p-adic heights on Jacobians of hyperelliptic curves II

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Coleman-Gross p-adic heights

Choices

We've seen the following: p-adic heights on elliptic curves over number fields beyond $\mathbf{Q}$ require a choice of idele class character.

However, going from elliptic curves to Jacobians of higher genus curves requires an additional choice: in particular, if our p-adic height is to be symmetric, we must choose a certain direct sum decomposition of

$$H^1_{dR}(X) = H^0(X, \Omega^1) \oplus W,$$

i.e., a choice of W such that W is isotropic with respect to the cup product pairing.

Since we have chosen p to be an ordinary prime, there is a canonical choice of W: the unit root subspace for the action of Frobenius.

Coleman-Gross p-adic height pairing

Then the Coleman-Gross p-adic height pairing is a symmetric bilinear pairing

$$h : \mathrm{Div}^0(X) \times \mathrm{Div}^0(X) \to \mathbf{Q}_p, \qquad \text{where}$$

- h can be decomposed into a sum of local height pairings $h = \sum_v h_v$ over all finite places v of $\mathbf{Q}$.
- $h_v(D, E)$ is defined for $D, E \in \mathrm{Div}^0(X \times \mathbf{Q}_v)$ with disjoint support.
- We have $h(D, \mathrm{div}(\beta)) = 0$ for $\beta \in k(X)^\times$, so h is well-defined on $J \times J$.
- The local pairings h_v can be extended (non-uniquely) such that $h(D) := h(D, D) = \sum_v h_v(D, D)$ for all $D \in \mathrm{Div}^0(X)$.
- We fix a certain extension and write $h_v(D) := h_v(D, D)$.

Local height pairings

We consider the global height pairing h as a sum of (finitely many) local height pairings $h = \sum h_v$; Coleman-Gross achieve a description of these local heights solely in terms of the curve.

Construction of h_v depends on whether $v = p$ or $v \neq p$.

- $v \neq p$: arithmetic intersection theory, as in Müller's lectures
- $v = p$: logarithms, Coleman integration of normalized differentials of the third kind (p-adic Green's functions); in particular,

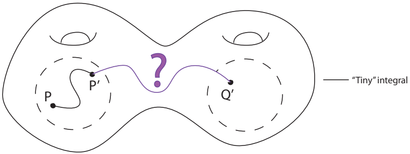
$$h_p(D, E) = \int_E \omega_D$$

for ω_D a certain differential of the third kind with $\mathrm{Res}(\omega_D) = D$. This is a Coleman integral.

Coleman integration

p-adic line integrals

A Coleman integral is a p-adic *line integral*.



p-adic line integration is difficult – how do we construct the correct path?

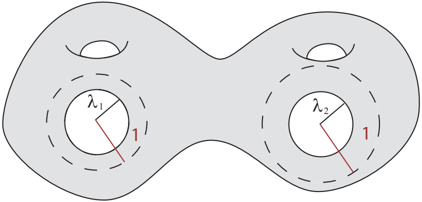
- We can construct local ("tiny") integrals easily, but extending them to the entire space is challenging.
- Coleman's solution: *analytic continuation along Frobenius*, giving rise to a theory of p-adic line integration satisfying the usual nice properties

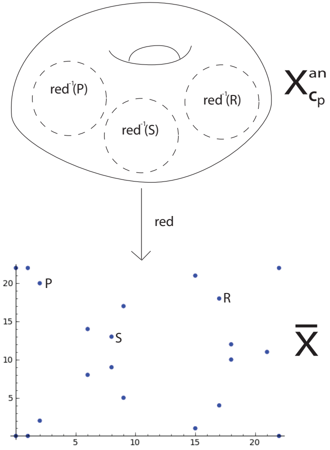
Notation and setup

- X: genus g hyperelliptic curve (of the form $y^2 = f(x)$, f monic of degree $2g+1$) over $K = \mathbf{Q}_p$
- $\overline{X}$: special fibre of X
- $X^{\mathrm{an}}_{\mathbf{C}_p}$: generic fibre of X (as a rigid analytic space)

Notation and setup, in pictures

- There is a natural reduction map from $X^{\mathrm{an}}_{\mathbf{C}_p}$ to $\overline{X}$; the inverse image of any point of $\overline{X}$ is a subspace of $X^{\mathrm{an}}_{\mathbf{C}_p}$ isomorphic to an open unit disk. We call such a disk a *residue disk* of X.
- A *wide open subspace* of $X^{\mathrm{an}}_{\mathbf{C}_p}$ is the complement in $X^{\mathrm{an}}_{\mathbf{C}_p}$ of the union of a finite collection of disjoint closed disks of radius $\lambda_i < 1$:





Warm-up: Computing "tiny" integrals

We refer to any Coleman integral of the form $\int_P^Q \omega$ in which P, Q lie in the same residue disk (so $P \equiv Q \pmod{p}$) as a *tiny integral*. To compute such an integral:

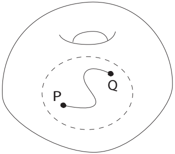
- Construct a linear interpolation from P to Q. For instance, in a non-Weierstrass residue disk, we may take

$$x(t) = (1-t)x(P) + tx(Q)$$
$$y(t) = \sqrt{f(x(t))},$$

where $y(t)$ is expanded as a formal power series in t.

- Formally integrate the power series in t:

$$\int_P^Q \omega = \int_0^1 \omega(x(t), y(t))\, dt.$$



Properties of the Coleman integral

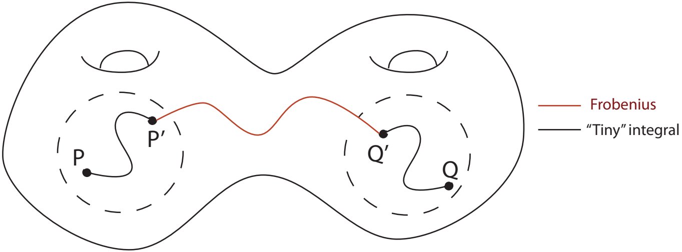
Coleman formulated an integration theory, allowing us to define $\int_P^Q \omega$ whenever ω is a meromorphic 1-form on X, and $P, Q \in X(\mathbf{Q}_p)$ are points where ω is holomorphic.
Properties of the Coleman integral include:

Theorem (Coleman)

- *Linearity:* $\int_P^Q (\alpha\omega_1 + \beta\omega_2) = \alpha \int_P^Q \omega_1 + \beta \int_P^Q \omega_2$.
- *Additivity:* $\int_P^R \omega = \int_P^Q \omega + \int_Q^R \omega$.
- *Change of variables: if X' is another such curve, and $f : U \to U'$ is a rigid analytic map between wide opens, then* $\int_P^Q f^*\omega = \int_{f(P)}^{f(Q)} \omega$.
- *Fundamental theorem of calculus:* $\int_P^Q df = f(Q) - f(P)$.

Coleman's construction

How do we integrate if P, Q aren't in the same residue disk?
Coleman's key idea: use Frobenius to move between different residue disks (Dwork's "analytic continuation along Frobenius")



So we need to calculate the action of Frobenius on differentials.

Frobenius, MW-cohomology

- X': affine curve ($X - \{$ Weierstrass points of $X\}$)
- A: coordinate ring of X'

To discuss the differentials we will be integrating, we recall:
The *Monsky-Washnitzer (MW) weak completion of A* is the ring $A^\dagger$ consisting of infinite sums of the form

$$\left\{ \sum_{i=-\infty}^{\infty} \frac{B_i(x)}{y^i},\ B_i(x) \in K[x], \deg B_i \leqslant 2g \right\},$$

further subject to the condition that $v_p(B_i(x))$ grows faster than a linear function of i as $i \to \pm\infty$. We make a ring out of these using the relation $y^2 = f(x)$.
These functions are holomorphic on wide opens, so we will integrate 1-forms

$$\omega = g(x,y)\frac{dx}{2y}, \quad g(x,y) \in A^\dagger.$$

Using the basis differentials

Any odd differential $\omega = h(x,y)\frac{dx}{2y}, h(x,y) \in A^\dagger$ can be written as

$$\omega = df_\omega + c_0\omega_0 + \cdots + c_{2g-1}\omega_{2g-1},$$

where $f_\omega \in A^\dagger$, $c_i \in \mathbf{Q}_p$ and

$$\omega_i = \frac{x^i\,dx}{2y} \qquad (i = 0, \ldots, 2g-1).$$

The set $\{\omega_i\}_{i=0}^{2g-1}$ forms a basis of the odd part of the de Rham cohomology of $A^\dagger$.

By linearity and the fundamental theorem of calculus, we reduce the integration of ω to the integration of the ω_i.

Integrals between points in different residue disks

Let ϕ denote a lift of p-power Frobenius:

- On a hyperelliptic curve $y^2 = f(x)$,

$$\phi : (x, y) \mapsto (x^p, \sqrt{f(x^p)}).$$

- A *Teichmüller point* of X is a point P fixed by Frobenius: $\phi(P) = P$.

Integrals between points in different residue disks

One way to compute Coleman integrals $\int_P^Q \omega_i$:

- Find the Teichmüller points P', Q' in the residue disks of P, Q.
- Use Frobenius to compute $\int_{P'}^{Q'} \omega_i$.
- Use additivity in endpoints to recover the integral: $\int_P^Q \omega_i = \int_P^{P'} \omega_i + \int_{P'}^{Q'} \omega_i + \int_{Q'}^{Q} \omega_i$.

The Frobenius step (Kedlaya's algorithm)

We have a p-power lift of Frobenius ϕ on $A^\dagger$:

$$\phi(x) = x^p,$$

$$\phi(y) = y^p\left(1 + \frac{f(x^p) - f(x)^p}{f(x)^p}\right)^{1/2} = y^p \sum_{i=0}^{\infty} \binom{1/2}{i} \frac{(f(x^p) - f(x)^p)^i}{y^{2pi}}$$

Now we use it on $H^1_{MW}(X')^-$; let $\omega_i = \frac{x^i dx}{2y}$.

$$\begin{aligned}
\phi^*(\omega_i) &= \phi^*\left(\frac{x^i dx}{2y}\right) = \frac{x^{pi} d(x^p)}{2\phi(y)} = \frac{x^{pi} p x^{p-1} dx}{2\phi(y)} \\
&= p x^{pi+p-1} y \left(y^{-p} \sum_{i=0}^{\infty} \binom{-1/2}{i} \frac{(f(x^p) - f(x)^p)^i}{y^{2pi}}\right) \frac{dx}{2y} \\
&= df_i + \sum_{j=0}^{2g-1} M_{ij} \omega_j,
\end{aligned}$$

where $f_i \in A^\dagger$.

Frobenius and Coleman integrals (B.-Bradshaw-Kedlaya ('10))

- Use Kedlaya's algorithm to calculate the action of Frobenius ϕ on each basis differential, letting

$$\phi^*\omega_i = df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j.$$

- Compute $\int_{P'}^{Q'} \omega_j$ by solving a linear system

$$\int_{P'}^{Q'} \omega_i = \int_{\phi(P')}^{\phi(Q')} \omega_i = \int_{P'}^{Q'} \phi^*\omega_i = \int_{P'}^{Q'} \left(df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j \right)$$

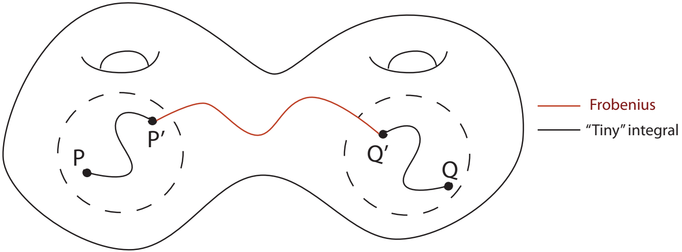
$$\int_{P'}^{Q'} \omega_i = f_i(Q') - f_i(P') + \sum_{j=0}^{2g-1} M_{ij} \int_{P'}^{Q'} \omega_j.$$

- The eigenvalues of M have $\mathbf{C}$-norm $p^{1/2} \neq 1$, so $M - I$ is invertible; solve the system to obtain the integrals $\int_{P'}^{Q'} \omega_i$.

Integrals via Teichmüller, continued

- The linear system gives us the integral between different residue disks.
- Then putting it all together, we have

$$\int_P^Q \omega_i = \int_P^{P'} \omega_i + \int_{P'}^{Q'} \omega_i + \int_{Q'}^{Q} \omega_i$$



Iterated Coleman integrals

There is a generalization to n-fold iterated line integrals:

$$\int_P^Q \omega_n \cdots \omega_1 = \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} f_n(t_n) \cdots f_1(t_1)\, dt_n \cdots dt_1$$

and an algorithm using Frobenius (B., 2013) to compute iterated Coleman integrals.

These iterated Coleman integrals play a key role in Kim's nonabelian Chabauty program.

We focus on the case $n = 2$, and we use the convention

$$\int_P^Q \omega_i \omega_j := \int_P^Q \omega_i(R) \int_P^R \omega_j$$

Tiny double integrals

"Tiny" double integration (points P, Q in the same non-Weierstrass residue disk)

- Compute local coordinate $(x(t), y(t))$ at P.
- Let $R = (a + x(Q),\ \sqrt{f(a + x(Q))})$.
- Write

$$\int_P^Q \omega_i \omega_j = \int_P^Q \omega_i(R) \int_P^R \omega_j$$
$$= \int_0^{x(Q)-x(P)} \left(\int_0^a \frac{x(t)^j dx(t)}{2y(t)} \right) \frac{x(R(a))^i}{2y(R(a))} \frac{dx(R(a))}{da}.$$

Moving between different disks

As before, we can link integrals between non-Weierstrass points via Frobenius.

To compute the integrals $\int_P^Q \omega_i\omega_k$ when P, Q are in different disks:

- Compute Teichmüller points P', Q' in the disks of P, Q.
- Use Frobenius to calculate $\int_{P'}^{Q'} \omega_i\omega_k$.
- Recover the double integral: $\int_P^Q \omega_i\omega_k = \int_{P'}^{Q'} \omega_i\omega_k - \int_{P'}^{P} \omega_i\omega_k - \left(\int_P^Q \omega_i\right)\left(\int_{P'}^{P} \omega_k\right) - \left(\int_Q^{Q'} \omega_i\right)\left(\int_{P'}^{Q'} \omega_k\right) + \int_{Q'}^{Q} \omega_i\omega_k$.

Expanding Frobenius

Suppose P, Q are Teichmüller. We have

$$\int_P^Q \omega_i\omega_k = \int_{\phi(P)}^{\phi(Q)} \omega_i\omega_k$$

$$\int_P^Q \omega_i\omega_k = \int_P^Q (\phi^*\omega_i)(\phi^*\omega_k)$$

$$\int_P^Q \omega_i\omega_k = \int_P^Q \left(df_i + \sum_{j=0}^{2g-1} M_{ij}\omega_j\right)\left(df_k + \sum_{j=0}^{2g-1} M_{kj}\omega_j\right)$$

The linear system

For all $0 \leqslant i,k \leqslant 2g-1$, define the constants c_{ik}:

$$
\begin{aligned}
c_{ik} = \int_P^Q & df_i(R)(f_k(R)) - f_k(P)(f_i(Q) - f_i(P)) \\
& + \int_P^Q \sum_{j=0}^{2g-1} M_{ij}\omega_j(R)(f_k(R) - f_k(P)) \\
& + f_i(Q) \int_P^Q \sum_{j=0}^{2g-1} M_{kj}\omega_j - \int_P^Q f_i(R)\Big(\sum_{j=0}^{2g-1} M_{kj}\omega_j(R)\Big).
\end{aligned}
$$

Then

$$
\begin{pmatrix} \int_P^Q \omega_0\omega_0 \\ \int_P^Q \omega_0\omega_1 \\ \vdots \\ \int_P^Q \omega_{2g-1}\omega_{2g-1} \end{pmatrix} = (I_{4g^2} - (M^t)^{\otimes 2})^{-1} \begin{pmatrix} c_{00} \\ \vdots \\ c_{2g-1,2g-1} \end{pmatrix}.
$$

Quadratic Chabauty

Kim's nonabelian Chabauty program

The aim is to generalize the Chabauty-Coleman method, which says that for a curve $X/\mathbf{Q}$ with rank $J(\mathbf{Q}) < g$, we have

$$X(\mathbf{Q}_p)_1 := \left\{ z \in X(\mathbf{Q}_p) : \int_b^z \omega = 0 \right\}$$

for some $\omega \in H^0(X_{\mathbf{Q}_p}, \Omega^1_X)$. Kim's program is to give further *iterated* p-adic integrals vanishing on rational or integral points on curves by studying *Selmer varieties*, with the hope of *precisely* cutting out rational or integral points.

Explicit examples have been worked out in the case of

- $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ (Dan-Cohen–Wewers, Dan-Cohen)
- Elliptic curve $E \setminus \{O\}$, rk $E(\mathbf{Q})$ is $0, 1$ (Kim, B.–Kedlaya–Kim, B.–Besser, B.–Dan-Cohen–Kim–Wewers, B.–Dogra)
- Genus g hyperelliptic curve $C \setminus \{\infty\}$ or C, where we have rank $J = g$ (B.–Besser–Müller, B.–Dogra)

Quadratic Chabauty

Let $X/\mathbf{Q}$ be a genus g hyperelliptic curve. Given a global p-adic height pairing h, we want to study it on integral points:

$$\underbrace{h}_{\substack{\text{quadratic form, rewrite as a} \\ p\text{-adic analytic function} \\ \text{using Coleman integrals}}} = \underbrace{h_p}_{\substack{p\text{-adic analytic function} \\ \text{via double Coleman integral}}} + \underbrace{\sum_{v \neq p} h_v}_{\substack{\text{takes on finite} \\ \text{number of values} \\ \text{on integral points}}}$$

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Local height at p

By Coleman-Gross, the local height h_p is given in terms of Coleman integration: for $D, E \in \mathrm{Div}^0(X)$ of disjoint support,

$$h_p(D, E) = \int_E \omega_D.$$

Theorem (B.-Besser-Müller)

If $P \in X(\mathbf{Q}_p)$, then $h_p(P - \infty) := h_p(P - \infty, P - \infty)$ is equal to a double Coleman integral

$$\tau(P) := h_p(P - \infty) = \sum_{i=0}^{g-1} \int_\infty^P \omega_i \bar{\omega}_i,$$

where $\{\bar{\omega}_0, \ldots, \bar{\omega}_{g-1}\}$ forms a dual basis to $\{\omega_0, \ldots, \omega_{g-1}\}$ with respect to the cup product pairing on $H^1_{dR}(X/\mathbf{Q}_p)$.

Local heights away from p

If $q \neq p$ then h_q is defined in terms of arithmetic intersection theory on a regular model of X over $\mathrm{Spec}(\mathbf{Z})$.

There is an explicitly computable finite set $T \subset \mathbf{Q}_p$ such that

$$-\sum_{q \neq p} h_q(P - \infty) \in T$$

for integral points $P \in X(\mathbf{Q})$.

Strategy of Quadratic Chabauty

Consider the $\mathbf{Q}_p$-valued functionals $f_i = \int_O \omega_i$ for $0 \leqslant i \leqslant g-1$ on $J(\mathbf{Q})$.

Idea when $\mathrm{rk}(J(\mathbf{Q})) = r = g$:

- Suppose the f_i are linearly independent functionals on $J(\mathbf{Q})$.
- Then $\{f_i f_j\}_{i \leqslant j \leqslant g-1}$ is a natural basis of the space of $\mathbf{Q}_p$-valued quadratic forms on $J(\mathbf{Q})$.
- The p-adic height h is also a quadratic form, so there must exist $\alpha_{ij} \in \mathbf{Q}_p$ such that

$$h = \sum_{i \leqslant j \leqslant g-1} \alpha_{ij} f_i f_j$$

- Linear algebra gives us the global p-adic height in terms of products of Coleman integrals.

Quadratic Chabauty

We use these double and single Coleman integrals to rewrite the global p-adic height pairing h and to study it on integral points:

$$\underbrace{h}_{\text{quadratic form, rewrite as a } p\text{-adic analytic function using Coleman integrals}} = \underbrace{h_p}_{p\text{-adic analytic function via double Coleman integral}} + \underbrace{\sum_{v \neq p} h_v}_{\text{takes on finite number of values on integral points}}$$

$$\underbrace{h_p}_{p\text{-adic analytic function via double Coleman integral}} - \underbrace{h}_{\text{quadratic form, rewrite as a } p\text{-adic analytic function using Coleman integrals}} = -\underbrace{\sum_{v \neq p} h_v}_{\text{takes on finite number of values on integral points}}$$

Quadratic Chabauty

Theorem (B.-Besser-Müller)

If $r = g \geqslant 1$ and the f_i are independent, then there is an explicitly computable finite set $T \subset \mathbf{Q}_p$ and explicitly computable constants $\alpha_{ij} \in \mathbf{Q}_p$ such that

$$\rho(P) := \tau(P) - \sum_{0 \leqslant i \leqslant j \leqslant g-1} \alpha_{ij} f_i f_j(P)$$

takes values in T on integral points.

Main strategy of quadratic Chabauty:

p-adic heights $\rightsquigarrow$ p-adic integrals $\rightsquigarrow$ p-adic power series (set equal to a finite set of constants)

Then solve and produce a finite set of points containing integral points!

Rational points for bielliptic genus 2 curves

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Let K be $\mathbf{Q}$ or a quadratic imaginary number field, X/K be given by

$$y^2 = x^6 + ax^4 + bx^2 + c$$

and let

$$E_1 : y^2 = x^3 + ax^2 + bx + c \qquad E_2 : y^2 = x^3 + bx^2 + acx + c^2,$$

with maps

$$f_1 : \begin{array}{ccc} X & \longrightarrow & E_1 \\ (x, y) & \mapsto & (x^2, y) \end{array} \qquad f_2 : \begin{array}{ccc} X & \longrightarrow & E_2 \\ (x, y) & \mapsto & (cx^{-2}, cyx^{-3}). \end{array}$$

Theorem (B.-Dogra '16)

Let X/K be as above and suppose E_1 and E_2 each have rank 1. We can carry out quadratic Chabauty to recover a finite set of p-adic points containing $X(K)$.

Details (*all* the p-adic heights)

Theorem (B.-Dogra '16)

Then X/K be a genus 2 bielliptic curve as before. Then $X(K)$ is contained in the finite set of z in $X(K_{\mathfrak{p}})$ satisfying

$$\begin{aligned}\rho(z) = & 2h_{E_2,\mathfrak{p}}(f_2(z)) - h_{E_1,\mathfrak{p}}(f_1(z) + (0, \sqrt{c})) - h_{E_1,\mathfrak{p}}(f_1(z) + (0, -\sqrt{c})) \\ & - 2\alpha_2 \log_{E_2}(f_2(z))^2 + 2\alpha_1(\log_{E_1}(f_1(z))^2 + \log_{E_1}((0, \sqrt{c}))^2) \\ & \in \Omega,\end{aligned}$$

where Ω is the finite set of values

$$\left\{ \sum_{v \nmid p} \left(h_{E_1,v}(f_1(z) + (0, \sqrt{c})) + h_{E_1,v}(f_1(z) + (0, -\sqrt{c})) - 2h_{E_2,v}(f_2(z)) \right) \right\},$$

for (z_v) in $\prod_{v \nmid p} X(K_v)$, and where $\alpha_i = \frac{h_{E_i}(P_i)}{[K:\mathbf{Q}] \log_{E_i}(P_i)^2}$.

Example : Computing $X_0(37)(\mathbf{Q}(i))$

Consider

$$X_0(37) : y^2 = -x^6 - 9x^4 - 11x^2 + 37.$$

We have $\mathrm{rk}(J_0(37)(\mathbf{Q}(i))) = 2$.

Change models and use

$$X : y^2 = x^6 - 9x^4 + 11x^2 + 37,$$

which is isomorphic to $X_0(37)$ over $K = \mathbf{Q}(i)$; we have $\mathrm{rk}(J(\mathbf{Q})) = \mathrm{rk}(J(\mathbf{Q}(i))) = 2$.

Define

$$E_1 : y^2 = x^3 - 16x + 16 \qquad E_2 : y^2 = x^3 - x^2 - 373x + 2813$$

and maps from X

$$\begin{array}{cccc} f_1 : & X & \longrightarrow & E_1 \\ & (x,y) & \mapsto & (x^2-3, y) \end{array} \qquad \begin{array}{cccc} f_2 : & X & \longrightarrow & E_2 \\ & (x,y) & \mapsto & (37x^{-2}+4, 37yx^{-3}). \end{array}$$

Take P_1 and P_2 to be points of infinite order in $E_1(\mathbf{Q})$ and $E_2(\mathbf{Q})$.

$X_0(37)(\mathbf{Q}(i))$, continued

We compute

$$\begin{aligned}\rho(z) = 2h_{E_2,\mathfrak{p}}(f_2(z)) - h_{E_1,\mathfrak{p}}(f_1(z) + (-3, \sqrt{37})) \\ - h_{E_1,\mathfrak{p}}(f_1(z) + (-3, -\sqrt{37})) \\ - 2\alpha_2 h_{E_2}(f_2(z)) + 2\alpha_1(h_{E_1}(f_1(z)) + \log_{E_1}((-3, \sqrt{37}))^2)\end{aligned}$$

and find that points $z \in X(\mathbf{Q}(i))$ satisfy

$$\rho(z) = \frac{4}{3}\log_p(37).$$

Taking $p = 41, 73, 101$, we use ρ to produce points in $X(\mathbf{Q}_{41}), X(\mathbf{Q}_{73}), X(\mathbf{Q}_{101})$.

Recovered points in $X(\mathbf{Q}_{41})$

$X(\mathbf{F}_{41})$	recovered $x(z)$ in residue disk	$z \in X(K)$
$\overline{(1,9)}$	$1 + 16 \cdot 41 + 23 \cdot 41^2 + 5 \cdot 41^3 + 23 \cdot 41^4 + O(41^5)$	
	$1 + 6 \cdot 41 + 23 \cdot 41^2 + 30 \cdot 41^3 + 14 \cdot 41^4 + O(41^5)$	
$\overline{(2,1)}$	$2 + O(41^5)$	$(2,1)$
	$2 + 19 \cdot 41 + 36 \cdot 41^2 + 15 \cdot 41^3 + 26 \cdot 41^4 + O(41^5)$	
$\overline{(4,18)}$		
$\overline{(5,12)}$	$5 + 25 \cdot 41 + 26 \cdot 41^2 + 26 \cdot 41^3 + 31 \cdot 41^4 + O(41^5)$	
	$5 + 14 \cdot 41 + 12 \cdot 41^3 + 33 \cdot 41^4 + O(41^5)$	
$\overline{(6,1)}$	$6 + 18 \cdot 41^2 + 31 \cdot 41^3 + 6 \cdot 41^4 + O(41^5)$	
	$6 + 30 \cdot 41 + 35 \cdot 41^2 + 11 \cdot 41^3 + O(41^5)$	
$\overline{(7,15)}$		
$\overline{(9,4)}$	$9 + 9 \cdot 41 + 34 \cdot 41^2 + 22 \cdot 41^3 + 24 \cdot 41^4 + O(41^5)$	$(i,4)$
	$9 + 39 \cdot 41 + 14 \cdot 41^2 + 6 \cdot 41^3 + 17 \cdot 41^4 + O(41^5)$	
$\overline{(12,5)}$		
$\overline{(13,19)}$	$13 + 10 \cdot 41 + 2 \cdot 41^2 + 15 \cdot 41^3 + 29 \cdot 41^4 + O(41^5)$	
	$13 + 7 \cdot 41 + 8 \cdot 41^2 + 32 \cdot 41^3 + 14 \cdot 41^4 + O(41^5)$	
$\overline{(16,1)}$	$16 + 13 \cdot 41 + 6 \cdot 41^3 + 18 \cdot 41^4 + O(41^5)$	
	$16 + 12 \cdot 41 + 8 \cdot 41^2 + 9 \cdot 41^3 + 32 \cdot 41^4 + O(41^5)$	
$\overline{(17,20)}$	$17 + 24 \cdot 41 + 37 \cdot 41^2 + 16 \cdot 41^3 + 28 \cdot 41^4 + O(41^5)$	
	$17 + 19 \cdot 41 + 20 \cdot 41^2 + 7 \cdot 41^3 + 7 \cdot 41^4 + O(41^5)$	
$\overline{(18,20)}$	$18 + 3 \cdot 41 + 7 \cdot 41^2 + 9 \cdot 41^3 + 38 \cdot 41^4 + O(41^5)$	
	$18 + 41 + 34 \cdot 41^2 + 3 \cdot 41^3 + 32 \cdot 41^4 + O(41^5)$	
$\overline{(19,3)}$		
$\overline{(20,6)}$	$20 + 7 \cdot 41 + 40 \cdot 41^2 + 22 \cdot 41^3 + 7 \cdot 41^4 + O(41^5)$	
	$20 + 23 \cdot 41 + 26 \cdot 41^2 + 17 \cdot 41^3 + 22 \cdot 41^4 + O(41^5)$	
$\overline{\infty^+}$	∞^+	∞^+
$\overline{(0,18)}$	$32 \cdot 41 + 13 \cdot 41^2 + 16 \cdot 41^3 + 8 \cdot 41^4 + O(41^5)$	
	$9 \cdot 41 + 27 \cdot 41^2 + 24 \cdot 41^3 + 32 \cdot 41^4 + O(41^5)$	

Recovered points in $X(\mathbf{Q}_{73})$

$X(\mathbf{F}_{73})$	recovered $x(z)$ in residue disk	$z \in X(K)$ (or $X(\mathbf{Q}(\sqrt{3}))$)
$\overline{(2,1)}$	$2 + 61 \cdot 73 + 50 \cdot 73^2 + 71 \cdot 73^3 + 56 \cdot 73^4 + O(73^5)$	
	$2 + O(73^5)$	$(2,1)$
$\overline{(5,26)}$	$5 + 63 \cdot 73 + 4 \cdot 73^2 + 42 \cdot 73^3 + 25 \cdot 73^4 + O(73^5)$	
	$5 + 39 \cdot 73 + 65 \cdot 73^2 + 33 \cdot 73^3 + 60 \cdot 73^4 + O(73^5)$	
$\overline{(7,16)}$	$7 + 62 \cdot 73 + 31 \cdot 73^2 + 33 \cdot 73^3 + 44 \cdot 73^4 + O(73^5)$	
	$7 + 29 \cdot 73 + 67 \cdot 73^2 + 69 \cdot 73^3 + 17 \cdot 73^4 + O(73^5)$	
$\overline{(9,34)}$		
$\overline{(10,30)}$	$10 + 53 \cdot 73 + 35 \cdot 73^2 + 21 \cdot 73^3 + 67 \cdot 73^4 + O(73^5)$	
	$10 + 39 \cdot 73 + 40 \cdot 73^2 + 17 \cdot 73^3 + 59 \cdot 73^4 + O(73^5)$	
$\overline{(18,17)}$		
$\overline{(19,2)}$		
$\overline{(20,15)}$		
$\overline{(21,4)}$	$21 + 17 \cdot 73 + 70 \cdot 73^2 + 42 \cdot 73^3 + 18 \cdot 73^4 + O(73^5)$	
	$21 + 52 \cdot 73 + 67 \cdot 73^2 + 20 \cdot 73^3 + 27 \cdot 73^4 + O(73^5)$	$(\sqrt{3},4)$
$\overline{(23,31)}$	$23 + 18 \cdot 73 + 59 \cdot 73^2 + 23 \cdot 73^3 + 2 \cdot 73^4 + O(73^5)$	
	$23 + 70 \cdot 73 + 53 \cdot 73^2 + 21 \cdot 73^3 + 50 \cdot 73^4 + O(73^5)$	
$\overline{(25,25)}$		
$\overline{(27,4)}$	$27 + 62 \cdot 73 + 28 \cdot 73^2 + 56 \cdot 73^3 + 58 \cdot 73^4 + O(73^5)$	$(i,4)$
	$27 + 24 \cdot 73 + 30 \cdot 73^2 + 20 \cdot 73^3 + 65 \cdot 73^4 + O(73^5)$	
$\overline{(29,8)}$	$29 + 70 \cdot 73 + 21 \cdot 73^2 + 56 \cdot 73^3 + 5 \cdot 73^4 + O(73^5)$	
	$29 + 34 \cdot 73 + 42 \cdot 73^2 + 19 \cdot 73^3 + 54 \cdot 73^4 + O(73^5)$	
$\overline{(30,20)}$		
$\overline{(36,17)}$	$36 + 70 \cdot 73 + 19 \cdot 73^2 + 11 \cdot 73^3 + 54 \cdot 73^4 + O(73^5)$	
	$36 + 32 \cdot 73 + 23 \cdot 73^2 + 23 \cdot 73^3 + 28 \cdot 73^4 + O(73^5)$	
$\overline{\infty^+}$	∞^+	∞^+
$\overline{(0,16)}$	$61 \cdot 73 + 63 \cdot 73^2 + 51 \cdot 73^3 + 16 \cdot 73^4 + O(73^5)$	
	$12 \cdot 73 + 9 \cdot 73^2 + 21 \cdot 73^3 + 56 \cdot 73^4 + O(73^5)$	

Recovered points in $X(\mathbf{Q}_{101})$

$X(\mathbf{F}_{101})$	recovered $x(z)$ in residue disk	$z \in X(K)$
$\overline{(2,1)}$	$2 + O(101^7)$	$(2,1)$
	$2 + 38 \cdot 101 + 11 \cdot 101^2 + 99 \cdot 101^3 + 26 \cdot 101^4 + O(101^5)$	
$\overline{(8,36)}$	$8 + 90 \cdot 101 + 39 \cdot 101^2 + 80 \cdot 101^3 + 70 \cdot 101^4 + O(101^5)$	
	$8 + 40 \cdot 101 + 84 \cdot 101^2 + 74 \cdot 101^3 + 15 \cdot 101^4 + O(101^5)$	
$\overline{(10,4)}$	$10 + 5 \cdot 101 + 29 \cdot 101^2 + 66 \cdot 101^3 + 10 \cdot 101^4 + O(101^5)$	$(i,4)$
	$10 + 49 \cdot 101 + 80 \cdot 101^2 + 74 \cdot 101^3 + 8 \cdot 101^4 + O(101^5)$	
$\overline{(12,7)}$	$12 + 12 \cdot 101 + 95 \cdot 101^2 + 55 \cdot 101^3 + 48 \cdot 101^4 + O(101^5)$	
	$12 + 36 \cdot 101 + 62 \cdot 101^2 + 97 \cdot 101^3 + 27 \cdot 101^4 + O(101^5)$	
$\overline{(14,21)}$	$14 + 62 \cdot 101 + 62 \cdot 101^2 + 41 \cdot 101^3 + 51 \cdot 101^4 + O(101^5)$	
	$14 + 80 \cdot 101 + 72 \cdot 101^2 + 32 \cdot 101^3 + 75 \cdot 101^4 + O(101^5)$	
$\overline{(15,11)}$		
$\overline{(17,18)}$	$17 + 65 \cdot 101 + 37 \cdot 101^2 + 80 \cdot 101^3 + 45 \cdot 101^4 + O(101^5)$	
	$17 + 50 \cdot 101 + 61 \cdot 101^2 + 89 \cdot 101^3 + 61 \cdot 101^4 + O(101^5)$	
$\overline{(18,45)}$		
$\overline{(20,47)}$		
$\overline{(22,3)}$	$22 + 59 \cdot 101 + 78 \cdot 101^2 + 43 \cdot 101^3 + 53 \cdot 101^4 + O(101^5)$	
	$22 + 96 \cdot 101 + 29 \cdot 101^2 + 43 \cdot 101^3 + 86 \cdot 101^4 + O(101^5)$	
$\overline{(24,19)}$		
$\overline{(27,39)}$		
$\overline{(28,37)}$	$28 + 30 \cdot 101 + 83 \cdot 101^2 + 5 \cdot 101^3 + 23 \cdot 101^4 + O(101^5)$	
	$28 + 37 \cdot 101 + 24 \cdot 101^2 + 78 \cdot 101^3 + 35 \cdot 101^4 + O(101^5)$	

Recovered points in $X(\mathbf{Q}_{101})$, continued

$X(\mathbf{F}_{101})$	recovered $x(z)$ in residue disk	$z \in X(K)$
$\overline{(30,46)}$		
$\overline{(31,23)}$	$31 + 23 \cdot 101 + 11 \cdot 101^2 + 67 \cdot 101^3 + 39 \cdot 101^4 + O(101^5)$	
	$31 + 29 \cdot 101 + 68 \cdot 101^2 + 29 \cdot 101^3 + 24 \cdot 101^4 + O(101^5)$	
$\overline{(34,45)}$	$34 + 91 \cdot 101 + 46 \cdot 101^2 + 28 \cdot 101^3 + 34 \cdot 101^4 + O(101^5)$	
	$34 + 51 \cdot 101 + 73 \cdot 101^2 + 34 \cdot 101^3 + 14 \cdot 101^4 + O(101^5)$	
$\overline{(37,22)}$		
$\overline{(38,28)}$		
$\overline{(39,46)}$	$39 + 76 \cdot 101 + 86 \cdot 101^2 + 18 \cdot 101^3 + 64 \cdot 101^4 + O(101^5)$	
	$39 + 31 \cdot 101 + 43 \cdot 101^2 + 10 \cdot 101^3 + 48 \cdot 101^4 + O(101^5)$	
$\overline{(46,6)}$		
$\overline{(47,32)}$		
$\overline{(48,27)}$	$48 + 43 \cdot 101 + 100 \cdot 101^2 + 47 \cdot 101^3 + 19 \cdot 101^4 + O(101^5)$	
	$48 + 21 \cdot 101 + 38 \cdot 101^2 + 80 \cdot 101^3 + 95 \cdot 101^4 + O(101^5)$	
$\overline{(50,5)}$	$50 + 59 \cdot 101 + 19 \cdot 101^2 + 64 \cdot 101^3 + 36 \cdot 101^4 + O(101^5)$	
	$50 + 74 \cdot 101 + 69 \cdot 101^2 + 80 \cdot 101^3 + 21 \cdot 101^4 + O(101^5)$	
$\overline{\infty^+}$	∞^+	∞^+
$\overline{(0,21)}$		

Putting it together and computing $X_0(37)(\mathbf{Q}(i))$

Steffen Müller carried out the Mordell-Weil sieve on the sets of points found in $X(\mathbf{Q}_{41}), X(\mathbf{Q}_{73})$, and $X(\mathbf{Q}_{101})$; conclusion:

$$X(\mathbf{Q}(i)) = \{(\pm 2 : \pm 1 : 1), (\pm i : \pm 4 : 1), (1 : \pm 1 : 0)\},$$

or in other words,

$$X_0(37)(\mathbf{Q}(i)) = \{(\pm 2i : \pm 1 : 1), (\pm 1 : \pm 4 : 1), (i : \pm 1 : 0)\}.$$

Note: the computation of points in $X(\mathbf{Q}_{73})$ recovered the points $(\pm\sqrt{-3}, \pm 4) \in X_0(37)(\mathbf{Q}(\sqrt{-3}))$ as well!