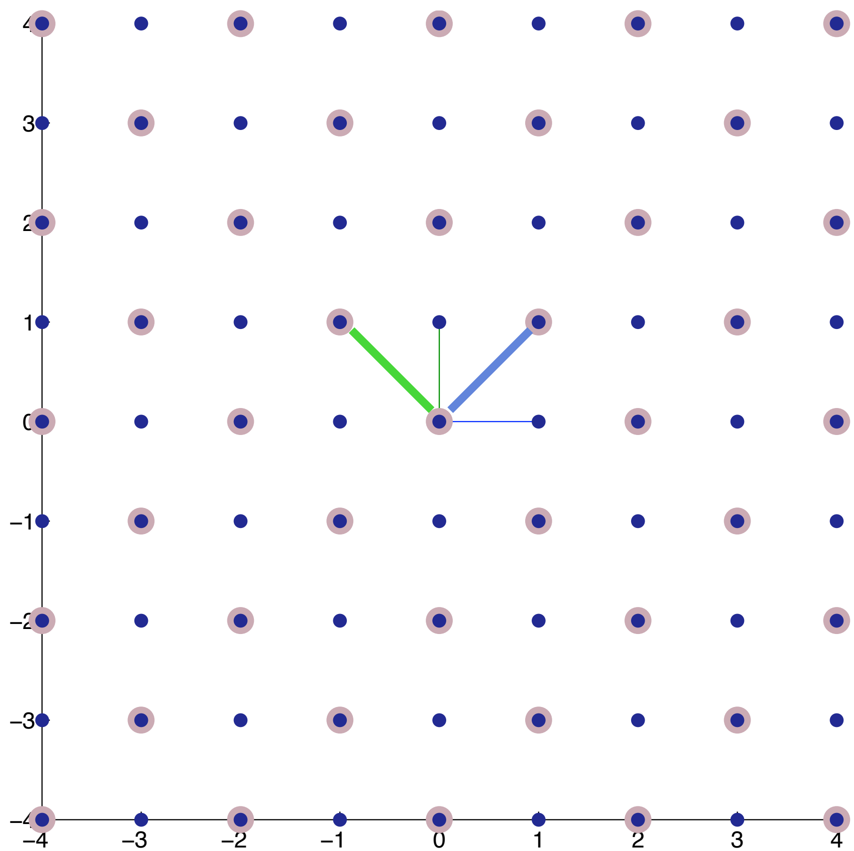

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However, possible for irrational $\mathbf{R}$!

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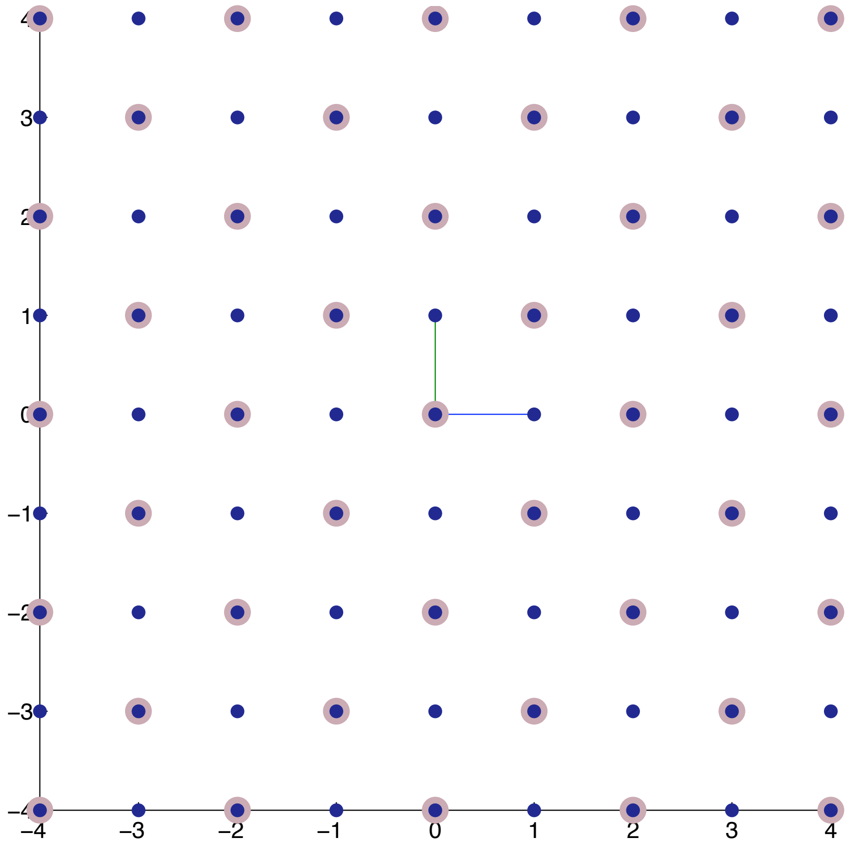

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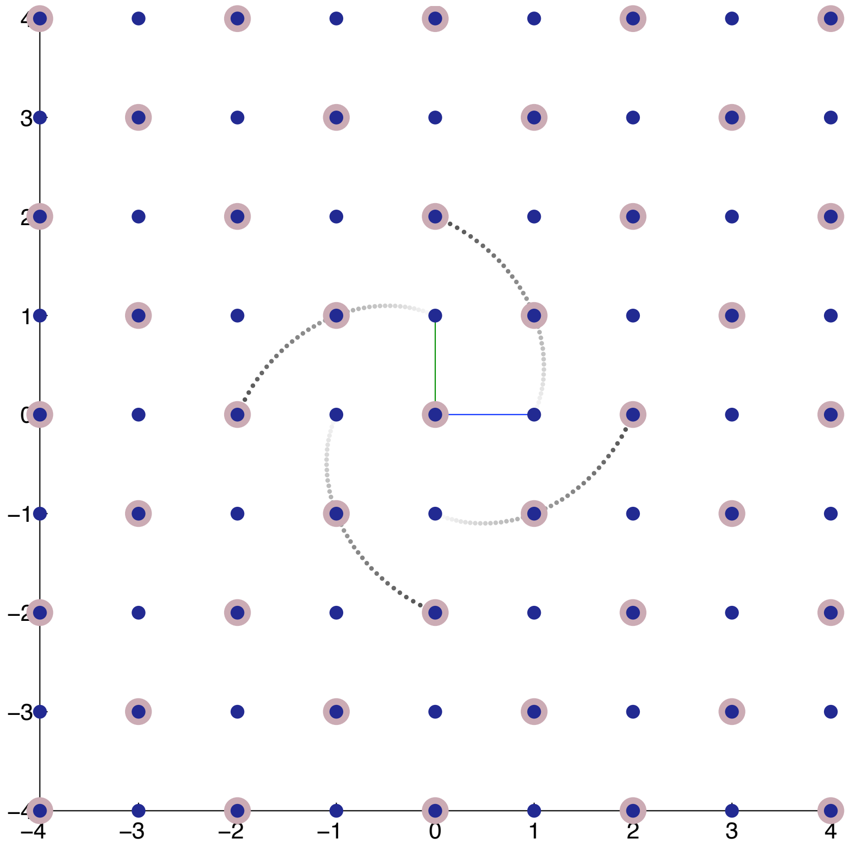

fractional subsampling

$\mathbf{R}\mathbf{K}^s$ for $s = 0..2$

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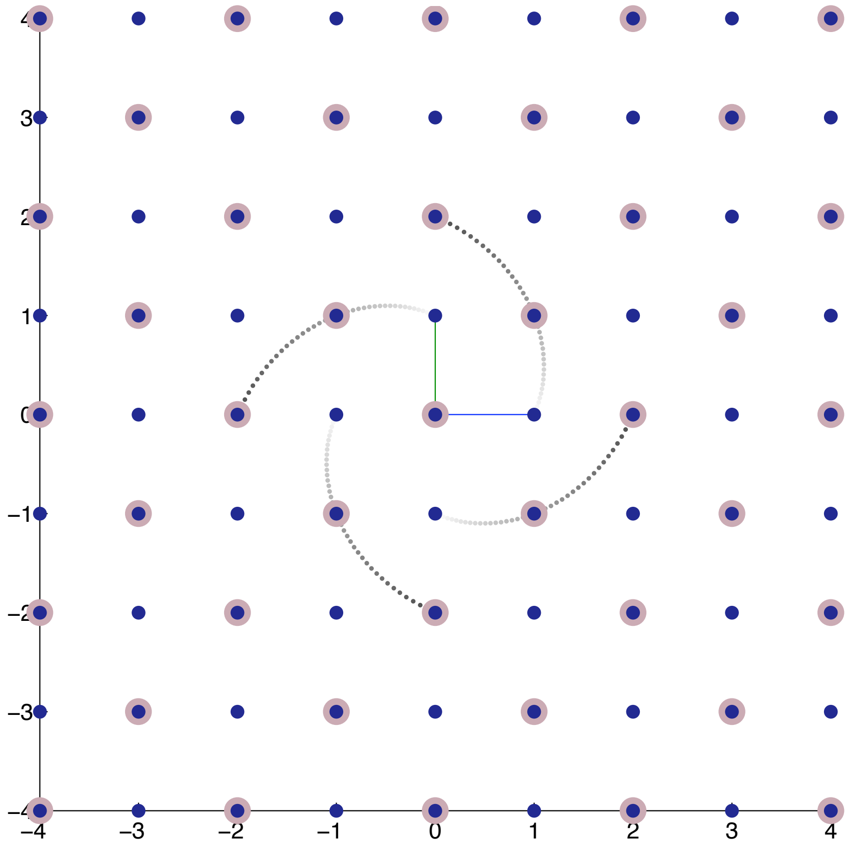

fractional subsampling

$\mathbf{R}\mathbf{K}^s$ for $s = 0..2$

acts like a scaled rotation $\mathbf{Q}\mathbf{R}$

with $\mathbf{Q}^T\mathbf{Q} = \alpha^2\mathbf{I}$

quincunx subsampling

Construction

Similarity of Q and K

$\mathbf{QR} = \mathbf{RK}$ with $\mathbf{Q}^T\mathbf{Q} = \alpha^2\mathbf{I}$

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- $\mathbf{K}$ and $\mathbf{Q}$ have same characteristic polynomial $d(\lambda) = \det(\mathbf{K} - \lambda\mathbf{I}) = \det(\mathbf{Q} - \lambda\mathbf{I})$

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$$= \sum_{k=0}^{n} c_k \lambda^k \in \mathbb{Z}[\lambda]$$

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  and thus agree in eigenvalues and determinant.

Diagonalizing rotation Q

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}\begin{bmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{bmatrix}\begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix}$$

$$= \mathbf{J}_2^{-1}\boldsymbol{\Delta}\mathbf{J}_2$$

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Analogue block-wise construction of $\mathbf{J}_n$

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Analogue block-wise construction of $\mathbf{J}_n$

Different eigenvalue structure for even and odd dimensionality restricts characteristic polynomial:

n even: $d(\lambda) = \lambda^n + C\lambda^{\frac{n}{2}} + \alpha^n$ with $C^2 < 4\alpha^n$

n odd: $d(\lambda) = \lambda^n - \alpha^n$

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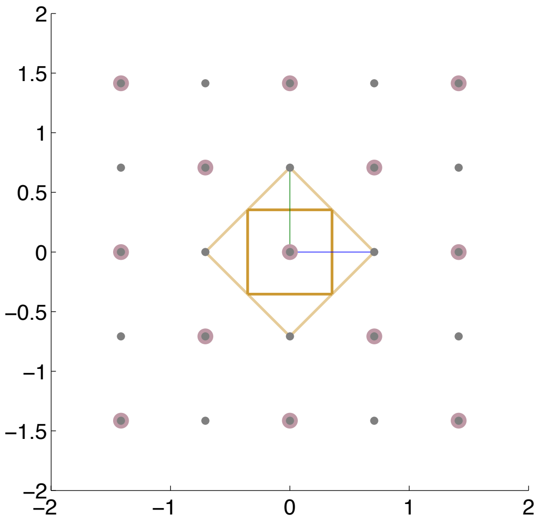
Results

2D

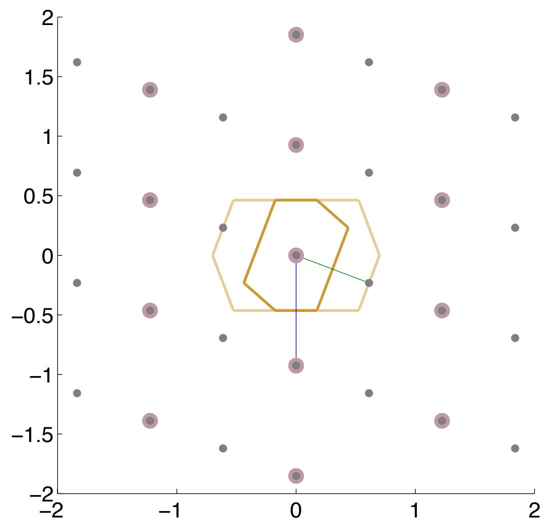
quincunx, $\theta = 45°$

$\mathbf{G} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}, \mathbf{K} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$



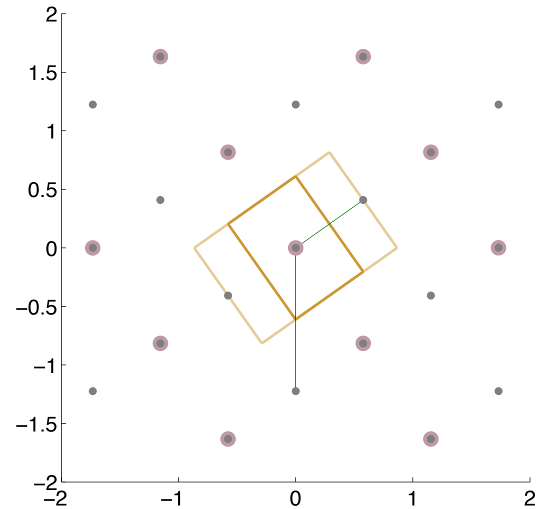
$\theta = \arccos \frac{\sqrt{2}}{2} \approx 69.3°$

$\mathbf{G} = \begin{bmatrix} 0 & 0.61 \\ -0.93 & -0.23 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix}$



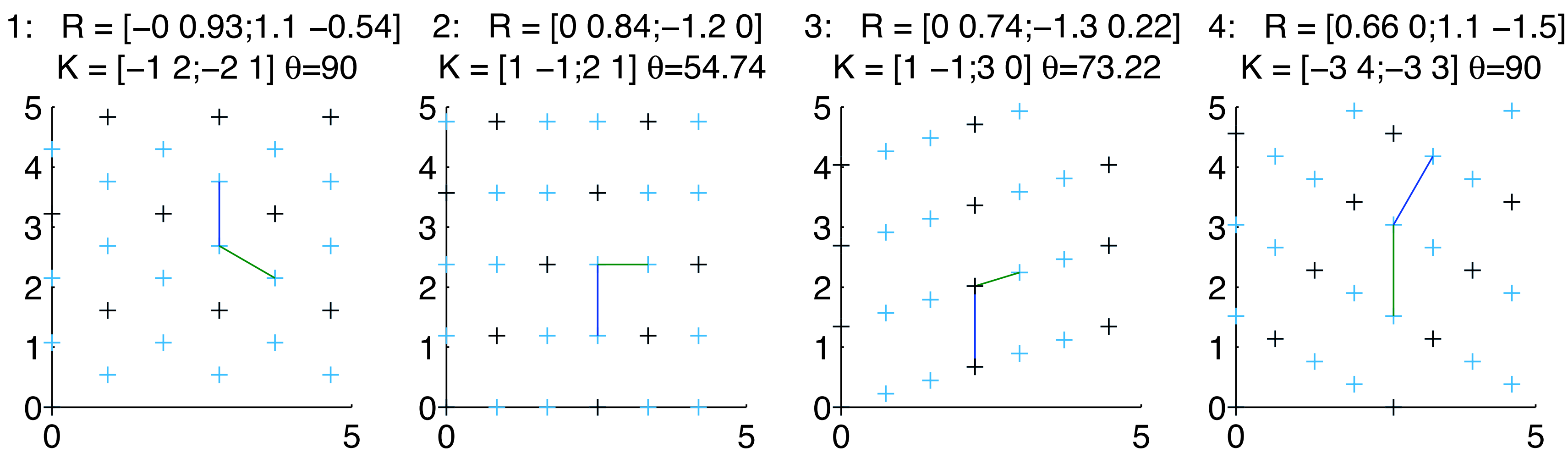
$\theta = 135°$

$\mathbf{G} = \begin{bmatrix} 0 & 0.58 \\ -1.22 & 0.41 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix}$



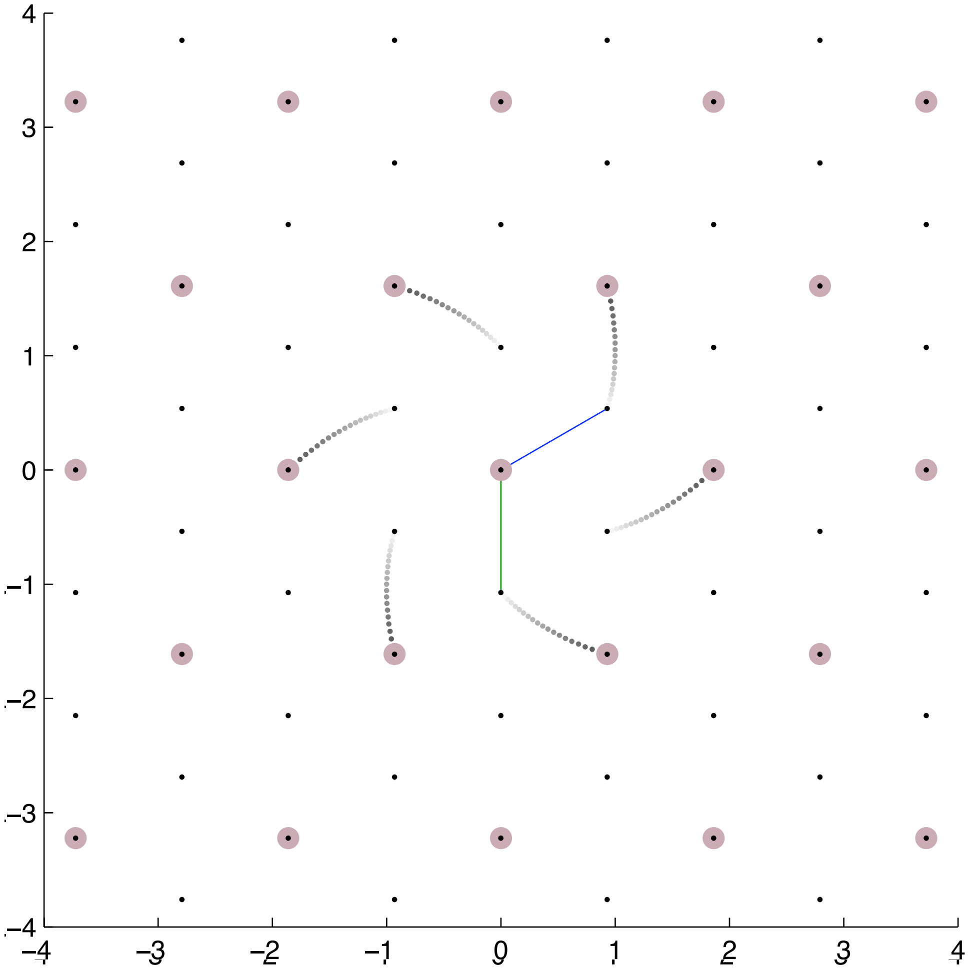
Dilation factor $|\det \mathbf{K}| = 2$

2D



Dilation factor $|\det \mathbf{K}| = 3$

$$\mathbf{R}_{hex} = \begin{bmatrix} 0.7071 & 0 \\ 0.4082 & -0.8165 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \quad \det \mathbf{K} = 3$$



3D

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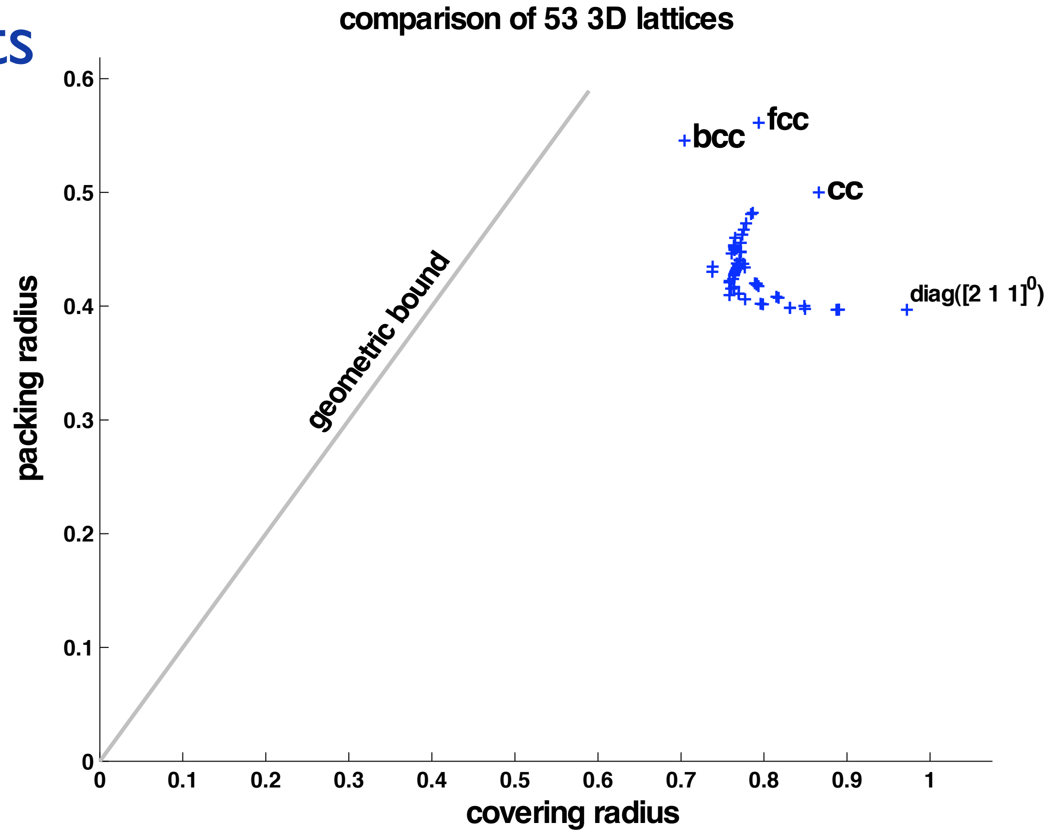
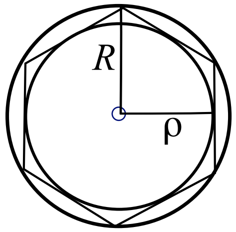
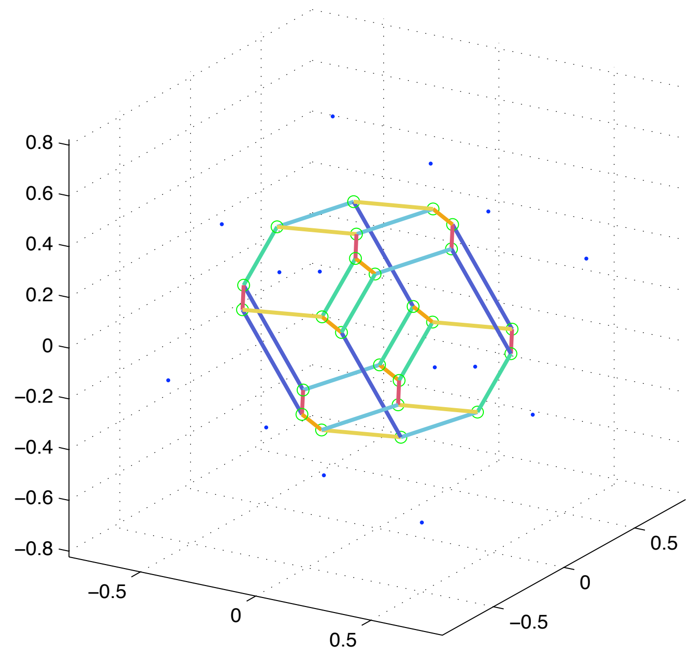




3D

- Lots of non-equivalent results
- Search over scaling S converged to one optimum



14 faces, 24 vertices, 6 zones
G(P) = 0.081904

Applications

- Optimize additional properties, e.g. runsize or uniform projections
- Uniform sampling of irregularly shaped regions
- n-D multi-resolution reconstruction pyramid

Questions?

- Thank you for your attention!