Summer School on Surgery and the Classification of Manifolds: Exercises

Monday

1. Let $n > 5$. Show that a smooth manifold homeomorphic to D^n is diffeomorphic to D^n.

2. For a topological space X, the *homotopy automorphisms* $\mathrm{hAut}(X)$ is the set of homotopy classes of self homotopy equivalences of X. Show that composition makes $\mathrm{hAut}(X)$ into a group.

3. Let M be a simply-connected manifold. Show $\mathcal{S}(M)/\,\mathrm{hAut}(M)$ is in bijection with the set $\mathcal{M}(M)$ of homeomorphism classes of manifolds homotopy equivalent to M.

4. *Let M^7 be a smooth 7-manifold with trivial tangent bundle. Show that every smooth embedding $S^2 \hookrightarrow M^7$ extends to a smooth embedding $S^2 \times D^5 \hookrightarrow M^7$. Are any two such embeddings isotopic?

5. *Make sense of the slogan: "The Poincaré dual of an embedded submanifold is the image of the Thom class of its normal bundle." Then show that the geometric intersection number (defined by putting two submanifolds of complementary dimension in general position) equals the algebraic intersection number (defined by cup product of the Poincaré duals). Illustrate with curves on a 2-torus.

6. Let M^n and N^n be closed, smooth, simply-connected manifolds of dimension greater than 4. Then M and N are diffeomorphic if and only if $M \times S^1$ and $N \times S^1$ are diffeomorphic.

7. A degree one map between closed, oriented manifolds induces a split epimorphism on integral homology. (So there is no degree one map $S^2 \to T^2$.)

Tuesday

8. Suppose $(W; M, M')$ is a smooth h-cobordism with $\dim W > 5$. Then $W - M'$ is diffeomorphic to $M \times [0, 1)$. (This is a version of the Eilenberg swindle.)

9. A degree one map between closed, oriented, connected manifolds induces a epimorphism on the fundamental group. Give an example where the epimorphism is not split.

10. Why does $S^2 = \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2$ not extend to a embedding $S^2 \times D^2 \hookrightarrow \mathbb{C}P^2$? Does $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^3$ have a trivial normal bundle?

11. *Suppose ξ is an $k + n$-dimensional vector bundle over a CW complex K of dimension k. Show that ξ is isomorphic to $\eta \oplus \underline{\mathbb{R}}^n$ for some k-dimensional vector bundle η over K. Show that for $n > 0$, if ξ is stably trivial, then ξ is trivial.

12. Let t generate a cyclic group C_5 of order 5.

    (a) Verify that $1 - t - t^4 \in \mathbb{Z}[C_5]$ is a unit.

    (b) Define a ring homomorphism $\phi : \mathbb{Z}[C_5] \to \mathbb{C}$ such that $\phi(1 - t - t^4) \notin S^1 \subset \mathbb{C}$.

    (c) Deduce that $\mathrm{Wh}(C_5)$ is infinite (Hint: $K_1(\mathbb{C})$ maps to $\mathbb{C}^\times$ via the determinant.)

13. $\mathbb{R}P^2$ is a nonorientable manifold. Let $\mathbb{Z}_w$ be the corresponding local coefficient system. Demonstrate Poincaré duality by computing $H_*(\mathbb{R}P^2; \mathbb{Z}), H^*(\mathbb{R}P^2; \mathbb{Z}_w)$ and $H_*(\mathbb{R}P^2; \mathbb{Z}_w), H^*(\mathbb{R}P^2; \mathbb{Z})$ using the free $\mathbb{Z}[\mathbb{Z}/2]$-complex given by the cellular homology of $\widetilde{\mathbb{R}P^2}$.

Wednesday

14. Let $f : S^k \to S^k$ be a homeomorphism. Show that $D^{k+1} \cup_f D^{k+1}$ is homeomorphic to S^{k+1}. This shows that a smooth closed manifold with a Morse function with two critical points is homeomorphic to a sphere (why?)

15. A degree one normal map restricted to the transverse inverse image of a submanifold is itself a degree one normal map.

16. Compute the topological structure set of $S^m \times S^n$.

17. Show there is a smooth, compact, parallelizable manifold Ω^8 with signature 8 and with boundary a homotopy sphere Σ^7 (use Wall Realization or plumbing). Show Σ^7 is homeomorphic to a sphere (use the h-cobordism theorem), but not diffeomorphic to a sphere (use the Hirzebruch Signature Theorem), and that $M^8 := \Omega^8 \cup \text{cone}\, \Sigma^7$ is a closed topological manifold with does not admit a smooth structure. (M^8 is often called the *Milnor 8-manifold*.)

18. *If Σ^k is an exotic k-sphere which bounds a parallelizable manifold, then $\Sigma^k \times S^2$ is diffeomorphic to $S^k \times S^2$. (This follows from a relative version of the π-π theorem).

19. *What is the order of $\mathcal{M}^{TOP}(S^2 \times S^2)$? What is the order of $\mathcal{M}^{TOP}(\mathbb{C}P^2)$?

20. *If $f : M^n \to N^n$ is a tangential homotopy equivalence (resp. a normal map, normally bordant to the identity), then $M^n \times D^{n+1}$ (resp. $M^n \times D^3$) is diffeomorphic to $N^n \times D^{n+1}$ (resp. $N^n \times D^3$).

Thursday

21. Any aspherical manifold has torsionfree fundamental group.

22. Compute the topological structure set of $S^m \times T^n$.

23. How many diffeomorphism classes of homotopy 7-spheres exist? How many diffeomorphism classes of homotopy 5-tori exist?

24. *The Borel Conjecture is true in dimension 1, 2, and 3. Discuss.

25. *Suppose the Novikov Conjecture holds for a closed aspherical manifold X whose dimension is greater than four. Then the number of homeomorphism classes of manifolds of the form $M \times D^3$ so that M is homotopy equivalent to X is finite.

26. Suppose the Novikov Conjecture holds for a closed, aspherical manifold X^k. Then for $f : M^{k+4n} \to X^k$ transverse to a point $* \in X$, sign $f^{-1}(*)$ is a homotopy invariant.