Open questions on Jacobians of curves over finite fields: supersingular curves

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Effective methods for abelian varieties
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Let $p$ be a prime number. Let $g$ be a natural number.

**Open question:**
Does there exist a supersingular curve of genus $g$ defined over a finite field of characteristic $p$, for every $p$ and $g$?

**Outline.** What is:
1. a supersingular elliptic curve;
2. a supersingular curve of higher genus;
3. known about this question already;
4. the next step?
Complex elliptic curves and \( p \)-torsion

Let \( E \) be a complex elliptic curve.

\[ E \cong \mathbb{C}/L \] for a lattice \( L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \).

(Thus \( E \) is an abelian group).

Torsion points: \( E[p](\mathbb{C}) = \{ Q \in E(\mathbb{C}) \mid pQ = 0_E \} \).

Then \( E[p](\mathbb{C}) \cong \frac{1}{p}L/L \cong (\mathbb{Z}/p)^2 \).

If \( X \) is a complex curve of genus \( g \geq 2 \), its Jacobian \( J_X \) is a p.p. abelian variety of dimension \( g \) and \( J_X[p](\mathbb{C}) \cong (\mathbb{Z}/p)^{2g} \).
Elliptic curves - algebraic version

Let $E : y^2 = h(x)$ be an elliptic curve over $k = \overline{\mathbb{F}}_p$ where

$h(x) = x^3 + ax^2 + bx + c = \prod_{i=1}^{3}(x - \lambda_i)$.

Algebraic group law on $E$:

The $\ell$-torsion of $E$ is $\text{Ker}[\ell]$ where $[\ell] : E \to E$ is mult.by-$\ell$.

$E[\ell](k) := \{Q \in E(k) \mid \ell Q = 0_E\} \simeq (\mathbb{Z}/\ell)^2$ if $p \nmid \ell$. 

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Let $E : y^2 = x^3 + ax^2 + bx + c$ and $\ell = 3$.

A point $Q$ has order 3 iff $2Q = -Q$ iff $x(2Q) = x(Q)$.

This occurs iff $x(Q)$ is a root of the $3$-division polynomial.

```python
P. < a, b, c > = PolynomialRing(ZZ,3)
E = EllipticCurve(P,[0,a,0,b,c])
d3 = E.division_polynomial(3,x=None)
```

$$3 \cdot x^4 + 4 \cdot a \cdot x^3 + 6 \cdot b \cdot x^2 + 12 \cdot c \cdot x - b^2 + 4 \cdot a \cdot c$$

If $p \neq 3$, then $d_3(x)$ has 4 distinct roots so $E$ has 8 points of order 3 and $|E[3](k)| = 9$. 
Collapsing torsion points - example

What if $p = 3$?

$$d_3 = 3 \cdot x^4 + 4 \cdot a \cdot x^3 + 6 \cdot b \cdot x^2 + 12 \cdot c \cdot x - b^2 + 4 \cdot a \cdot c.$$
Collapsing torsion points - example

What if \( p = 3 \)?
\[
d_3 = 3 \cdot x^4 + 4 \cdot a \cdot x^3 + 6 \cdot b \cdot x^2 + 12 \cdot c \cdot x - b^2 + 4 \cdot a \cdot c.
\]

\( P3. < a, b, c > = \text{PolynomialRing}(GF(3), 3) \)
\[
r_3 = d_3.\text{change\_ring}(P3)
\]
\[
+ a \cdot x^3 - b^2 + a \cdot c
\]

\textbf{Mod p binomial thm:} In \( k[x] \), \((x + \alpha)^p = x^p + \alpha^p\).

So \( r_3 = a \cdot x^3 - b^2 + a \cdot c \) has
\[
\begin{cases}
\text{one (triple) root} & a \not\equiv 0 \mod 3 \\
\text{no roots} & a \equiv 0 \mod 3
\end{cases}
\]

So \(|E[3](k)|\) divides 3 when \( p = 3 \).
Ordinary and supersingular elliptic curves

<table>
<thead>
<tr>
<th>$p$</th>
<th>$r_p$ reduction of $p$ – division polynomial of $y^2 = x^3 + bx + c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$+2 \cdot b \cdot x^{10} - b^2 \cdot c \cdot x^5 + b^6 - 2 \cdot b^3 \cdot c^2 - c^4$</td>
</tr>
<tr>
<td>7</td>
<td>$+3 \cdot c \cdot x^{21} + 3 \cdot b^2 \cdot c^2 \cdot x^{14} + (-b^7 \cdot c - 2 \cdot b^4 \cdot c^3 + 3 \cdot b \cdot c^5) \cdot x^7$ $- b^{12} - b^9 \cdot c^2 + 3 \cdot b^6 \cdot c^4 - b^3 \cdot c^6 + 2 \cdot c^8$</td>
</tr>
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Then $r_p$ has at most $(p - 1)/2$ roots. The $p$-torsion points on $E : y^2 = f(x)$ collapse to either $p$ points or 1 point modulo $p$.

**Def:**

$$E \text{ is } \begin{cases} \text{ordinary} & \text{if } |E[p](k)| = p \\ \text{supersingular} & \text{if } |E[p](k)| = 1 \end{cases}$$
If \( E/\mathbb{F}_q \) is elliptic curve, then \( \#E(\mathbb{F}_q) = q + 1 - a. \)
The zeta function of \( E \) is \( Z(t) = (1 - at + qt^2)/(1 - t)(1 - qt). \)

Fact: \( p | a \) iff \( E \) supersingular.

\( E \) supersingular, Newton polygon of \( 1 - at + qt^2 \) has slopes 1/2.

called \( G_{1,1} \).

\( E \) ordinary, then Newton polygon has slopes 0 and 1.

called \( G_{0,1} \oplus G_{1,0} \).
$E = \textit{EllipticCurve}(GF(5), [0, 1, 0, 2, 0])$

Elliptic Curve defined by $y^2 = x^3 + x^2 + 2 \times x$ over Finite Field of size 5

$E\text{.is_supersingular}()$

True

$E\text{.hasse_invariant}()$

0

$E\text{.trace_of_frobenius}()$

0

$F = E\text{.frobenius}()$

$C = F\text{.absolute_charpoly}()$

$x^2 + 5$

$C\text{.newton_slopes}(5)$

$[1/2, 1/2]$
Examples of supersingular elliptic curves

For all $p$, there exists a supersingular elliptic curve $E/\mathbb{F}_p^2$ (Igusa).

The number of isomorphism classes of ss elliptic curves is $\left\lfloor \frac{p}{12} \right\rfloor + \varepsilon$.

$p = 2$: $y^2 + y = x^3$ (unique)

$p \equiv 3 \text{ mod } 4$: $y^2 = x^3 - x$

$p \equiv 2 \text{ mod } 3$: $y^2 = x^3 + 1$

$p$ odd: $y^2 = h(x)$, where $h(x)$ cubic with distinct roots, is supersingular iff the coefficient $c_{p-1}$ of $x^{p-1}$ in $h(x)^{(p-1)/2}$ is zero.

This coefficient vanishes iff Cartier operator trivializes $\frac{dx}{y} \in H^0(E, \Omega^1)$.

$$C\left(\frac{dx}{y}\right) = C\left(\frac{y^{p-1}dx}{y^p}\right) = \frac{1}{y} C\left(h(x)^{(p-1)/2}\right)dx = \frac{c_{p-1}^{1/p}dx}{y}.$$  

$y^2 = x(x - 1)(x - \lambda)$ is supersingular for $\frac{p-1}{2}$ choices of $\lambda \in \overline{\mathbb{F}}_p$ (Igusa).
Let $E$ be a smooth elliptic curve over $k = \overline{k}$, with $\text{char}(k) = p$. Let $E[p]$ be the kernel of the inseparable multiplication-by-$p$ morphism.

$E$ is **supersingular** if it satisfies the following equivalent conditions:

A. The only $p$-torsion point is the identity: $E[p](k) = \{\text{id}\}$.

B. The Newton polygon of $E$ is a line segment of slope $\frac{1}{2}$.

C. The Cartier operator annihilates $H^0(E, \Omega^1)$.

D. $\text{End}(E)$ non-commutative (order inquat. algebra)
Introduction: different properties when $g > 1$

Let $A$ be a p.p. abelian variety of dimension $g$ over $k = \bar{k}$, $\text{char}(k) = p$. Let $A[p]$ be the kernel of the inseparable multiplication-by-$p$ morphism.

The following conditions are all different for $g \geq 3$.

A. $p$-rank 0 - The only $p$-torsion point is the identity: $A[p](k) = \{\text{id}\}$.

B. supersingular - The Newton polygon of $A$ is a line of slope $\frac{1}{2}$.

C. superspecial - The Cartier operator annihilates $H^0(X, \Omega^1)$.

Then $C \Rightarrow B \Rightarrow A$ but $A \nRightarrow B \nRightarrow C$

Question: if $g \geq 2$, do these occur for Jacobian of smooth $k$-curve?
Curves of higher genus

Let $k = \overline{\mathbb{F}}_p$ (an algebraically closed field of char. $p$).

Let $X$ be a (smooth projective connected) curve over $k$.

Recall: everything you learned about Riemann surfaces ($\mathbb{C}$-curves).

Analogous structures: e.g., functions, differentials, Jacobians.

More complicated definitions: e.g., genus is $g = \dim(H^0(X, \Omega_1))$ rather than 'the number of holes'.

Guideline:

Most facts not involving the number $p$ are still true.
Most facts involving the number $p$ are now false.
Suppose $X$ is a curve. The genus is $g = \dim(H^0(X, \Omega_1))$.

If $g \geq 2$, there is no natural group law on the points of $X$.

(Recall, define group structure on points of a complex curve by integrating holomorphic differentials and taking quotient by lattice of periods: $J_x = \Omega^1(X)^*/H^1(X, \mathbb{Z}) \cong \mathbb{C}^g/\mathbb{L}$. Its $p$-torsion points satisfy $J_x[p](\mathbb{C}) \cong (\mathbb{Z}/p)^{2g}$.)

Now Jacobian $J_X$ of $X$ is $\text{Pic}^0(X)$ (line bundles of deg 0) or $\text{Div}^0(X)/\text{PDiv}(X)$ (divisors of deg 0 mod principal divisors).

Then $J_X$ is a principally polarized abelian variety of dimension $g$. 
B. Definition of Newton polygon

Let $X$ be a smooth projective curve defined over $\mathbb{F}_q$, with $q = p^a$. Zeta function of $X$ is $Z(X/\mathbb{F}_q, t) = L(X/\mathbb{F}_q, t)/(1 - t)(1 - qt)$

where $L(X/\mathbb{F}_q, t) = \prod_{i=1}^{2g}(1 - w_i t) \in \mathbb{Z}[t]$ and $|w_i| = \sqrt{q}$.

The Newton polygon of $X$ is the NP of the $L$-polynomial $L(t)$. Find $p$-adic valuation $v_i$ of coefficient of $t^i$ in $L(t)$. Draw lower convex hull of $(i, v_i/a)$ where $q = p^a$.

**Facts:** The NP goes from $(0, 0)$ to $(2g, g)$. NP line segments break at points with integer coefficients; If slope $\lambda$ occurs with length $m_\lambda$, so does slope $1 - \lambda$.

**Definition**

$X/\mathbb{F}_q$ is **supersingular** if the Newton polygon of $L(X/\mathbb{F}_q, t)$ is a line segment of slope $1/2$. 
B. Definition of Newton polygon

Let $A$ be a p.p. abelian variety of dimension $g$ over $k$.

**Manin:** for $c, d$ relatively prime s.t. $\lambda = \frac{c}{d} \in \mathbb{Q} \cap [0, 1]$, define a $p$-divisible group $G_{c,d}$ of dimension $c$ and height $d$.

The Dieudonné module $D_{\lambda}$ for $G_{c,d}$ is a $W(k)$-module. Over $\text{Frac}(W(k))$, there is a basis $x_1, \ldots, x_d$ for $D_{\lambda}$ s.t. $F^d x_i = p^c x_i$.

There is an isogeny of $p$-divisible groups $A[p^\infty] \sim \bigoplus_{\lambda} G_{c,d}^{m_\lambda}$.

Newton polygon: lower convex hull - line segments slope $\lambda$, length $m_\lambda$.

**Definition:** A *supersingular* iff $\lambda = \frac{1}{2}$ is the only slope.

There is a partial ordering on NPs; the supersingular NP is ’smallest’. 

Let $X$ be a smooth projective curve defined over $\mathbb{F}_q$, with $q = p^a$. The following are equivalent:

1. $X$ is supersingular;
2. the Newton polygon of $L(X/\mathbb{F}_q, T)$ is a line segment of slope $1/2$;
3. each eigenvalue of the relative Frobenius morphism equals $\zeta \sqrt{q}$ for some root of unity $\zeta$;
4. $X$ is minimal (satisfies lower bound in Hasse-Weil bound for number of points) over $\mathbb{F}_{q^r}$ for some $r$;
5. Tate: $\text{End}(\text{Jac}(X \times_{\mathbb{F}_q} k)) \otimes \mathbb{Q}_p \simeq M_g(D_p)$, $D_p$ quat alg ram at $p, \infty$;
6. Oort: $\text{Jac}(X)$ is geometrically isogenous to a product of supersingular elliptic curves.
Motivation for studying supersingular curves

* maximal and minimal curves (supersingular) yield good error-correcting Goppa codes;

* abelian varieties with complex multiplication are often supersingular, useful in cryptography;

* good signature schemes built using supersingular curves;

* supersingular curves play a key role in geometric proofs about stratifications of $A_g$ by Newton polygon type (or EO type).
Example: Hermitian curves are supersingular

Let $q = p^n$. The *Hermitian curve* $X_q$ has affine equation $y^q + y = x^{q+1}$.

It has genus $g = q(q - 1)/2$.

It is maximal over $\mathbb{F}_{q^2}$ because $\#X_q(\mathbb{F}_{q^2}) = q^3 + 1$.

**Ruck/Stichtenoth:** $X_q$ is unique curve of genus $g$ maximal over $\mathbb{F}_{q^2}$.

**Hansen:** $X_q$ is the Deligne-Lusztig variety for $\text{Aut}(X_q) = \text{PGU}(3, q)$.

The $L$-polynomial of $X_q$ is $L(X_q/\mathbb{F}_q, t) = (1 + qt^2)^g$.

The only slope of the Newton polygon of $L(X_q/\mathbb{F}_q, t)$ is $1/2$.

Thus $\text{Jac}(X_q)$ is supersingular.
Which Newton polygons occur for Jacobians?

For all $p$ and $g$, there exists:
a supersingular p.p. *abelian variety* of dimension $g$, namely $E^g$;
and a supersingular *singular* curve of genus $g$.

Open question:
Does there exist a supersingular smooth curve of genus $g$ defined over a finite field of characteristic $p$, for every $p$ and $g$?

More generally, which Newton polygons occur for Jacobians of smooth curves?

For $g = 1$ both, $g = 2$ all three, $g = 3$ all five.

Let $\mathcal{A}_g$ be the moduli space of p.p. abelian varieties of dimension $g$.
The image of $M_g$ in $\mathcal{A}_g$ is open and dense for $g \leq 3$. 
Open question for $g = 4$:

For all $p$, does there exist a smooth curve of genus 4 which is supersingular? or whose NP has slopes $1/3, 1/2, 2/3$?

\[ \sigma_4 \leftarrow \nu_3^0 \oplus \sigma_1 \quad \nu_1^1 \oplus \sigma_3 \leftarrow \nu_1^1 \oplus \nu_3^0 \leftarrow \nu_4^2 \leftarrow \nu_4^3 \leftarrow \nu_4^4 \]

*: don’t know if this NP occurs for Jacobian of smooth curve for all $p$
*: this NP occurs but some components may have problems
*: each component has good geometric properties.

(Katz, Oort, Faber/Van der Geer, Pries, Achter-Pries)
Do all NPs occur for Jacobians? Guess - unlikely?

Observation (Oort 2005) \( \dim(A_g) = g(g + 1)/2 \) and the dimension of the supersingular locus \( A_g[\sigma_g] \) is \( \lfloor g^2/4 \rfloor \).

The difference \( \delta_g \) is length of longest chain of NPs connecting the supersingular NP \( \sigma_g \) to the ordinary NP \( \nu_g \).

If \( g \geq 9 \), then \( \delta_g > 3g - 3 = \dim(M_g) \).

Either (i) \( M_g \) does not admit a perfect stratification by NP (i.e., there are two NPs \( \xi_1 \) and \( \xi_2 \) such that \( A_g[\xi_1] \) is in the closure of \( A_g[\xi_2] \) but \( M_g[\xi_1] \) is not in the closure of \( M_g[\xi_2] \).)

or (ii) some NPs do not occur for Jacobians of smooth curves.

Test case: \( g = 11 \) with NP \( G_{5,6} \oplus G_{6,5} \) having slopes of \( 5/11, 6/11 \) (does occur when \( p = 2 \) - Blache).
Do all NPs occur for Jacobians? Evidence?

Only non-existence results are for curves with automorphisms:
Bouw 2001: Not all \( p \)-ranks occur for cyclic degree \( d > 2 \) covers

Especially, not all NPs occur for wildly ramified covers:
Deuring-Shafarevich formula restricts \( p \)-rank.
Oort: If \( p = 2 \), there does not exist a hyp. ss curve of genus 3.
Scholten/Zhu: \( p = 2, n \geq 2 \), there is no hyp. ss curve with \( g = 2^n - 1 \).
(for odd \( p \), generalized for Artin-Schreier covers \( X \xrightarrow{\mathbb{Z}/p} \mathbb{P}^1 \) by Blache)

But.....

Van der Geer & Van der Vlugt: If \( p = 2 \), then there exists a supersingular curve of every genus.
Step one of proof by VdG/VdV

Def: \( R[x] \in k[x] \) is an additive polynomial if \( R(x_1 + x_2) = R(x_1) + R(x_2) \). Then \( R[x] = c_0 x + c_1 x^p + c_2 x^{p^2} + c_h x^{p^h} \).

Supersingular Artin-Schreier curves

If \( R[x] \in k[x] \) is an additive polynomial of degree \( p^h \), then \( X : y^p - y = xR[x] \) is supersingular with genus \( p^h(p - 1)/2 \).

Proof: Induction on \( h \), starting with \( h = 0 \).
Key fact: \( \text{Jac}(X) \) is isogenous to a product of Jacobians of Artin-Schreier curves for additive polynomials of smaller degree.

Remark: Bouw et al studied \( L \)-polynomials, automorphism groups of \( X \).
Remark: Blache studied first slope of NP of more general AS curves.
Existence of supersingular curves when \( p = 2 \)

Van der Geer and Van der Vlugt

If \( p = 2 \), then there exists a supersingular curve over \( \overline{F}_2 \) of every genus.

**Proof sketch:** Expand \( g \) as (with \( s_i \leq s_{i-1} + r_{i-1} + 2 \))

\[
g = 2^{s_1}(1 + 2 + \cdots + 2^{r_1}) + 2^{s_2}(1 + 2 + \cdots 2^{r_2}) + \cdots + 2^{s_t}(1 + 2 + \cdots + 2^{r_t}).
\]

Let \( L = \bigoplus_{i=1}^t L_i \) for \( L_i \) subspace of dim \( d_i := r_i + 1 \) in vector space of additive polynomials of deg \( 2^{u_i} \), with \( u_i = (s_i + 1) - \sum_{j=1}^{i-1} (r_j + 1) \).

If \( f \in L \), let \( C_f : y^p - y = xf \). Let \( Y \) be fiber product of \( C_f \to \mathbb{P}^1 \) for all \( f \in L \). Then \( J_Y \sim \bigoplus_{f \neq 0} J_{C_f} \) (thus supersingular). Also, \( g_Y = \sum_{f \neq 0} g_{C_f} \).

The number of \( f \in L \) which have a non-zero contribution from \( L_i \), but not from \( L_j \) for \( j > i \), is \( (2^{d_i} - 1) \prod_{j=1}^{i-1} 2^{d_j} \). Each adds \( 2^{u_i-1} \) to \( g \).

So \( g_Y = \sum_{i=1}^t (2^{d_i} - 1) \prod_{j=1}^{i-1} 2^{d_j} 2^{u_i-1} = \sum_{i=1}^t 2^{s_i}(1 + \cdots + 2^{r_i}) = g \).
Here is what VdG/VdV’s method produces for odd $p$.

**Karemaker/P**

Let $g = Gp(p - 1)^2/2$ where $G = \sum_{i=1}^{t} p^{s_i}(1 + p + \cdots p^{r_i})$. Then there exists a supersingular curve over $\mathbb{F}_p$ of genus $g$.

Can this be improved?

VdG/VdV also prove that there exists a supersingular curve defined over $\mathbb{F}_2$ of every genus. The construction is a little more complicated.
Related question: the $p$-rank of $X$

If $X$ is a smooth $k$-curve of genus $g$,

**Fact/Def:**

then $|J_X[p](k)| = p^f$ for some integer $0 \leq f \leq g$ called the $p$-rank of $X$.

Also, $f = \dim_{\mathbb{F}_p} \text{Hom}(\mu_p, J_X[p])$ where

$\mu_p \simeq \text{Spec}(k[x]/(x^p - 1))$ is the kernel of Frobenius on $\mathbb{G}_m$.

Let $L(t)$ be the $L$-polynomial of the zeta function of an $\mathbb{F}_q$-curve $X$.

The $p$-rank of $X$ is the length of the slope 0 portion of $\text{NP}(X)$.

$X$ is supersingular if all slopes of $\text{NP}(X)$ equal $1/2$. $X$ supersingular implies $X$ has $p$-rank 0 but converse false for $g \geq 3$. 

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Existence of curves with given genus and \( p \)-rank

Let \( g \in \mathbb{N}, 0 \leq f \leq g \) and \( p \) prime.

The moduli space \( \mathcal{M}_g \) (resp. \( \mathcal{H}_g \)) of (hyperelliptic) curves of genus \( g \) can be stratified by \( p \)-rank into strata \( \mathcal{M}_g^f \) (resp. \( \mathcal{H}_g^f \)) whose points represent (hyperelliptic) curves of genus \( g \) and \( p \)-rank \( f \).

Theorem: Faber/Van der Geer

Every component of \( \mathcal{M}_g^f \) has dimension \( 2g - 3 + f \); there exists a smooth curve over \( \overline{\mathbb{F}}_p \) with genus \( g \) and \( p \)-rank \( f \).

Theorem: Glass/P (\( p \) odd), P/Zhu (\( p \) even)

Every component of \( \mathcal{H}_g^f \) has dimension \( g - 1 + f \); there exists a smooth hyp. curve over \( \overline{\mathbb{F}}_p \) with genus \( g \) and \( p \)-rank \( f \).

In most cases, it is not known whether \( \mathcal{M}_g^f \) and \( \mathcal{H}_g^f \) are irreducible.
Let $A/k$ be a p.p. abelian variety of dimension $g$.

**Fact:** If $A$ is supersingular, then $A$ has $p$-rank 0.

If $g \in \{1, 2\}$ and $A$ has $p$-rank 0, then $A$ is supersingular. If $g \geq 3$ and $A$ has $p$-rank 0, then $A$ usually not supersingular.

**Example:** Let $j \in \mathbb{N}$ with $p \nmid j$ and $h(x) \in k[x]$ of degree $j$. The curve $X : y^q + y = h(x)$ has genus $g = (q - 1)(j - 1)/2$. Deuring-Shafarevich formula: $\text{Jac}(X)$ has $p$-rank 0.

**Zhu:** Let $q = 2$ and $j = 2^{n+1} - 1$, none of the 2-rank 0 curves $y^2 + y = h(x)$ are supersingular.
Moduli of curves: supersingular versus $p$-rank 0

**Oort:** There exists a hyperelliptic curve of genus 3 with $p$-rank 0 which is not supersingular.

proof: study intersection of two codim 1 conditions in $M_3^0$.

**Application - Achter/P.** Let $g \geq 3$ and $p \neq 2$ for hyperelliptic

The generic point of any component of the $p$-rank 0 strata $M_g^0$ and $H_g^0$ is not supersingular.

$A \nRightarrow B$ for curves:
if $g \geq 3$, there exists a (hyperelliptic) curve of genus $g$ with $p$-rank 0 which is not supersingular.
Newton polygon results for $f = g - 3$ and $f = g - 4$

For $g \geq 4$ and $g - 2 \leq f \leq g$, the $p$-rank determines the Newton polygon (and so that Newton polygon occurs, open and dense in $\mathcal{M}_f^g$).

Let $\nu_{g,f} = f(\mathcal{G}_{0,1} + \mathcal{G}_{1,0}) + (\mathcal{G}_{1,g-f-1} + \mathcal{G}_{g-f-1,1})$.

**Application - Achter/P.** Let $g \geq 3$ and $f = g - 3$.

The generic point of each component of $\mathcal{M}_g^{g-3}$ has Newton polygon $\nu_{g,g-3}$ (slopes $0, \frac{1}{3}, \frac{2}{3}, 1$).

**Application - Achter/P.** Let $g \geq 4$ and $f = g - 4$.

The generic point of at least one component of $\mathcal{M}_g^f$ has Newton polygon $\nu_{g,g-4}$ (slopes $0, \frac{1}{4}, \frac{3}{4}, 1$).

**Note:** When $g = 4$, there is at most one component of $\mathcal{M}_4^0$ whose generic NP is not $\nu_{4,0}$. If so, the NP has slopes $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$.
Proof: inductive strategy, reduce to \( p \)-rank \( f = 0 \)

Let \( \nu_r \) be a NP type with \( p \)-rank 0 occurring in dimension \( r \).

Let \( c_r = \text{codim}(\mathcal{A}_g[\nu_r], \mathcal{A}_g) \).

For \( g \geq r \), let \( \nu_g \) be the NP type with \( p \)-rank \( g - r \) \('containing'\) \( \nu_r \)

\( (\nu_g = (G_{0,1} \oplus G_{1,0})^{g-r} \oplus \nu_r) \), add \( g - r \) slopes of 0, 1.

Proposition P

If there exists a component \( S_r \) of \( \mathcal{M}_r[\nu_r] \) s.t. \( \text{codim}(S_r, \mathcal{M}_r) = c_r \),

then, for all \( g \geq r \),

there exists a component \( S_g \) of \( \mathcal{M}_g[\nu_g] \) s.t. \( \text{codim}(S_g, \mathcal{M}_g) = c_r \).