The structure and classification of nuclear C^*-algebras

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Contents

1 An introduction to classification and the beginnings of K-theory

Classification is a natural and prominent theme in mathematics, and there are many approaches one can take to problem of classifying a collection or category of objects.

1. Classify by brute force; that is, make a list.

   e.g. Finite simple groups.
   e.g. Hyperfinite factors.

2. Classify by invariants.

   e.g. Vector spaces are determined up to isomorphism by their dimension.
   e.g. Infinitely generated subgroups G of $\mathbb{Q}$ $\rightarrow$ are determined by their associated "supernatural number", that is, the symbol $n = p_1^{n_1} p_2^{n_2} \cdots$ (where the p_is are primes and $n_j \in \mathbb{N} \cup \{\infty\}$) with the property that

$$G = \{x \in \mathbb{Q} \mid \text{denominator of } x \text{ in lowest terms divides } n\}$$

3. Classify by functor. Evidently, this requires categories to begin with. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two categories and $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ a functor. If F is to be a "classification functor", then which properties should F have?

   i. Isomorphism of invariants should imply isomorphism between objects in $\mathcal{C}_1$:

$$F(a) \cong F(b) \implies a \cong b.$$

   ii. F should (at least appear) to forget something
   iii. Strengthening **i.**, we could ask that if $\phi : F(a) \rightarrow F(b)$ is an isomorphism, then there should be an isomorphism $\Phi : a \rightarrow b$ such that $F(\Phi) = \phi$; that is, isomorphisms in the classifying category lift. Of course, we could ask the same of any morphism.
   iv. Ideally, $\mathcal{C}_2$ should have a concrete interpretation.

4. Classification via Borel complexity. A problem with the approach in **iii.** is that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are Borel spaces and F is Borel, then F being a classification functor really says that $\mathcal{C}_2$ is 'more complicated' than $\mathcal{C}_1$! To formalize this notion of complexity, descriptive set theorists have developed a theory of classification for Borel equivalence relations.

We call a Borel space *standard* if it is Borel isomorphic to a Polish space. Let E be a Borel equivalence relation on X. Declare $B \subseteq X/E$ to be Borel if its pre-image in X is Borel; with this Borel structure X/E may not be standard. This failure to be standard can be measured as follows: let Y be another standard Borel space and F a Borel relation on Y. A *Borel reduction* of E to F is a Borel map $\theta : X \to Y$ such that $xEy \iff \theta(x)F\theta(y)$. We denote this by $E \leq_B F$. With this notion, we can measure the complexity of the isomorphism relation for various classes of operator algebras.

1.1 K-theory

1.1.1 The K_0-group

The K_0-group of a C*-algebra records the structure of projections in matrices over the algebra up to a relativized notion of dimension called *Murray-von Neumann equivalence*. Let A be a C*-algebra and suppose that p and q are projections. Say that p is Murray-von Neumann equivalent to q (written $p \sim_{MvN} q$) if there exists a partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$. Let $Proj(M_\infty(A))$ denote the set of all projections that lie in $M_n(A)$ for some $n \in \mathbb{N}$, and set

$$V(A) = \{Proj(M_\infty(A))\}/\sim_{MvN} \qquad [\text{note} : M_\infty(A) = \bigcup_{n=1}^{\infty} M_n(A)].$$

- We can equip $V(A)$ with addition via
$$[p] + [q] = [p \oplus q].$$
- There is a pre-order given by $[p] \leq [q]$ if there exists a projection r such that $p \sim_{MvN} r \leq q$ (in the sense that $q - r$ is a positive element).

$V(A)$ is called the Murray-von Neumann semigroup. A *dimension function* on A is an additive, order-preserving map $d : V(A) \to \mathbb{R}^+ \cup \{\infty\}$.

(Aside: dimension functions lead to Murray and von Neumann's type classification of factors. Let M be a von Neumann factor, and let $V(M)_M$ denote $[p] \in V(M)$ where p is a projection in M. The possible images of $V(M)_M$ under d, up to normalization, are as follows:

Type I. $\{0, 1, \cdots, n\}$ or $\{0, 1, \cdots, \infty\}$
Type II. $[0, 1]$ or $[0, \infty]$
Type III. $\{0, \infty\}$.)

Let A be a unital C^*-algebra. $K_0(A)$ is the Grothendieck enveloping group of $V(A)$:

$$K_0(A) = \{[p] - [q] \mid [p], [q] \in V(A)\}/\sim$$

where $[p]-[q] \sim [e]-[f]$ if and only if there exists $[r]$ such that $[r]+[p]+[f]=[e]+[q]+[r]$ in $V(A)$. Set $K_0(A)^+$ to be the subset of $K_0(A)$ consisting of classes represented by elements of $V(A)$ (as opposed to formal differences of such).Then $(K_0(A), K_0(A)^+, [1_A])$ is a pointed, pre-ordered Abelian group. If $V(A)$ is cancellative then $V(A)$ is faithfully represented in $K_0(A)$ as $K_0(A)^+$. We have the following facts:

1. K_0 is a functor: if $\phi : A \to B$ is a unital $*$-homomorphism, then $K_0(\phi)[p] = [\phi(p)]$.
2. K_0 is homotopy invariant: if $\phi, \psi : A \to B$ are homotopic $*$-homomorphisms, then $K_0(\phi) = K_0(\psi)$.
3. K_0 is half exact: if

   $$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

   is an exact sequence of C^*-algebras, then

   $$K_0(I) \longrightarrow K_0(A) \longrightarrow K_0(B)$$

   is an exact sequence of Abelian groups.

Examples:

1. $A = \mathbb{C} \to (\mathbb{Z}, \mathbb{Z}^+, 1)$
2. $A = C[0,1]^n \to (\mathbb{Z}, \mathbb{Z}^+, 1)$
3. $A = M_n(\mathbb{C}) \to (\mathbb{Z}, \mathbb{Z}^+, n)$
4. $A = C(S^{2N})$, $N \geq 1 \to (\mathbb{Z} \oplus \mathbb{Z}, G, (1,0))$ where G is the perforated group

$$G = \{(0,0), (1,0), \cdots (N-1, 0)\} \cup \{(m,n) \mid m \geq N \text{ and } m \in \mathbb{Z}\}.$$

Finally, we note that if A is non-unital then we adjoin a unit to get $A^\sim$. This gives the exact sequence

$$0 \longrightarrow A \longrightarrow A^\sim \xrightarrow{\pi} \mathbb{C} \longrightarrow 0.$$

We define $K_0(A) = ker(K_0(\pi))$.

1.1.2 The K_1-group

We now move on to the K_1-group which, roughly, describes unitary operators in a C*-algebra up to homotopy. Suppose A is a unital C^*-algebra and define $\mathcal{U}(A)$ to be the unitary group of A ($u^{-1} = u^*$). Let $\mathcal{U}_0(A)$ be the connected component of the identity, 1_A, in $\mathcal{U}(A)$.

Exercise. Check that $\mathcal{U}_0(A)$ is normal in $\mathcal{U}(A)$.

Define $G_n = \mathcal{U}(M_n(A))/\mathcal{U}_0(M_n(A)$. The map

$$u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1_A \end{pmatrix}$$

taking $M_n(A)$ into $M_{n+1}(A)$ induces a map $\phi : G_n \to G_{n+1}$. Let $K_1(A)$ be the inductive limit of these groups; that is,

$$K_1(A) = \varinjlim (G_n, \phi_n).$$

Let us use $[u]_1$ to denote the K_1-class of $u \in \mathcal{U}(M_n(A))$.

K_1 has the following properties:

1. $K_1(A)$ is endowed with the structure of an Abelian group via

$$[u]_1 + [v]_1 = [uv]_1;$$

2. K_1 is a functor;
3. K_1 is homotopy invariant;
4. if A is non-unital, then we define $K_1(A) := K_1(A^{\sim})$.

Exercise. Show that $[uv]_1 = [vu]_1$.

1.1.3 Six term exact sequence

K-theory has a six term exact sequence: if

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

is an exact sequence, then the following six term sequence of groups is also exact:

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(B) \\ \uparrow & & & & \downarrow \\ K_1(B) & \longrightarrow & K_1(A) & \longrightarrow & K_1(I) \end{array}$$

The horizontal maps are those induced from the exact sequence, while the vertical maps are more complicated.

1.1.4 Cross products and the Pimsner - Voiculescu sequence

Suppose A is a C^*-algebra and $\alpha \in Aut(A)$. Then we define $A \rtimes_\alpha \mathbb{Z}$ to be the cross product; that is, for every representation $\pi : A \to \mathcal{B}(\mathcal{H})$ and $u \in \mathcal{B}(\mathcal{H})$ such that $u\pi(a)u^* = \pi(\alpha(a))$, for all $a \in A$, there exists a covariant representation of $A \rtimes_\alpha \mathbb{Z}$ into $C^*(\pi(A), u)$.

Exercise. Show that if α is inner, then $A \rtimes_\alpha \mathbb{Z} \cong A \otimes C(S^1)$.

Cross products give rise to the following exact sequence of K-groups:

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{id - K_0(\alpha)} & K_0(A) & \xrightarrow{K_0(\iota)} & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \xrightarrow[K_1(\iota)]{} & K_1(A) & \xrightarrow[id - K_1(\alpha)]{} & K_1(A) \end{array}$$

This exact sequence is called the Pimsner - Voiculescu sequence.

1.1.5 Universal Coefficient Theorem

There is a bivariant K-theory for pairs of separable C^*-algebras given by abelian groups, denoted $KK(A,B)$. We say that a C^*-algebra A satisfies the universal coefficient theorem (UCT) if, for all separable C^*-algebras B, the following sequence is exact:

$$0 \longrightarrow Ext(K_*(A), K_*(B)) \longrightarrow KK^*(A,B) \longrightarrow Hom(K_*(A), K_*(B)) \longrightarrow 0.$$

1.1.6 Kunneth Theorem

Let A and B be C^*-algebras and suppose A is in the 'bootstrap class'. Then the following sequence is exact:

$$0 \longrightarrow K_*(A) \otimes K_*(B) \longrightarrow K_*(A \otimes B) \longrightarrow Tor(K_*(A), K_*(B)) \longrightarrow 0.$$

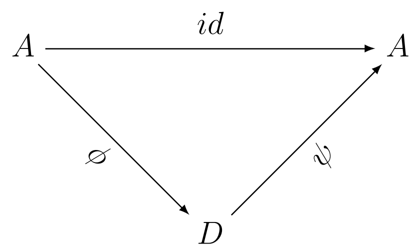
2 More K-theory: Elliott's conjecture

Let A be a unital C^*-algebra. The Elliott invariant of A consists of four pieces:

1. $(K_0(A), K_0(A)^+, [1_A])$;
2. $K_1(A)$;
3. $TA = \{\tau \text{ is a pos. lin. func. on A} \mid \ \tau(xy) = \tau(yx) \forall x, y \in A \text{ and } \tau(1_A) = 1\}$. The elements of TA are called tracial states, and TA is a compact, metrizable Choquet simplex for separable A;
4. A pairing $\rho_A : K_0(A) \times TA \to \mathbb{R}$ is defined by

$$\rho_A([p] - [q], \tau) = \tau(p) - \tau(q).$$

A C^*-algebra A is *nuclear* if for any finite subset F of A and $\epsilon > 0$, there exist a finite dimensional C^*-algebra D and contractive, completely positive maps $\phi : A \to D$ and $\psi : D \to A$ such that



commutes, up to ϵ on F ($\|f - \psi \circ \phi(f)\| < \epsilon$ for all $f \in F$).

2.1 Elliott Conjecture. *Let A and B be simple, unital, nuclear, separable C^*-algebras. If there exists an isomorphism*

$$\phi : Ell(A) \to Ell(B),$$

then there exists a $*$-isomorphism

$$\Phi : A \to B$$

such that $Ell(\Phi) = \phi$.

Exercise. Clarify what isomorphism means for $Ell(\cdot)$.

Even if one proves Elliott's conjecture for some class $\mathcal{C}$, one still needs the range of the invariant for satisfactory classification. Which instances of $Ell(A)$ occur for simple C^*-algebras? This question remains open, but we have an answer if $(K_0(A), K_0(A)^+)$ is weakly unperforated and A is separable. (An ordered group (G, G^+) is said to be weakly unperforated if whenever $n \cdot x \in G^+ \setminus \{0\}$, then $x \in G^+$.) In this case any 4-tuple in which K_0 and K_1 are countable can occur.

2.0.7 *UHF*-algebras

A C^*-algebra A is *UHF* if

$$A = \varinjlim (M_{n_i}(\mathbb{C}), \phi_i) \quad \text{where} \quad \phi_i : M_{n_i}(\mathbb{C}) \to M_{n_{i+1}}(\mathbb{C})$$

is a unital $*$-homomorphism. Note that, for the map ϕ_i to be unital, it is required that n_i divides n_{i+1}. From the n_i one produces a supernatural number $n = p_1^{k_1} p_2^{k_2} \cdots$, the least supernatural number such that every n_i divides n. As explained in Section 1, n corresponds to a subgroup of $\mathbb{Q}$, say G. At the level of K-theory, ϕ_i must act as follows:

$$\begin{array}{ccc} (K_0(M_{n_i}(\mathbb{C})), K_0(M_{n_i}(\mathbb{C}))^+, [1_{M_{n_i}(\mathbb{C})}]) & \cong & (\mathbb{Z}, \mathbb{Z}^+, n_i) \\ & & \Big\downarrow K_0(\phi_i) = \times \frac{n_{i+1}}{n_i} \\ (K_0(M_{n_{i+1}}(\mathbb{C})), K_0(M_{n_{i+1}}(\mathbb{C}))^+, [1_{M_{n_{i+1}}(\mathbb{C})}]) & \cong & (\mathbb{Z}, \mathbb{Z}^+, n_{i+1}) \end{array}$$

Fact. K_0 is a continuous functor, i.e., it commutes with inductive limits.

For us, this means

$$K_0(A = \varinjlim M_{n_i}(\mathbb{C})) = \varinjlim(K_0(M_{n_i}(\mathbb{C}))).$$

This shows that

$$(K_0(A), K_0(A)^+, [1_A]) \cong (G, G \cap \mathbb{Q}^+, 1).$$

2.2 Theorem (Glimm)**.** *If A and B are UHF, then $A \cong B$ if and only if*

$$(K_0(A), [1_A]) \cong (K_0(B), [1_B])$$

.

2.0.8 AF-algebras

AF stands for 'approximately finite dimensional' algebras. These were introduced by Bratelli in 1972. A C^*-algebra A is an AF-algebra if

$$A = \varinjlim(F_i, \phi_i) \quad \text{where} \quad \phi_i : F_i \to F_{i+1}$$

and F_i is a finite dimensional C*-algebra. Bratteli diagrams classify AF-algebras and if we restrict our attention to $F_i = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_k}$ and $F_{i+1} = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_l}$ then the map $\phi_i : F_i \to F_{i+1}$ has the form of a matrix B_i where the b_{ij} entry denotes the number of copies of M_{m_i} are mapped into M_{n_j}.

2.3 Theorem (Elliott, 1976)**.** *AF-algebras are determined, up to isomorphism, by their scaled, ordered Murray - von Neumann semigroup $(V(A), \Sigma V(A))$.*

It was later observed that ordered, scaled K_0 suffices.

2.0.9 $A\mathbb{T}$-algebras

A C^*-algebra A is an $A\mathbb{T}$-algebra if

$$A = \varinjlim(C(\mathbb{T}) \otimes F_i, \phi_i) \quad \text{where} \quad \phi_i : C(\mathbb{T}) \otimes F_i \to C(\mathbb{T}) \otimes F_{i+1}$$

and F_i is a finite dimensional C*-algebra.

2.4 Theorem (Elliott, 1989)**.** *$A\mathbb{T}$-algebras are determined, up to isomorphism, by*

$$K_0(A), K_0(A)^+, \Sigma K_0(A), K_1(A))$$

provided that A has real rank zero (lots of projections).

A is an AI-algebra if

$$A = \varinjlim (C(I) \otimes F_i, \phi_i) \quad \text{where} \quad \phi_i : C(I) \otimes F_i \to C(I) \otimes F_{i+1}$$

and F_i is a finite dimensional algebra. In 1991, Thomsen showed that traces and the associated pairing with K_0 matters for simple AI-algebras; that is, the invariant of the theorem above is not sufficient for a general classification.

2.0.10 Other confirmations of the conjecture

1. Simple AI-algebras, Elliott showed that $Ell(\cdot)$ classifies.
2. $A\mathbb{T}$-algebras with real rank zero, Elliott showed that $Ell(\cdot)$ classifies.
3. The irrational rotation algebra $A_\theta = C(\mathbb{T}) \rtimes_{\alpha_\theta} \mathbb{Z}$, where α_θ is rotation by $2\pi\theta$ for θ irrational. The K-theory is given by

   $$(K_0(A_\theta), K_0(A_\theta)^+, [1_{A_\theta}]) \cong (\mathbb{Z} + \theta\mathbb{Z}, \mathbb{Z} + \theta\mathbb{Z} \cap \mathbb{R}^+, 1),$$

   2.5 Theorem (Elliott-Evans, Putnam)**.** *A_θ is a simple $A\mathbb{T}$-algebra.*

4. Purely infinite algebras: A projection $p \in Proj(A)$ is *infinite* if $p \sim_{MvN} q \lneq p$. A simple C^*-algebra is said to be purely infinite if every non-zero hereditary sub-C^*-algebra of the given algebra contains an infinite projection.

   2.6 Theorem (Kirchberg-Phillips)**.** *Let A and B be simple, unital, separable, nuclear, purely infinite C^*-algebras which satisfy the UCT. Then $A \cong B$ if and only if $(K_0(A), [1_A], K_1(A)) \cong (K_0(B), [1_B], K_1(B))$.*

5. AH-algebras are inductive limits of C*-algebras of the form

   $$H_i = \{\oplus_{i=1}^n p_i M_{n_i}(C(X_i))p_i \mid X_i \text{ compact }, p_i \text{ a projection}\}.$$

   2.7 Theorem (Elliott, Gong-Li, Gong)**.** *Unital, simple AH-algebras for which the dimensions of the X_i above are uniformly bounded satisfy Elliott's conjecture.*

6. Tracially AF-algebras [Lin, 2000]. Note that Nate Brown will cover these in subsequent lectures.

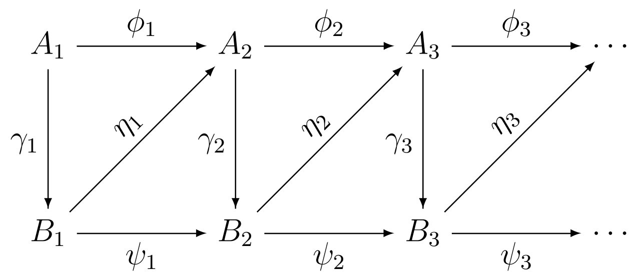
2.0.11 A counterexample

2.8 Theorem (Rørdam, 2002)**.** *There exists a simple, nuclear, separable C^*-algebra with a finite and an infinite projection.*

Exercise. Convince yourself, in light of the Kirchberg - Phillips Theorem, that the above theorem gives a counterexample to Elliott's conjecture.

3 Elliott's Intertwining Argument

Most confirmations of Elliott's conjecture involve, at some point and in some form, Elliott's Intertwining Argument, a technique for proving isomorphism of inductive limits. Let $A = \varinjlim(A_i, \phi_i)$ and $B = \varinjlim(B_i, \psi_i)$ be inductive limit C*-algebras (we may assume that the ϕ_i and ψ_i are injective; equivalently, the A_is and B_is are nested). Assume that we have $*$-homomorphisms γ_i and η_i together with the following (not necessarily commutative) diagram:



Suppose that we have sequences of nested finite sets (F_i) and (G_i) such that

- $F_i \subseteq A_i$ and $G_i \subseteq B_i$, and
- $\cup_i F_i$ is dense in A and $\cup_i G_i$ is dense in B.

Assume that the triangle formed by A_1, A_2, and B_1 at least commutes up to $\epsilon_1 > 0$ of F_1. Next assume that the triangle formed by B_1, B_2, and A_2 commutes up to $\epsilon_2 > 0$ on $G_1 \cup \gamma_1(F_1)$. Continuing in this manner defines a sequence of almost commuting triangles up to tolerances $\{\epsilon_i\}_{i=1}^{\infty}$. Now if $\sum \epsilon_i < \infty$, then for $j < i$,

$$\lim_{i\to\infty} \gamma_{i+1} \circ (\phi_i \circ \cdots \circ \phi_j)$$

defines a map on F_j; the sequence of these maps is Cauchy on $\cup_j F_j$.

Exercise. Check that this sequence defines a map on $\cup F_i$ which extends to a $*$-homomorphism from A to B.

The usual application of this argument is the question of isomorphism.

3.0.12 Intertwining of UHF-algebras

We now examine Glimm's Theorem for UHF algebras to illustrate Elliott's Intertwining Argument. Recall that a UHF-algebra A is an inductive limit of matrix algebras $\varinjlim(M_{n_i}, \phi_i)$. We re-state and sketch the proof of Theorem 2.2.

3.1 Theorem (Glimm)**.** *If A and B are UHF, then $A \cong B$ if and only if*

$$(K_0(A), [1_A]) \cong (K_0(B), [1_B])$$

.

Sketch of proof. At the level of K-theory we can (after perhaps compressing our inductive sequences) intertwine the inductive sequences to give the following commuting diagram:

$$
\begin{array}{ccccccccc}
K_0(M_{n_1}) & \longrightarrow & K_0(M_{n_2}) & \longrightarrow & K_0(M_{n_3}) & \longrightarrow & \cdots\cdots & \longrightarrow & K_0(A) \\
\downarrow & \nearrow & \downarrow & \nearrow & \downarrow & & & & \downarrow \cong \\
K_0(M_{m_1}) & \longrightarrow & K_0(M_{m_2}) & \longrightarrow & K_0(M_{m_3}) & \longrightarrow & \cdots\cdots & \longrightarrow & K_0(B)
\end{array}
$$

More explicitly:

$$
\begin{array}{ccccccccc}
(\mathbb{Z},\mathbb{Z}^+,n_1) & \longrightarrow & (\mathbb{Z},\mathbb{Z}^+,n_2) & \longrightarrow & (\mathbb{Z},\mathbb{Z}^+,n_3) & \longrightarrow & \cdots\cdots & \longrightarrow & K_0(A) \\
\times\frac{m_1}{n_1}\downarrow & \nearrow +\frac{n_2}{m_1} & \times\frac{m_2}{n_2}\downarrow & \nearrow +\frac{n_3}{m_2} & \times\frac{m_3}{n_3}\downarrow & & & & \downarrow \cong \\
(\mathbb{Z},\mathbb{Z}^+,m_1) & \longrightarrow & (\mathbb{Z},\mathbb{Z}^+,m_2) & \longrightarrow & (\mathbb{Z},\mathbb{Z}^+,m_3) & \longrightarrow & \cdots\cdots & \longrightarrow & K_0(B)
\end{array}
$$

It is straightforward to lift the above diagram to the level of C^*-algebras and $*$-homomorphisms in a not necessarily commuting way:

$$
\begin{array}{ccccccccc}
M_{n_1} & \xrightarrow{\phi_1} & M_{n_2} & \xrightarrow{\phi_2} & M_{n_3} & \xrightarrow{\phi_3} & \cdots\cdots & \longrightarrow & A \\
\gamma_1\downarrow & \nearrow \eta_1 & \gamma_2\downarrow & \nearrow \eta_2 & \gamma_3\downarrow & & & & \\
M_{m_1} & \xrightarrow{\psi_1} & M_{m_2} & \xrightarrow{\psi_2} & M_{m_3} & \xrightarrow{\psi_3} & \cdots\cdots & \longrightarrow & B
\end{array}
$$

Exercise. Show that if $\phi, \psi : M_k \to M_l$ are unital $*$-homomorphisms, then there exists a unitary in M_l, say u, such that $\phi = u\psi u^*$.

With the exercise in hand, the γ_i and η_i can be conjugated by unitaries which will make the diagram above commute. Then Elliott's Intertwining Argument gives isomorphism. □

4 Infinite dimensional phenomena in simple C^*-algebras

Question. Does there exist a simple C^*-algebra whose ordered K-theory group is *not* weakly unperforated?

Jesper Villadsen answered this question affirmatively in 1996. To understand what he did and why it matters for classification, we need the classical view of K-theory. Recall the following from the talks of Heath Emerson: in $C(X)$ projections are given by complex topological vector bundles and Murray-von Neumann equivalence of projections corresponds to isomorphism of vector bundles.

We wish to 'see' perforation in $(K_0(C(X)), K_0(C(X))^+)$. Our tool is the Chern class, a map $c : Vect(H) \to H^{2*}(X;\mathbb{Z})$ with properties

1. $c(\omega) = 1 + c_1(\omega) + c_2(\omega) + \cdots + c_{dim(\omega)}(\omega)$ where $c_i(\omega) \in H^{2i}(X;\mathbb{Z})$
2. $c(\theta) = 1$ when θ is trivial
3. $c(\gamma + \omega) = c(\gamma)c(\omega)$
4. $c(\gamma) = c(\omega)$ whenever $\gamma \cong \omega$.

4.1 Lemma (Villadsen)**.** *Suppose ω is a vector bundle over X and that $c_{dim(\omega)}(\omega) \neq 0$. If $l < dim(\omega)$, then*

$$[\omega] - [\theta_l] \notin K_0(C(X))^+.$$

Proof. If $[\omega] - [\theta_l]$ is positive, then there exists $r \geq 0$ and γ a vector bundle over X such that

$$\begin{aligned} & \omega \oplus \theta_l \cong \gamma \oplus \theta_{l+r} \\ \Longrightarrow \quad & c(\omega \oplus \theta_l) = c(\gamma \oplus \theta_{l+r}) \\ \Longrightarrow \quad & c(\omega)c(\theta_l) = c(\gamma)c(\theta_{l+r}) \\ \Longrightarrow \quad & c(\omega) = c(\gamma) \end{aligned}$$

which is a contradiction, since $c_{dim(\omega)}(\omega) \neq 0$ and $c_{dim(\omega)}(\gamma) = 0$. □

Fact. If ω and γ are vector bundles over X, $dim(X) < d < \infty$ and $dim(\omega) \geq dim(\gamma) + d/2$, then γ is isomorphic to a sub-bundle of ω and

$$[\omega] - [\gamma] \in K_0(C(X))^+.$$

Thus, if $[\omega] - [\theta_l]$ is not positive and $0 < l < dim(\omega)$, then

$$n([\omega] - [\theta_l]) = [\omega \oplus \cdots \oplus \omega] - [\theta_{nl}]$$

is positive for some n and therefore $K_0(C(X))^+$ is not weakly unperforated.

Now we have a tool for finding perforation. But we want a simple algebra. Start with $M_2(C(S^2 \times S^2))$ (because there exists a line bundle H^* over S^2 such that $c(H^*) = 1$). Let $\pi_i : S^2 \times S^2 \to S^2$ be coordinate projections. Define $H^* \times H^* = \pi_1^*(H^*) \oplus \pi_2^*(H^*)$. By naturality of $c(\cdot)$ and the ring structure of $H^{2*}(S^2 \times S^2)$, we can show that $c_2(H^* \times H^*) \neq 0$, whence $[H^* \times H^*] = [\theta_1] \notin K_0(C(S^2 \times S^2))^+$. Now define $\phi : M_2(C(S^2)^{\times 2}) \to M_4(C(S^2)^{\times 4})$ as follows: let $\eta_1, \eta_2 : (S^2 \times S^2)^{\times 2} \to S^2 \times S^2$ be coordinate projections. Then

$$\phi(f) = \begin{pmatrix} f \circ \eta_1 & 0 \\ 0 & f \circ \eta_2 \end{pmatrix}.$$

One calculates:

$$\begin{aligned} K_0(\phi)[H^* \times H^*] &= [H^* \times H^* \times H^* \times H^*] \\ K_0(\phi)[\theta_l] &= [\theta_{2l}] \end{aligned}$$

$$[H^* \times H^*] - [\theta_1] \mapsto [H^* \times H^* \times H^* \times H^*] - [\theta_2].$$

Perforation is preserved! By iterating and perturbing slighlty a sequence of such maps, one gets a simple limit with perforated K_0-group. This algebra is AH, but does not have bounded dimension growth.

Idea of things to come. It was thought when Villadsen's proof arrived that there might be a counterexample to Elliott's conjecture in it. The problem was that the K-theory of Villadsen's algebra was incomputable. One wants the same sort of phenomenon with computable K-theory. To this end, we replace projections with positive elements... this leads us to the Cuntz semigroup.

5 The Cuntz semigroup

5.1 Definition (Cuntz)**.** Let A be a C^*-algebra and a, b in $(A \otimes \mathcal{K})^+$. We say $a \precsim b$ if there exists a sequence $\{v_i\}_{i\in\mathbb{N}}$ in $A \otimes \mathcal{K}$ such that $v_i b v_i^* \to a$ in norm. We say $a \sim b$ if $a \precsim b$ and $b \precsim a$. Let

$$W(A) = \{(A \otimes \mathcal{K})^+\}/\sim .$$

Let $< a >$ denote the class of a in $W(A)$. $W(A)$ can be made into an ordered semigroup, the Cuntz semigroup, by equipping it with the binary operation

$$< a > + < b > = < a \oplus b > .$$

and the order

$$< a > \leq < b > \Leftrightarrow a \precsim b.$$

Examples:

1. $A = \mathbb{C}$, $W(A) = \mathbb{N} \cup \{\infty\}$
2. $A = C([0, 1])$, $W(A) = \{f : [0, 1] \to \mathbb{N} \cup \infty \mid \text{f is lower semi-cts}, f \geq 0\}$
3. $A = C(X)$, $W(A)$ can be a disaster
   - Let us show this for $X = [0, 1]^3$. Notice that we have $S^2 \subset [0, 1]^3$ and S^2 has the projection H^* which is the dual to the Hopf bundle. Now we can extend H^* to the open superset U of S^2 which we continue to denote by H^*. Multiply H^* by a continuous function $f : [0, 1]^3 \to [0, 1]$ such that $f|_{S^2} = 1$, $f|_{U^c} = 0$. Then $f \cdot H^*$ is positive in $M_2(C([0, 1]^3))$. Of course, we could do the same trick for any projection p over S^2, to get a positive element $f \cdot p$. Now we observe that $f \cdot p \sim f \cdot q$ if and only if $p \sim_{MvN} q$.
     The idea of this type of argument is that $W(C(X))$ 'contains' the K-theory of the closed subsets of X.

5.2 Definition (Rørdam)**.** $W(A)$ is called *almost unperforated (AUP)* if $x \leq y$ whenever $(n + 1)x \leq ny$ for $x, y \in W(A)$.

Now observe that Villadsen's Lemma 4.1 can also be a tool for witnessing a failure of almost unperforation. To wit, apply Villadsen's argument for $[H^* \times H^*] - [\theta_1]$ to $< f \cdot H^* \times f \cdot H^* >$ and $< f \cdot \theta_1 >$ to see the failure of almost unperforation in $M_4(C([0,1]^3 \times [0,1]^3))$.

5.3 Theorem. *There exists a simple, unital AH-algebra A and a UHF algebra B such that $Ell(A) = Ell(A \otimes B)$ such that $A \not\cong A \otimes B$.*

Sketch of proof. Take Villadsen's construction for the inductive limit of $(M_{2^n}(C(S^2)^{2n}), \phi_n)$ and replace S^2 with $[0,1]^3$. Let A be the inductive limit, then $W(A)$ fails to be almost unperforated for the same reason that the K-theory of Villadsen's example failed to be weakly unperforated. On the other hand, $A \otimes B$ is almost unperforated by a result of Rørdam, and some standard results about the Elliott Invariant and tensor products takes care of the rest. □

5.0.13 States on the Cuntz semigroup

5.4 Definition. An additive, order preserving map $d : W(A) \to \mathbb{R}^+$ is a *state* on $W(A)$.

Example: If τ is a tracial state on a unital C^*-algebra A, then $d_\tau(< a >) = \varinjlim \tau(a^{1/n})$ is a state on $W(A)$. (This should be thought of as giving rank with respect to τ.)

5.5 Definition. A unital C^*-algebra A has *strict comparison* if $x \lesssim y$ in $W(A)$ whenever $d_\tau(x) \leq d_\tau(y)$ for all $\tau \in T(A)$.

5.6 Theorem (Rørdam)**.** *If A is simple then $W(A)$ is almost unperforated if and only if A has strict comparison.*

5.7 Definition. $W(A)$ is *almost divisible* if, for all x in $W(A)$ and for all n in $\mathbb{N}$, there exists y in $W(A)$ such that $ny \leq x \leq (N+1)y$.

5.8 Theorem. *Let A be a simple, unital, tracial C^*-algebra. Suppose that A has strict comparison and $W(A)$ is almost divisible. Then,*

$$W(A) = V(A) \bigsqcup SAFF_{>0}(TA)$$

where $V(A)$ is the Murray - von Neumann semigroup and $SAFF_{>0}(TA)$ is the suprema of sequences of continuous, affine, strictly positive functions on TA.

Note that we can define a map $\iota : W(A) \to SAFF_{>0}(TA)$ via

$$\iota(x)(\tau) = d_\tau(x).$$

Question. Is it always true for a simple, unital C^*-algebra A that the image of $\iota : W(A) \to SAFF_{>0}(TA)$ is onto?

Remarks.

- Suppose $\mathcal{Z}$ is the Jiang - Su algebra, and $A \cong A \otimes \mathcal{Z}$, then the question has a positive answer.
- If A has strict comparison and extreme boundary of the tracial state space is both compact and finite dimensional, then the question has positive answer.

Exercise. Prove that the question has positive answer if A has strict comparison and unique trace.