## 1. Pressure and codons (Problem 85)

Dan Thompson gave a "Mini-presentation on Bowen Notebook Problem 85". This is the succinct problem/question:

Codon frequencies via equilibrium states for "some potential"?

Very briefly: in a recent paper, Thompson and Koslicki used a finite, small-in-large-scale version of topological pressure pressure to distinguish coding sequences in the genome of humans (and some other species) from the so-called "junk DNA" not directly transcribed to code proteins. Thompson also cited related work of a group including Mike Shub, and of Bruno Cessac.

See the mini-presentation for more.

## 2. The Entropy Conjecture (Problem 12)

Shub's famous and longstanding Entropy Conjecture holds that for any $C^1$ diffeomorphism of a compact manifold, the entropy is bounded below by $\log(\rho)$, where $\rho$ is the spectral radius of the induced action on homology.

Mike Shub gave at the blackboard a presentation which followed fairly closely his succinct file, "Remarks on the history of the Entropy Conjecture". He described how the conjecture was born, and a significant comment of Bowen, which facilitated a key transition in the formulation of his conjecture.

See Shub's "Remarks ... " for more.

## 3. "Classify symbolic systems with specifications" (Problem 32)

"Classify symbolic systems with specification": this was Bowen's Problem 32. Vaughn Climenhaga and Dominic Kwietniak gave presentations on this.

Climenhaga described some key advances on systems with specification since Bowen, due to Bertrand (1985), Fiebig and Fiebig (1992) and Thomsen (2006).

Kwietniak offered a program to show that classification of symbolic systems with specification up to topological conjugacy is maximal among the Borel equivalence relations with countable equivalence classes (under the preordering of Borel reducibility). Informally, this would imply that any dynamical classification problem with countable equivalence classes can be reduced to the problem of classifying systems with specification up to topological conjugacy. The idea is to find a class of systems with specification for which classification of their Markov boundaries (introduced by Klaus Thomsen) implies topological conjugacy.

In contrast, by Mike Hochman's work (as Kwietniak noted), given Bernoullicity of the unique measure of maximal entropy for subshifts with specification (with entropy then being a complete invariant of measurable conjugacy w.r.t. these measures), topological entropy is a complete invariant for the following relation: Borel conjugacy of the systems restricted to the complement of the periodic points. The Bernoullicity statement for the maximal measures does not appear to be in the literature, although from the references provided by Climenhaga, there is a plausible proof scheme to be checked (use the synchronizing property to cover the specification subshift with a countable state equal entropy positive recurrent Markov shift, such that the factor map onto the subshift sends the Markov shift measure of maximal entropy to an equal entropy measure).

Discussion was lively. For example, a free thinker asked why specification was of interest. Another offered a classification program which could not succeed if Kwietniak's program can be carried through.

## 4. Relative equilibrium states

Jisang Yoo gave a presentation describing progress on the understanding of relative equilibrium staets for a factor code $\pi : X \to Y$ from an irreducible shift of finite type: $\pi$ is a composition of a class degree one code followed by a finite to one code from a sofic shift. Consequetly, any potential function of sufficient regularity lifts to a unique measure of maximal relative entropy. Yoo asked if there is any generalization of this result for $\mathbb{Z}^d$ SFTs for $d > 1$.

For more detail, see Yoo's presentation, "Generalizing the uniqueness of equilibrium states in a conditional setting", the references cited there, and his paper "Decomposition of infinite-to-one factor codes and uniqueness of relative equilibrium states" (https://arxiv.org/abs/1705.00448).

## 5. Geodesic flows as Smale flows (Problem 6)

Bowen's Problem 6 ("Zeta function for Axiom A flows and systems") included a request for connection with geodesic results. Dan Thompson described his joint work with Constantine and Lafont studying geodesic flows of CAT(-1) spaces as Smale flows. A key technical issue is to make a construction for which the regularity of the roof function suffices to apply Pollicott's work on Smale flows. Thompson posed the general problem of establishing the mixing property for such flows, and extending the work to other types of spaces.

For more, see the Constantine-Lafont-Thompson paper "The weak specification property for geodesic flows on CAT(-1) spaces" (https://arxiv.org/abs/1704.00857).

## 6. Almost specification and unique measures of maximal entropy

Ronnie Pavlov gave background and (see below) one precise version of the question: how much can the specification property be relasxed and still guarantee there is a unique measure of maximal entropy (MME)?

A subshift $X$ has *almost specfication* for a mistake function $g : \mathbb{N} \to \mathbb{N}$ if for all words $w_1, \ldots, w_k$ in the $L(X)$ (the language of $X$) there exist words $v_1, \ldots, v_k$ in $L(X)$ such that

(1) For all $i$, $v_i$ and $w_i$ have equal length.
(2) For all $i$, $v_i$ and $w_i$ differ in at most $g(|w_i|)$ letters (i.e., $v_i$ is copied from $w_i$ with at most $g(|w_i|)$ "mistakes").
(3) The word $v_1 v_2 \ldots v_k$ is in $L(X)$.

For example, $\beta$-shifts have almost specification with $g = 1$ (i.e., $g$ is the constant function 1). Pavlov has shown there exist subshifts with $g = 4$ (!) which do not have a unique MME. Climenhga and Pavlov have shown $g = 1$ does guarantee a unique MME.

Pavlov asked for the boundary constant: does $g = 2$ guarantee a unique MME? $g = 3$?

Answering a question, Pavlov asserted he could adapt his construction to produce an arbitrarily large finite number of ergodic MMEs for a subshift satisfying

almost specification with $g = 4$. This leaves another open question: which mistake functions $g$ guarantee there are only finitely many ergodic MMEs?

For more, see the Climenhaga-Pavlov paper "One-sided almost specification and intrinsic ergodicity" (https://arxiv.org/abs/1605.05354) and its references.

## 7. Symbolic codings for Vershik maps

Karl Petersen posed the problem: when does a Vershik map on a Bratteli diagram $X$ admit a symbolic coding?

Here, the lexicographic (Vershik) map is not required to have unique maximal and minimal paths, or to be continuous. A symbolic coding here means a factor map from the Vershik map to a subshift which is injective on the complement of a set which has measure zero for every nonatomic invariant Borel probability. Such a coding maps a point to its itinerary through a finite Borel partition. Variants of the question might require the finite partition to be a partitio of $X$ into clopen sets, or to be adapted only to a single measure on $X$,

Petersen described related work by Xavier Mela (coding the Pascal graph with a left-right ordering); by Frick-Petersen-Shields; and by Downarowicz and Maass, who showed that a continuous Vershik rank on a Bratelli diagram of finite rank (i.e., uniformly bounded number of vertices at each level) defines either a subshift or an odometer.

## 8. Hochman's Speed Date

Mike Hochman gave several open problems in a "speed date", giving not so much definition and background so as to give more problems.

(1) For $k < d$, the automorphism group of a full $\mathbb{Z}^k$ shift on two symbols embeds as a subgroup in the automorphism group of a full $\mathbb{Z}^d$ shift on two symbols. Is there an embedding if $k > d$?

(2) The set of nonexpansive directions for an infinite $\mathbb{Z}^2$ subshift can be any nonempty closed set of directions. But, what can this set be if the subshift is required to be minimal?

(3) (Question of B. Weiss) Suppose $T$ is a homeomorphism of a compact metric space $X$, and for all $(x, y)$ in $X \times X$, the point $(x, y)$ is forward or backward recurrent under $T \times T$. Does this force the topological entropy of $T$ to be zero? (If "forward or backward" is replaced with "forward", the answer is Yes.)

(4) Let $X$ be a $\mathbb{Z}^2$ SFT with block gluing at seperation $n^\alpha$: that is there exists a positive constant $C$ such that for any pair of $n \times n$ words of $X$, there is a point of $X$ in which they occur with separation at most $Cn^\alpha$. In an arxiv post, Gangloff and Sablik produce a positive constant $\kappa$ such that $\alpha < \kappa$ implies the language of $X$ is decidable (there is a Turing machine which for all words on the alphabet of $X$ will give a definite answer as to whether the word is in the language of $X$). Gangloff and Sablik show for $\alpha = 1$ this property is lost.

What is the largest number $\kappa$ such that for $\alpha < \kappa$, a $\mathbb{Z}^2$ SFT with block gluing at seperation $n^\alpha$ must have a decidable language ?

(5) Let $X$ be the full $\mathbb{Z}$ shift on two symbols. Let $Y$ be the $\mathbb{Z}$ SFT on symbols $a, b, c$ defined by disallowing the words $aa, bb, cc$. Let $X'$ be the complement in $X$ of the periodic points. Let $Y'$ be the complement in $Y$ of the periodic

points. With the restriction of the shift, these are self homeomorphisms of Polish spaces. There is a Borel isomorphism $\phi : X' \to Y'$ which conjugates these actions.

Can $\phi$ be made continuous? (i.e., a topological conjugacy)

## 9. Oriented expansive lines

John Franks noted that there are oriented and nonoriented notions of an expansive line for a $\mathbb{Z}^2$ action on a compact metric space, and that there are good reasons for considering the oriented version. He asked if there could be a classification of $Z^2$ SFTS for which there are finitely many nonexpansive oriented lines. (There are many algebraic systems satisfying this condition; the Ledrappier example is an examplar.)

Given the technology for constructions involving Turing machines, some pessism was expressed by other participants about prospects for such a classification.

## 10. Bowen's Dream (Problem 7)

Bowen's Problem 7, "Structure of basic sets", has a part (a): "Classification via (R,A)". Mike Boyle referred to this as 'Bowen's Dream', still largely unrealized. What is it?

A square nonnegative integer matrix $A$ can be used to define an SFT. Significant invariants of the SFT can be computed from $A$. Bowen noted that any expansive quotient (factor) of an SFT can be presented (up to topological conjugacy) by a pair $(R, A)$. Here $R$ is a relation $\sim$ on symbols of the SFT: under the factor map to the quotient system, points $x$ and $w$ have the same image if and only if for every $n$ in $\mathbb{Z}$, $x_n \sim w_n$. The relation is reflexive and symmetric, but not transitive (when the quotient is not zero dimensional). The dream would be to compute properties of the quotient from $(R, A)$ – classification? fundamental group? ... This is done for the zeta function, but (apart from the case of a zero dimensional quotient–i.e., a sofic shift) not for much more. David Fried studied these quotients in his paper "Finitely presented dynamical systems". See the comments on Problem 7 in the notebook for more, and for references.

Another question (of Kitchens?) asks if the classification up to topological conjugacy from $(R, A)$ is undecidable. This is open even in the case $R$ is trivial – i.e., it is not known if the classification of SFTs is undecidable.